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POLYNOMIAL HULLS AND ANALYTIC DISCS

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ABSTRACT. The goal of the present note is to construct a class of examples for connected compact sets $K \subset \mathbb{C}^n$ whose polynomial hull $\widehat{K}$ cannot be covered by analytic discs with boundaries contained in an arbitrarily small neighborhood of K. This gives an answer to a recent question raised by B. Drinovec Drnovšek and F. Forstnerič.

1. INTRODUCTION

At least since Wermer's example of polynomial hulls without complex structure, the intriguing problem to give a *geometric description* of the polynomial hull of a compact $K \subset \mathbb{C}^n$ has been continuously attracting the interest of complex analysts and symplectic geometers. Today two general approaches are known. One due to Duval and Sibony [4] relies on plurisuperharmonic currents, and the other, due to Poletsky [12] and independently to Bu and Schachermayer [2], uses analytic discs (see also [16] for a link between the two approaches).

The main point in Poletsky's construction is to work with discs mapping the unit circle to a given neighborhood of K except for some set of arbitrarily small length. This leads to the question to which extent one can restrict to discs with full boundary close to K. To this end, we define for a compact $K \subset \mathbb{C}^n$ its *disc hull* $\widehat{K}_{\mathsf{disc}}$ as the set of all points $z \in \mathbb{C}^n$ such that for every $\varepsilon > 0$ there is an analytic disc $f \in A(\mathbb{D}, \mathbb{C}^n)$ satisfying $f(0) = z$ and $f(\mathbb{T}) \subset K_\varepsilon$. Here $A(\mathbb{D}, \mathbb{C}^n)$ is the space of mappings $f : \overline{\mathbb{D}} \to \mathbb{C}^n$ which are holomorphic on the unit disc $\mathbb{D}$ and continuous up to the boundary $\mathbb{T}$, and $K_\epsilon = \bigcup_{z \in K} B_\varepsilon(z)$ is the ε-neighborhood of K. Obviously $\widehat{K}_{\mathsf{disc}}$ is contained in the polynomial hull

$$\widehat{K} = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |p(z)| \le \max_K |p| \text{ for every } p \in \mathbb{C}[z_1, \dots, z_n]\},$$

but the converse does not hold. As observed in [3] (see [13] for another example), it suffices to consider the union T of two circles T_1, T_2 lying in the unit sphere S in $\mathbb{C}^2$ and bounding a holomorphic annulus $A \subset \mathbb{B}^2$ (for example $T = \{z_1 z_2 = 1/2\} \cap S$). Indeed, if $f_j \in A(\mathbb{D}, \mathbb{C}^2)$ is a sequence satisfying $f_j(0) = z_0 \in A$ and $f_j(\mathbb{T}) \subset T_{1/j}$, then a subsequence has its boundaries close to one of the circles, say T_1, meaning that z_0 lies in the polynomial hull of T_1. This contradicts the polynomial convexity of T_1, which follows from Stolzenberg's theorem [14]. Hence we have that

$$\widehat{T}_{\mathsf{disc}} = T \subsetneq \widehat{T}. \tag{1}$$

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The question which compacts satisfy $\widehat{K}_{\mathsf{disc}} = \widehat{K}$ turns out to be subtle. In [3], B. Drinovec Drnovšek and F. Forstnerič show equality for those connected compacts K that are invariant with respect to the natural action of the unit circle $\mathbb{T} \subset \mathbb{C}$ on $\mathbb{C}^n$, and raise the question whether this remains true for every *connected* compact K [3, Question 5.4]. The essence of this note is a class of counter-examples leading to a negative answer to the question.

Theorem 1.1. *Let K be a compact set in the boundary S of the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$, $n \geq 2$, with finitely many connected components $K_1, \ldots, K_m$. Then there is a connected compact $L \subset \mathbb{C}^n$ such that $L \cap \overline{\mathbb{B}^n} = K$, $\widehat{L} = L \cup \widehat{K}$ and $\widehat{L}_{\mathsf{disc}} = L \cup \widehat{K}_{\mathsf{disc}}$.*

The theorem together with the example T in (1) yields the existence of a connected compact which coincides with its disc hull but has nontrivial polynomial hull.

In Theorem 1.1, the assumption that K is contained in the unit sphere can be weakened: Our proof goes through if it lies in a smooth level set of a strictly pseudoconvex exhaustion function of $\mathbb{C}^n$.

2. Proof of Theorem 1.1

Step 1 (Construction of L with required polynomial hull). We start by connecting the components $K_1, \ldots, K_m$ of K by $\mathcal{C}^1$-arcs: For $\mu = 1, \ldots, m-1$, pick points $z_\mu^- \in K_\mu$ and $z_\mu^+ \in K_{\mu+1}$, and choose $\mathcal{C}^1$-smooth imbeddings $\gamma_\mu : [0,1] \to \mathbb{C}^n \setminus \mathbb{B}^n$ satisfying $\gamma_\mu(0) = z_\mu^-$, $\gamma_\mu(1) = z_\mu^+$ and $\Gamma_\mu = \gamma_\mu((0,1)) \subset \mathbb{C}^n \backslash \overline{\mathbb{B}^n}$. For dimensional reasons we may arrange the open arcs Γ_μ to be pairwise disjoint.

By construction $L = K \cup \bigcup_{\mu=1}^{m-1} \Gamma_\mu$ satisfies $L \cap \overline{\mathbb{B}^n} = K$. By the following lemma we may, after an appropriate deformation of the Γ_μ, also assume that $\widehat{L} = L \cup \widehat{K}$.

Lemma 2.1. *Given $0 < a_\mu < b_\mu < 1$, $\mu = 1, \ldots, m-1$, there are $\mathcal{C}^1$-embeddings $\gamma_{\mu,1} : [0,1] \to \mathbb{C}^n \backslash \mathbb{B}^n$ arbitrarily $\mathcal{C}^1$-close to γ_μ, coinciding with γ_μ on $[0,1] \backslash (a_\mu, b_\mu)$, and such that*

$$L_1 = K \cup \bigcup_{\mu=1}^{m-1} \gamma_{\mu,1}([0,1])$$

satisfies $\widehat{L}_1 = L_1 \cup \widehat{K}$.

Generic polynomial convexity for totally real submanifolds of dimension $k < n$ is studied in [5, 6, 10]. Lemma 2.1 is a relative variant of these results for $k = 1$. Our proof exploits the method of "separation of hulls" introduced in [5]; see also [8] for a similar construction.

Proof. If $\gamma_\mu(t) = \gamma_\mu(t_\mu) + \gamma'_\mu(t_\mu)(t - t_\mu) + o(|t - t_\mu|)$ holds for some $t_\mu \in (a_\mu, b_\mu)$, we can $\mathcal{C}^1$-approximate γ_μ by an embedding $\gamma_{\mu,0}(t)$ that coincides with $\gamma_\mu(t)$ for $t \in [0,1] \backslash (a_\mu, b_\mu)$ and with $\gamma_\mu(t_\mu) + \gamma'_\mu(t_\mu)(t - t_\mu)$ for t close to t_μ. Let

$$H_\mu = \{z \in \mathbb{C}^n : \langle v_\mu, z - \gamma_\mu(t_\mu) \rangle = 0\}, \text{ where } 0 \neq v_\mu \in \mathbb{C}^n,$$

be a complex hyperplane containing the line parametrised by $\gamma_\mu(t_\mu) + \gamma'_\mu(t_\mu)(t - t_\mu)$. Picking $a'_\mu < b'_\mu < a''_\mu < b''_\mu$ sufficiently close to t_μ and $c_\mu \in \mathbb{C}^*$ of small modulus, we can $\mathcal{C}^1$-approximate $\gamma_{\mu,0}(t)$ by $\gamma_{\mu,1}(t)$ satisfying $\gamma_{\mu,1}(t) \in H_\mu$ for $a'_\mu < t < b'_\mu$ and $\gamma_{\mu,1}(t) \in \{z \in \mathbb{C}^n : \langle v_\mu, z - \gamma_\mu(t_\mu) \rangle = c_\mu\}$ for $a''_\mu < t < b''_\mu$.

The inclusion $L_1 \cup \widehat{K} \subset \widehat{L}_1$ being obvious, we consider an arbitrary $z_0 \in \widehat{L}_1$. Let $\mathcal{A}$ be the uniform algebra of those continuous functions on L_1 that are obtained as uniform limits of restrictions of entire functions. There is a Jensen measure for z_0, i.e. a probability measure on L_1, such that

$$\log|f(z_0)| \le \int \log|f|\,d\lambda \tag{2}$$

holds for every $f \in \mathcal{A}$; see [15]. Applying this to $f_\mu(z) = \langle v_\mu, z - \gamma_\mu(t_\mu)\rangle$ we get that $z_0 \in \{\langle v_\mu, z - \gamma_\mu(t_\mu)\rangle = 0\}$ holds if $\lambda(\gamma_{\mu,1}([a'_\mu, b'_\mu])) > 0$ (meaning that the right side of (2) becomes $-\infty$). Together with the analogous argument for $\langle v_\mu, z - \gamma_\mu(t_\mu)\rangle - c_\mu$, it follows that λ can charge at most one of the open arcs $\gamma_{\mu,1}((a'_\mu, b'_\mu))$ and $\gamma_{\mu,1}((a''_\mu, b''_\mu))$. For every μ, we let J_μ be one of these arcs on which λ vanishes (note that this choice may depend on z_0) and set

$$L'_1 = K \cup \bigcup_{\mu=1}^{m-1} (\gamma_{\mu,1}([0,1]) \setminus J_\mu).$$

Since $L'_1 \cup \widehat{K}$ retracts to $\widehat{K}$, the inclusion $\widehat{K} \hookrightarrow L'_1 \cup \widehat{K}$ induces an isomorphism between cohomology groups $H^1(L'_1 \cup \widehat{K}, \mathbb{Z}) \to H^1(\widehat{K}, \mathbb{Z})$ (here Čech cohomology is adequate; see [15] for detailed information). Hence $L'_1 \cup \widehat{K}$ is polynomially convex by the cohomological part of the theorem of Stolzenberg [14], and we get $z_0 \in L'_1 \cup \widehat{K}$. The proof of Lemma 2.1 is complete. □

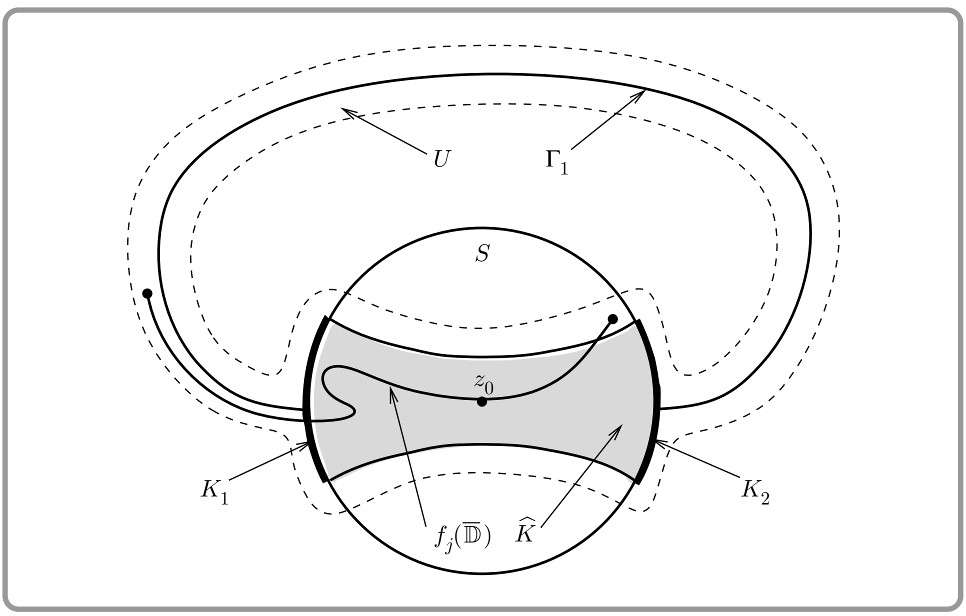

FIGURE 1. $m = 2$, $L = K_1 \cup K_2 \cup \Gamma_1$

Step 2 (Proof that $\widehat{L}_{\mathsf{disc}} = L \cup \widehat{K}_{\mathsf{disc}}$). Obviously $\widehat{L}_{\mathsf{disc}} \supset L \cup \widehat{K}_{\mathsf{disc}}$. To prove the converse, consider $z_0 \in \widehat{L}_{\mathsf{disc}} \setminus L$. Then there is a sequence $f_j \in A(\mathbb{D}, \mathbb{C}^n)$ with $f_j(0) = z_0$ and $f_j(\mathbb{T}) \subset L_{1/j}$. The idea is to construct another sequence with boundaries tending to K by selecting suitable restrictions of the f_j. Viewed geometrically, we will cut away the pieces of $f_j(\overline{\mathbb{D}})$ that leave $\mathbb{B}^n$. Of course, the main point is to make sure that we get a topological disc.

By general position arguments we may assume that

a) the f_j are holomorphic near $\overline{\mathbb{D}}$,
b) the derivatives f_j' do not vanish along $\mathbb{T}$,
c) the images $f_j(\mathbb{T})$ are embedded circles transverse to $S = \partial\mathbb{B}^n$,
d) the full images $f_j(\overline{\mathbb{D}})$ intersect S transversally along the image of the set $\{\zeta \in \overline{\mathbb{D}} : f_j'(\zeta) \neq 0\}$.

Actually we could simplify a little more by pushing the circles $f_j(\mathbb{T})$ off $\overline{\mathbb{B}^n}$, but this would make the argument depend more on the convexity of $\mathbb{B}^n$.

Lemma 2.2. *Let $H \subset \mathbb{C}^n$ be compact and polynomially convex, and let $g_j \in A(\mathbb{D}, \mathbb{C}^n)$ be a sequence such that $g_j(\mathbb{T}) \subset H_{\varepsilon_j}$ and $\varepsilon_j \to 0$. Then for every neighborhood U of H, the images $g_j(\overline{\mathbb{D}})$ are contained in U provided j is large enough.*

Proof. Using an exhaustion function $\phi \in PSH^\infty(\mathbb{C}^n)$ which is zero on H and positive elsewhere, we find a Runge neighborhood basis $\{U_j\}$ of H. Since a disc with boundary in some U_j is contained in U_j, the lemma follows. □

Step 2 will be complete as soon as we have constructed, for prescribed $\varepsilon > 0$, a disc f_ε with $f_\varepsilon(0) = z_0$ and $\operatorname{dist}(f_\varepsilon(\mathbb{T}), K) < \varepsilon$. Figure 1 gives an overview of the geometry.

Note first that $\widehat{L} \cap S = \widehat{K} \cap S = K$, where the first equality follows from Step 2 and the second from the existence of peak functions for every boundary point of the ball. As in the proof of the preceding lemma, we can thus choose a Runge neighborhood U of $\widehat{L}$ such that

$$U \cap S \subset K_\varepsilon \cap S.$$

By Lemma 2.2 there is $f = f_{j_0}$ satisfying $f(\overline{\mathbb{D}}) \subset U$ and $f(\mathbb{T}) \subset L_\varepsilon$.

Since $\widehat{L}_{\mathsf{disc}} \subset \widehat{L}$ and $\widehat{L} \setminus L = \widehat{K} \cap \mathbb{B}$ holds by Step 1, we have $z_0 \in \mathbb{B}^n$. Let G be the connected component of $(f|_{\mathbb{D}})^{-1}(\mathbb{B}^n)$ containing the origin. Then the boundary of G is a finite collection of analytic curves by **(a-d)**, and G is simply connected by the maximum principle (applied to $|f|^2$). By the Riemann and Carathéodory mapping theorems there is a conformal mapping $\varphi : G \to \mathbb{D}$, $\varphi(0) = 0$, extending continuously to the boundary. Hence the disc $f_\varepsilon = f \circ \varphi^{-1}$ has the desired properties, and the proof of Theorem 1.1 is complete. □

3. Hulls of open sets

In conclusion, we briefly relate our result to hulls of *open* sets. For a domain $D \subset \mathbb{C}^n$, its *Runge hull* is defined in [11] by

$$\widehat{D} = \bigcup_{K \subset D,\ K \text{ compact}} \widehat{K}.$$

A modification of the constructions above shows that $\widehat{D}$ is in general not the union of analytic discs with boundary in D. More precisely, we start again from the example $T \subset \partial\mathbb{B}^2$ discussed before Theorem 1.1 and connect the two circles by an arc Γ as in the proof of the theorem. For every sufficiently thin neighborhood D of the resulting set, the annulus $\{z_1 z_2 = 1/2\} \cap \mathbb{B}^2$ is contained in $\widehat{D}$ but cannot be covered by discs with boundary in D. For further work on related questions we refer to the recent article [9].

Note that the situation becomes very different for *envelopes of holomorphy*: For the envelope of holomorphy $\pi : \mathsf{E}(D) \to \mathbb{C}^n$ of a domain $D \subset \mathbb{C}^n$, it was proved in [7] that $\mathsf{E}(D)$ is covered by the liftings of holomorphic discs with boundary in D (actually [7] even proves the existence of discs that can be contracted to a point in D via discs projecting to discs with boundary in D).

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