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QUANTITATIVE RECURRENCE FOR GENERIC HOMEOMORPHISMS

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Abstract. In this article we study quantitative recurrence for generic measure preserving homeomorphisms on euclidian spaces with respect to Lebesgue measure and compact manifolds with respect to Oxtoby-Ulam measures (i.e., it is nonatomic and positive on each open set). As an application we show that the decay of correlations of generic measure preserving homeomorphisms on compact manifolds is slow.

1. Introduction

Poincaré's recurrence theorem states that, in a dynamical system preserving a probability measure on a Polish metric space (i.e., a complete separable metric space), almost every orbit returns as closely as you wish to the initial point. To be more precise, for $\mu-$ almost ever $x \in X$, one has

$$\liminf_{n\to+\infty} d(T^n(x), x) = 0.$$

Now, if a dynamical system preserves a probability measure μ on a Polish metric space and is ergodic, then almost every orbit hit as closely as you wish on a fixed point y on the support of the measure. To be more precise, for $\mu-$ almost ever $x \in X$, one has

$$\liminf_{n\to+\infty} d(T^n(x), y) = 0.$$

Later, Boshernitzan [Bo] proved that in very general conditions

$$\liminf_{n\to\infty} n^\beta d(T^n(x), x) < +\infty,$$

for almost every point with respect to an invariant probability measure and for some $\beta > 0$. This result means that the speed of recurrence is not too slow with respect to n^β. So, a natural question is if the recurrence can be arbitrarily fast and the answer is yes as we will see in Theorem A. On the other hand, Bonanno, Galatolo, and Isola [Bon] proved under very general conditions that

$$\liminf_{n\to\infty} n^\alpha d(T^n(x), y) = +\infty$$

for almost every point with respect to an invariant probability measure and for some $\alpha > 0$. This means that the speed of hitting is not too fast with respect to the sequence n^α. So, a natural question is if the hitting can be arbitrarily slow

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and the answer is yes as we will see in Theorem B. We also investigate the same question but in a noncompact setting as we will see in Theorem C.

2. Preliminaries

2.1. **Compact manifolds.** Let M be a compact connected Riemannian manifold with no boundary. Let us define the measures to be used throughout this paper.

Definition 2.1. We say that a Borel probability measure on M is an OU(Oxtoby-Ulam) *measure* if:

(1) μ is *nonatomic*. This means that it is zero on singleton sets.
(2) μ is *locally positive*. This means that it is positive on every nonempty open set.

If μ is a Borel probability measure on a Polish metric space X and $T : X \to X$ is a measurable map, then we say that T preserves μ (or μ is invariant under T) if $\mu(T^{-1}(A)) = A$ for all Borel sets A. We say that μ is *ergodic* (with respect to T) if $T^{-1}(A) = A$ implies that $\mu(A) = 0$ or $\mu(A^c) = 0$. Given Borel sets A, B in M, we say that $A \subset B(mod\ \mu)$ if $\mu(A - B) = 0$. If μ is a probability measure, then we define the *support* of μ as:

$$supp(\mu) = \{x \in X : \mu(U) > 0 \text{ for some open set U containing } x\}.$$

Now, let us define the main space of this work.

Definition 2.2. If μ is an OU probability measure on M, then we define:

$$\mathcal{M}[M, \mu] := \{T : M \to M : T \text{ is a homeomorphism preserving } \mu\}$$

with the *uniform topology* induced by the metric:

$$||T - S|| := ess\, sup_{x \in M} d(T(x), S(x)).$$

The space $\mathcal{M}[M, \mu]$ is not complete with this metric but there exists an equivalent metric making it complete. Then we can apply Baire's theorem for $\mathcal{M}[M, \mu]$. See [AP] for more details. A set which is the countable intersection of open sets is called a G_δ set. We shall call a subset $\mathcal{R} \subset \mathcal{M}[M, \mu]$ *generic* if it contains a dense G_δ set. It follows from Baire's theorem that a generic subset is dense. Now, let us remember the classical Poincaré recurrence theorem.

Theorem 2.3 (Poincaré)**.** *Let (X, d) be a Polish metric space and $T : X \to X$ a measurable map preserving a Borel probability measure μ on X. Then:*

(1) $\liminf_{n \to +\infty} d(T^n(x), x) = 0$ *for μ a.e. $x \in X$.*
(2) *If $y \in supp(\mu)$, then* $\liminf_{n \to +\infty} d(T^n(x), y) = 0$ *for μ a.e. $x \in X$.*

So, an important problem is to study the speed of those limits.

2.2. **Euclidian spaces.** Let us denote by $\mathbb{R}^n$ the *n-dimensional* Euclidian space and λ the Lebesgue measure.

Definition 2.4. Let us define the following space:

$$\mathcal{M}[\mathbb{R}^n, \lambda] := \{T : \mathbb{R}^n \to \mathbb{R}^n : T \text{ is a homeomorphism preserving } \lambda\}.$$

In this space we put the compact-open topology, where the basic open sets are given by:

$$\mathcal{C}(T, K, \delta) := \{S \in \mathcal{M}[\mathbb{R}^n, \lambda] : |T(x) - S(x)| < \delta \text{ for } \lambda \text{ a.e. } x \in K\},$$

where $K \subset \mathbb{R}^n$ is a compact set, $\delta > 0$, and $T \in \mathcal{M}[\mathbb{R}^n, \lambda]$. This space can be metrized as follows: Let K_i be a sequence of compact sets whose union is X. Then the compact-open topology is induced by the complete metric

$$\Gamma(T, S) = \sum_{i=1}^{+\infty} \frac{\gamma_{K_i}(T, S)}{1 + \gamma_{K_i}(T, S)},$$

where

$$\gamma_K(T, S) = \max\left(\max_{x \in K} |T(x) - S(x)|, \max_{x \in K} |T^{-1}(x) - S^{-1}(x)|\right).$$

Remark 2.5. The metric space $\mathcal{M}[\mathbb{R}^n, \lambda]$ is complete and then we can apply Baire's theorem. In particular we can define generic sets as before. Ergodicity of Lebesgue measure also can be defined as before.

Poincaré's recurrence theorem also holds in this setting but we need an additional hypothesis.

Definition 2.6. We say that a measurable map T on $(\mathbb{R}^n, \lambda)$ is conservative if E is always a Borel set such that $\{T^{-n}(E)\}_{n \geq 0}$ are disjoint. Then we have that $\lambda(E) = 0$.

Now, let us remember Poincaré's recurrence theorem in this setting.

Theorem 2.7 ([Aa]). *Let T be a conservative measurable map and invariant with respect to Lebesgue measure on $\mathbb{R}^n$. Then:*

$$\liminf_{n \to +\infty} d(T^n(x), x) = 0,$$

for λ a.e. $x \in \mathbb{R}^n$.

It is a classical result that ergodicity is a generic property in $\mathcal{M}[\mathbb{R}^n, \lambda]$. To be more precise we have:

Theorem 2.8 ([AP]). *There exists a residual set $\mathcal{U} \subset \mathcal{M}[\mathbb{R}^n, \lambda]$ such that if $T \in \mathcal{U}$, then λ is ergodic with respect to T.*

It is possible that Poincaré's recurrence theorem does not hold for ergodic measure preserving transformations when the measure is infinite (see e.g. [Aa, p. 22]). In our setting, however, this does not happen. To be more precise we have:

Theorem 2.9 ([AP]). *An invertible ergodic measure preserving transformation of a nonatomic measure space (i.e., each singleton set has measure 0) is necessarily conservative.*

Then, we have that Poincaré's recurrence theorem holds in a generic set of $\mathcal{M}[\mathbb{R}^n, \lambda]$. So, it is interesting to study the speed of recurrence as before.

2.3. **Decay of correlations and recurrence.** Let us begin with the definition of local dimension of a measure.

Definition 2.10. Let μ be a Borel probability measure on M. Then we define the local dimension of μ in $x \in M$ as:

$$d_\mu(x) := \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r},$$

if this limit there exists.

Now, let us recall the definition of superpolynomial decay of correlations. This notion is deeply connected with quantitative recurrence of dynamical systems; see [Ga] for more details.

Definition 2.11. Let $\phi, \psi : M \to \mathbb{R}$ be Lipschitz functions on M. If (M, T, μ) is a measure preserving system, then we say that T has a superpolynomial decay of correlations if:

$$\left| \int_X (\phi \circ T^n) \psi d\mu - \int_X \phi d\mu \int_X \psi d\mu \right| \leq ||\phi|| ||\psi|| \theta_n,$$

where $\lim_{n \to +\infty} \theta_n n^p = 0$ for all $p > 0$ and $||.||$ is the Lipschitz norm given by:

$$||\phi|| = \max\{||\phi||_\infty, ||\phi||_\eta\},$$

where $||\phi||_\infty$ is the supremum norm,

$$||\phi||_\eta = sup \left\{ \frac{|\phi(s) - \phi(t)|}{\eta(s,t)} : s, t \in M \text{ and } s \neq t \right\},$$

and η is the Riemannian metric of M.

We need to state Galatolo's result that will be important later. Before that let us remember the definition of strongly Borel-Cantelli. This notion also is connected with quantitative recurrence; see [GK].

Definition 2.12. If (M, T, μ) is a measure preserving system, then we say that a sequence of sets $S_n \subset M$ is strongly Borel-Cantelli if $\sum_{n=1}^{+\infty} \mu(S_n) = +\infty$ and for $\mu-$a.e. $x \in X$ we have:

$$\lim_{N \to +\infty} \frac{\sum_{n=1}^{N} 1_{T^{-n}(S_n)}(x)}{\sum_{n=1}^{N} \mu(S_n)} = 1.$$

Remark 2.13. Note that if $S_n \subset M$ is strongly Borel-Cantelli, then:

$$\mu(\{x \in M : T^n(x) \in S_n(x) \text{ for infinitely many } n\}) = 1.$$

This property is known as Borel-Cantelli.

Theorem 2.14 ([Ga])**.** *Let us suppose that (M, T, μ) has superpolynomial decay of correlations and there exists $y \in M$ such that $d_\mu(y) > 0$. If $0 < \beta < \frac{1}{d_\mu(y)}$, then the sequence of balls $S_n = B(y, n^{-\beta})$ is strongly Borel-Cantelli.*

3. Main statements

Throughout this section, M will denote a compact connected Riemannian manifold with no boundary and μ an OU measure on M. In what follows, $Y = (Y, d)$ will denote a metric space and $f : M \to Y$ a continuous map. Such maps f are known as observables. They are important because in many situations, experimentalists are interested in the evolution of some quantities (temperature, pressure, mass,...) under a dynamical system. Such quantities can be thought of as an observable. Quantitative recurrence under an observable has recently been studied (see e.g. [Ro] and [RS]).

Theorem A. *Let (r_n) be a sequence in $\mathbb{R}^+$ such that $\lim_{n\to+\infty} r_n = +\infty$, (Y, d) a metric space, and $f : M \to Y$ a continuous map. Then there exists a generic set $\mathcal{R} \subset \mathcal{M}[M, \mu]$ such that if $T \in \mathcal{R}$, then:*

$$\liminf_{n\to+\infty} r_n d(f(T^n(x)), f(x)) < +\infty \text{ for } \mu \text{ a.e. } x \in M.$$

Theorem B. *Let (r_n) be a sequence in $\mathbb{R}^+$ such that $\lim_{n\to+\infty} r_n = +\infty$, (Y, d) a metric space, $y \in M$, and $f : M \to Y$ a continuous map such that $\mu(f^{-1}(\{f(y)\}) = 0$. Then there exists a generic set $\mathcal{G} \subset \mathcal{M}[M, \mu]$ such that if $T \in \mathcal{G}$, then:*

$$\liminf_{n\to+\infty} r_n d(f(T^n(x)), f(y)) = +\infty \text{ for } \mu \text{ a.e. } x \in M.$$

Theorem C. *Let (r_n) be a sequence in $\mathbb{R}^+$ such that $\lim_{n\to+\infty} r_n = +\infty$. Then there exists a generic set $\mathcal{M} \subset \mathcal{M}[\mathbb{R}^n, \lambda]$ such that if $T \in \mathcal{M}$, then:*

$$\liminf_{n\to+\infty} r_n d(T^n(x), x) < +\infty \text{ for } \lambda \text{ a.e. } x \in \mathbb{R}^n.$$

To finish we have a consequence on the decay of correlations.

Corollary A. *Let us suppose that there exists $y \in M$ such that $d_\mu(y) > 0$. Then there exists a generic set $\mathcal{T} \subset \mathcal{M}[M, \mu]$ such that if $T \in \mathcal{T}$, then the decay of correlations of T with respect to μ is not superpolynomial.*

4. Proofs

Proof (Theorem A). Let us begin with the following lemma:

Lemma 4.1. *Let μ be an OU measure on M. Then, for every $\epsilon > 0$ the set of homeomorphisms $T \in \mathcal{M}(M, \mu)$ such that $\mu(Per(T)) > 1 - \epsilon$ is dense in $\mathcal{M}[M, \mu]$, where $Per(T)$ denotes the set of periodic points of T.*

Proof. Let $\epsilon > 0$ and $\delta > 0$ be given. Using Vitali's covering lemma we can find a finite collection of disjoint balls $\{U_i\}_{i=1}^N$ such that $Diam(U_i) < \delta$ for each i and

$$\mu(\bigcup_{i=1}^N U_i) > 1 - \frac{\epsilon}{2}.$$

Now, we take balls $V_i \subset U_i$ such that:

$$\mu(\bigcup_{i=1}^N V_i) > 1 - \epsilon.$$

Now, let $T \in \mathcal{M}[X, \mu]$ be given and $V \subset U$ balls in X such that $Diam(U) < \delta$. Then it follows from corollary 9 in [DF] that there exists $g \in \mathcal{M}[X, \mu]$ and $p \in \mathbb{N}$

such that g^p is the identity on V, $g = T$ on U^c and $||g - T|| \leq Diam(V) \leq \delta$. So, there exists $g_1 \in \mathcal{M}[X, \mu]$ and $p_1 \in \mathbb{N}$ such that $g_1^{p_1}$ is the identity on V_1, $g_1 = T$ on U_1^c and $||g_1 - T|| \leq Diam(V_1) \leq \delta$. If we repeat the argument for g_1 in place of T and so on we obtain $g_2, \ldots, g_N$ in $\mathcal{M}[M, \mu]$ and if we define $g := g_N$, then we have that $g \in \mathcal{M}[X, \mu]$, g^{p^i} is the identity on V_i for each i, $g = T$ on $(U_1 \cup \cdots \cup U_N)^c$, and $||T - g|| < \delta$. As we have that $\mu(Per(g)) > 1 - \epsilon$, the proof of Lemma 4.1 is complete. □

Now, let us return to the proof of Theorem A. Given $T \in \mathcal{M}[M, \mu]$, we define:

$$\mathcal{R}(T) := \{x \in M : \liminf_{n \to +\infty} r_n d(f(T^n(x)), f(x)) < +\infty\}.$$

If we fix $T \in \mathcal{M}[M, \mu]$, $k > 0$, and $n \in \mathbb{N}$, then we define:

$$X_{n,k}^f(T) = \{x \in M : r_n d(f(T^n(x)), f(x)) < k\}.$$

Then, we have the following claim:

Claim 1.

$$\mathcal{R}(T) = \bigcup_{k>0} \bigcap_{m \geq 1} \bigcup_{n \geq m} X_{n,k}^f(T).$$

If $x \in \mathcal{R}(T)$, then $\liminf_{n \to +\infty} r_n d(f(T^n(x)), f(x)) = c \in \mathbb{R}$. If we take $k > c$, then there exists a sequence (n_j) in $\mathbb{N}$ such that $r_{n_j} d(f(T^{n_j}(x)), f(x)) < k$ for all $j \in \mathbb{N}$ and this shows that $x \in \bigcup_{k>0} \bigcap_{m \geq 1} \bigcup_{n \geq m} X_{n,k}^f(T)$. On the other hand if $x \in \bigcup_{k>0} \bigcap_{m \geq 1} \bigcup_{n \geq m} X_{n,k}^f(T)$, then there exists $k > 0$ and a sequence n_j in $\mathbb{N}$ such that $r_{n_j} d(f(T^{n_j}(x)), f(x)) < k$ for all $j \in \mathbb{N}$. Then we have that:

$$\liminf_{n \to +\infty} r_n d(f(T^n(x)), f(x)) < k < +\infty.$$

Then $x \in \mathcal{R}(T)$ and this proves Claim 1.

As a consequence of Claim 1 we get the following claim:

Claim 2. $\mu(\mathcal{R}(T)) > 1 - \epsilon$ if and only if there exists $k > 0$ such that for all $m \in \mathbb{N}$, there exists a positive integer $l > m$ such that $\mu(\bigcup_{n=m}^{l} X_{n,k}^f(T)) > 1 - \epsilon$.

In fact, let us suppose that $\mu(\mathcal{R}(T)) > 1 - \epsilon$. Then, we have that

$$\mu(\mathcal{R}(T)) = \lim_{k \to +\infty} \mu(\bigcap_{m \geq 1} \bigcup_{n \geq m} X_{n,k}^f(T)) > 1 - \epsilon,$$

and then there exists $k > 0$ such that $\mu(\bigcap_{m \geq 1} \bigcup_{n \geq m} X_{n,k}^f(T)) > 1 - \epsilon$. So, there exists $k > 0$ such that for all $m \in \mathbb{N}$, $\mu(\bigcup_{n \geq m} X_{n,k}^f(T)) > 1 - \epsilon$. It follows that there exists $l > m$ such that $\mu(\bigcup_{n=m}^{l} X_{n,k}^f(T)) > 1 - \epsilon$. On the other hand, let us suppose that there exists $k > 0$ such that for all $m \in \mathbb{N}$, there exists a positive integer $l > m$ such that

$$\mu(\bigcup_{n=m}^{l} X_{n,k}^f(T)) > 1 - \epsilon.$$

This implies that

$$\mu(\bigcup_{n\geq m} X^f_{n,k}(T)) > 1-\epsilon,$$

and letting $m \to \infty$ we have

$$\mu(\bigcap_{m\geq 1}\bigcup_{n\geq m} X^f_{n,k}(T)) > 1-\epsilon.$$

Now, it is clear that $\mu(\mathcal{R}(T)) > 1-\epsilon$ and this finishes the proof of Claim 2.

For each $\epsilon > 0$ and $l > m$ in $\mathbb{N}$, let us define:

$$\mathcal{R}^{m,l}_\epsilon := \{T \in \mathcal{M}[M,\mu] : \text{ there exists } k > 0 \text{ such that } \mu(\bigcup_{n=m}^{l} X^f_{n,k}(T)) > 1-\epsilon\}.$$

Then we have:

Claim 3. $\mathcal{R}^{m,l}_\epsilon$ is open in the uniform topology.

Let us fix $n, k \in \mathbb{N}$ and $T \in \mathcal{M}[M,\mu]$. If $S \in \mathcal{M}[M,\mu]$, then by triangle inequality we have that

$$r_n d(f(S^n(x)), f(x)) \leq r_n d(f(S^n(x)), f(T^n(x))) + r_n d(f(T^n(x)), f(x)).$$

By continuity of f we have that if S is sufficiently close to T in the uniform topology (see Definition 2.2), then $X^f_{n,k}(T) \subset X^f_{n,k}(S)$. By the same argument, if we fix $k > 0$, $l > m$ and $T \in \mathcal{M}[M,\mu]$, then $\bigcup_{n=m}^{l} X^f_{n,k}(T) \subset \bigcup_{n=m}^{l} X^f_{n,k}(S)$ for every $S \in \mathcal{M}[M,\mu]$ sufficiently close to T. This shows that $\mathcal{R}^{m,l}_\epsilon$ is open and this finishes the proof of Claim 3.

For each $\epsilon > 0$ let us define:

$$\mathcal{R}^\epsilon := \{T \in \mathcal{M}[M,\mu] : \mu(\mathcal{R}(T)) > 1-\epsilon\}.$$

By Claim 1 we have that $\mathcal{R}(T) = \bigcup_{k>0}\bigcap_{m\geq 1}\bigcup_{n\geq m} X^f_{n,k}(T)$. By Claim 2 we get that

$$\mathcal{R}^\epsilon = \bigcap_{m\in\mathbb{N}}\bigcup_{l>m} \mathcal{R}^{m,l}_\epsilon,$$

which shows that $\mathcal{R}^\epsilon$ is a G_δ set for each $\epsilon > 0$. If we show that $\mathcal{R}^\epsilon$ is dense in $\mathcal{M}[M,\mu]$ for each $\epsilon > 0$, then $\mathcal{R} := \bigcap_{n\in\mathbb{N}} \mathcal{R}^{1/n}$ will be the generic of Theorem A. Note that $Per(T) \subset \mathcal{R}(T)$ and then we have that

$$\{T \in \mathcal{M}[M,\mu] : \mu(Per(T)) > 1-\epsilon\} \subset \mathcal{R}^\epsilon.$$

By Lemma 4.1 we have that $\mathcal{R}^\epsilon$ is dense and this proves Theorem A. □

Proof (Theorem B). Given $T \in \mathcal{M}[M,\mu]$ and $p \in \mathbb{N}$ we define:

$$\mathcal{W}^f_p(y, \{r_n\}, T) := \{x \in M : d(f(T^n(x)), f(y)) < \frac{p}{r_n} \text{ for infinitely many } n\}.$$

Note that:

$$\mathcal{W}^f_p(y, \{r_n\}, T) = \bigcap_{m\geq 1}\bigcup_{n\geq m} T^{-n}(f^{-1}(B(f(y), \frac{p}{r_n}))).$$

Then we have the following claim:

Claim 4. Given $\epsilon > 0$ we have that $\mu(\mathcal{W}_p^f(y, \{r_n\}, T)) < \epsilon$ if and only if for every $m \in \mathbb{N}$ (sufficiently large) there exists $l > m$ (sufficiently large) such that

$$\mu\left(\bigcup_{n=m}^{l} T^{-n}(f^{-1}(B(f(y), \frac{p}{r_n}))))\right) < \epsilon.$$

In fact, if $\mu(\mathcal{W}_p^f(y, \{r_n\}, T)) < \epsilon$, then we get that

$$\lim_{m\to+\infty} \mu\left(\bigcup_{n\geq m} T^{-n}(f^{-1}(B(f(y), \frac{p}{r_n}))))\right) < \epsilon,$$

which implies that $\mu\left(\bigcup_{n\geq m} T^{-n}(f^{-1}(B(f(y), \frac{p}{r_n}))))\right) < \epsilon$ for all $m \in \mathbb{N}$ sufficiently large. Then we have that $\lim_{l\to+\infty} \mu\left(\bigcup_{n=m}^{l} T^{-n}(f^{-1}(B(f(y), \frac{p}{r_n}))))\right) < \epsilon$ for all $m \in \mathbb{N}$. So, we have that for all $m \in \mathbb{N}$ there exists $l > m$ (sufficiently large) such that:

$$\mu\left(\bigcup_{n=m}^{l} T^{-n}(f^{-1}(B(f(y), \frac{p}{r_n}))))\right) < \epsilon.$$

On the other hand let us suppose that for all $m \in \mathbb{N}$ there exists $l > m$ (sufficiently large) such that:

$$\mu\left(\bigcup_{n=m}^{l} T^{-n}(f^{-1}(B(f(y), \frac{p}{r_n}))))\right) < \epsilon.$$

Then, letting $l \to +\infty$ we get that $\mu\left(\bigcup_{n\geq m} T^{-n}(f^{-1}(B(f(y), \frac{p}{r_n}))))\right) < \epsilon$ for all $m \in \mathbb{N}$ and then we get that $\mu(\mathcal{W}_p^f(y, \{r_n\}, T)) < \epsilon$. This proves Claim 4.

For each $\epsilon > 0$ and $p \in \mathbb{N}$ let us define:

$$\mathcal{G}_{\epsilon,p} = \{T \in \mathcal{M}[M, \mu] : \mu(\mathcal{W}_p^f(y, \{r_n\}, T)) < \epsilon\},$$

and for each $\epsilon > 0$, $p \in \mathbb{N}$, and $l > m$ let us define:

$$\mathcal{G}_{\epsilon,p}^{m,l} := \{T \in \mathcal{M}[M, \mu] : \mu\left(\bigcup_{n=m}^{l} T^{-n}(f^{-1}(B(f(y), \frac{p}{r_n}))))\right) < \epsilon\}.$$

Then, the next claim is the following:

Claim 5. Each set $\mathcal{G}_{\epsilon,p}^{m,l}$ is open in the uniform topology and $\mathcal{G}_{\epsilon,p} = \bigcap_{m\in\mathbb{N}} \bigcup_{l>m} \mathcal{G}_{\epsilon,p}^{m,l}$. In particular, $\mathcal{G}_{\epsilon,p}$ is a G_δ set. Furthermore, $\mathcal{G}_{\epsilon,p}$ is dense in $\mathcal{M}[M, \mu]$.

Equality follows from Claim 4. The opening follows from the same argument of Claim 3 and will be omitted. Let us prove the density. Using that $\mu(f^{-1}(\{f(y)\}) = 0$ we get that

$$Per(T) \subset (X - W_p^f(y, \{r_n\}, T))(mod\ \mu)$$

and then:

$$\{T \in \mathcal{M}[M, \mu] : \mu(Per(T)) > 1 - \epsilon\} \subset \mathcal{G}_{\epsilon,p}.$$

By Lemma 4.1 we have that $\mathcal{G}_{\epsilon,p}$ is dense. The proof of Claim 5 is complete.

If we define:

$$\mathcal{G}_p := \bigcap_{q\in\mathbb{N}} \mathcal{G}_{1/q,p},$$

then $\mathcal{G}_p$ is a generic set and if $T \in \mathcal{G}_p$, then $\mu(\mathcal{W}_p^f(y, \{r_n\}, T) = 0$, which shows that $\liminf_{n\to+\infty} r_n d(f(T^n(x), f(y))) \geq p$ for μ-a.e. $x \in M$ if $T \in \mathcal{G}_p$.

If we define:

$$\mathcal{G} := \bigcap_{p\in\mathbb{N}} \mathcal{G}_p,$$

then $\mathcal{G}$ is a generic set and $\liminf_{n\to+\infty} r_n d(f(T^n(x)), f(y)) = +\infty$ for μ-a.e. $x \in M$ if $T \in \mathcal{G}$. This finishes the proof of Theorem B. □

Proof (Theorem C). Let us begin with the following lemma:

Lemma 4.2. *Let λ be the Lebesgue measure on $\mathbb{R}^n$. Then, for every $\epsilon > 0$ the set of homeomorphisms $T \in \mathcal{M}(\mathbb{R}^n, \lambda)$ such that $\lambda((Per(T))^c) < \epsilon$ is dense in $\mathcal{M}[\mathbb{R}^n, \lambda]$ with respect to compact open topology, where $Per(T)$ denotes the set of periodic points of T.*

Proof. Let $\mathcal{C}(f, K, \delta)$ be a basic open set with respect to compact open topology. We can suppose without loss of generality that K is a compact cube. Let C be a compact cube containing $K \cup f(K)$ in its interior. Then it follows from Lemma 12.2 of [AP] that there exists $\hat{f} \in \mathcal{M}[\mathbb{R}^n, \lambda]$ which leaves C invariant, and agrees with f on K. Now, it follows from the proof of Lemma 4.1 that there exists $g \in \mathcal{M}(C, \lambda)$, where $\mathcal{M}(C, \lambda)$ denotes the set of homeomorphisms of C such that the Lebesgue measure is invariant, such that $\lambda(Per(g)) > \lambda(C) - \epsilon$ and $d(g(x), \hat{f}(x)) < \delta$ for λ a.e. $x \in C$. Let $C_1 \supset C$ be a cube concentric to C such that $\lambda(C_1 - C) < \epsilon$ and extend g to a homeomorphism of C_1 onto itself such that g is equal to the identity on the boundary of C_1. Let $A := C_1 - C$ and define for each Borel set $B \subset A$ the measures:

$$\mu_1(B) = \lambda(B) \quad \text{and} \quad \mu_2(B) = \lambda(g(B)).$$

Therefore by the Homeomorphic Measures Theorem (see Corollary A2.6 on [AP]) there exists a homeomorphism $h : A \to A$ such that $\mu_2(h(B)) = \mu_1(B)$ for each Borel set $B \subset A$ and such that h is the identity on the boundary of A. Define $k : C_1 \to C_1$ such that:

$$k(x) = g(x) \text{ in } C \text{ and } k(x) = g(h(x)) \text{ in } C_1 - C,$$

and define $\hat{g} : \mathbb{R}^n \to \mathbb{R}^n$ such that:

$$\hat{g}(x) = k(x) \text{ in } C_1 \text{ and } \hat{g}(x) = Id \text{ in } C_1^c.$$

Note that $\hat{g} \in \mathcal{M}(\mathbb{R}^n, \lambda)$, $\lambda(Per(\hat{g})^c) < 2\epsilon$ and $\hat{g} \in \mathcal{C}(f, K, \delta)$ which shows the density. □

To complete the proof of Theorem C we follow the same ideas of Theorem A. In fact let us define for each $T \in \mathcal{M}[\mathbb{R}^n, \lambda]$ the set:

$$\mathcal{M}(T) = \{x \in \mathbb{R}^n : \liminf_{n\to\infty} r_n |T^n(x) - x| < +\infty\}.$$

For each $n, k \in \mathbb{N}$ let us define:

$$Y_{n,k}(T) = \{x \in \mathbb{R}^n : r_n |T^n(x) - x| < k\}.$$

The following two claims hold in the same way as the compact case:

Claim 6.

$$\mathcal{M}(T) = \bigcup_{k>0} \bigcap_{m\geq 1} \bigcup_{n\geq m} Y_{n,k}(T)$$

Claim 7. $\lambda([\mathcal{M}(T)]^c) < \epsilon$ if and only if there exists $k > 0$ such that for all $m \in \mathbb{N}$, there exists a positive integer $l > m$ such that $\lambda(\bigcap_{n=m}^{l} [Y_{n,k}(T)]^c) < \epsilon$.

For each $\epsilon > 0$ and $l > m$ in $\mathbb{N}$ let us define:

$$\mathcal{M}_\epsilon^{m,l} := \{T \in \mathcal{M}[\mathbb{R}^n, \lambda] : \text{ there exists } k > 0 \text{ such that } \lambda(\bigcap_{n=m}^{l} (Y_{n,k}(T))^c) < \epsilon\}.$$

Then we have:

Claim 8. $\mathcal{M}_\epsilon^{m,l}$ is open in the compact-open topology.

Proof (Claim 8). Let $T \in \mathcal{M}_\epsilon^{m,l}$. Then we have to prove that if S is sufficiently close to T, then $S \in \mathcal{M}_\epsilon^{m,l}$. Note that it is enough to prove that if S is sufficiently close to T, then $Y_{n,k}(T) \subset Y_{n,k}(S)$. By triangle inequality

$$r_n|S^n(x) - x| \leq r_n|S^n(x) - T^n(x)| + r_n|T^n(x) - x|.$$

Using this inequality and the metric Γ of the compact-open topology we get that if S is close to T, then $Y_{n,k}(T) \subset Y_{n,k}(S)$ and this proves the opening. □

For each $\epsilon > 0$ we define:

$$\mathcal{M}_\epsilon = \{T \in \mathcal{M}[\mathbb{R}^n, \lambda] : \mu((\mathcal{M}(T))^c) < \epsilon\}.$$

By Claim 7 we get

$$\mathcal{M}_\epsilon = \bigcap_{m\in\mathbb{N}} \bigcup_{l>m} \mathcal{M}_\epsilon^{m,l}.$$

The residual set is then given by

$$\mathcal{M} := \bigcap_{n\in\mathbb{N}} \mathcal{M}_{1/n}$$

which proves Theorem C. □

Proof (Corollary A). Let $T \in \mathcal{M}[M, \mu]$ be a homeomorphism with superpolynomial decay of correlations and $y \in M$ such that $d_\mu(y) > 0$. If $0 < \beta < \frac{1}{d_\mu(y)}$ and $r_n := n^{-\beta}$, by Theorem 2.14 we have that the sequence of balls $\{B(y, r_n)\}$ is strongly Borel-Cantelli, which implies that (see Remark 2.13)

$$\mu(\{x \in M : d(T^n(x), y) < r_n \text{ for infinitely many } n\}) = 1.$$

This implies that

$$\liminf_{n\to+\infty} r_n d(T^n(x), y) \leq 1,$$

for μ-a.e. $x \in M$.

On the other hand, by Theorem B with $f = Id$ and r_n as above we get a residual set $\mathcal{T} \subset \mathcal{M}[M, \mu]$ such that $T \in \mathcal{T}$ implies:

$$\liminf_{n\to+\infty} n^{\frac{1}{\beta}} d(T^n(x), y) = +\infty \text{ for } \mu \text{ a.e. } x \in M.$$

This shows that in this generic set the decay of correlations is not superpolynomial and the proof of Corollary A is complete. □

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