CASTELNUOVO-MUMFORD REGULARITY AND BRIDGELAND STABILITY OF POINTS IN THE PROJECTIVE PLANE

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ABSTRACT. In this paper, we study the relation between Castelnuovo-Mumford regularity and Bridgeland stability for the Hilbert scheme of $n$ points on $P^2$. For the largest $\lfloor \frac{n}{2} \rfloor$ Bridgeland walls, we show that the general ideal sheaf destabilized along a smaller Bridgeland wall has smaller regularity than one destabilized along a larger Bridgeland wall. We give a detailed analysis of the case of monomial schemes and obtain a precise relation between the regularity and the Bridgeland stability for the case of Borel fixed ideals.

1. INTRODUCTION

In this paper, we consider the relation between the Castelnuovo-Mumford regularity and the Bridgeland stability of zero-dimensional subschemes of $P^2$. Our study is motivated by the following result which relates geometric invariant theory (GIT) stability and Castelnuovo-Mumford regularity.

Theorem ([HH13, Corollary 4.5]). Let $C \subset P^{g-4}$ be a $c$-semistable bicanonical curve. Then $\mathcal{O}_C$ is 2-regular.

Note that $c$-semistability of curves [HH13, Definition 2.6] is a purely geometric notion concerning singularities and subcurves, whereas Castelnuovo-Mumford regularity is an algebraic notion regarding the syzygies of ideal sheaves.

For points in $P^2$, a similar but weaker statement holds. A set of $n$ points in $P^2$ is GIT semistable if and only if at most $2n/3$ of the points are collinear, in which case the regularity is at most $2n/3$. However, the regularities of semistable points cover a broad spectrum. Our goal in this paper is to use Bridgeland stability to obtain a closer relationship between stability and regularity.

There is a distinguished half-plane $H = \{(s,t) | s > 0, t \in \mathbb{R}\}$ of Bridgeland stability conditions for $P^2$. Let $\xi$ be a Chern character. The half-plane $H$ admits a wall-and-chamber decomposition, where in each chamber the set of Bridgeland semistable objects with Chern character $\xi$ remains constant.

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The Bridgeland walls where an ideal sheaf of points is destabilized consist of the vertical line \( s = 0 \) and a finite set of nested semicircular walls \( W_c \), centered along the \( s \)-axis at \( s = -c - \frac{3}{2} < 0 \). \textsuperscript{[ABCH13 Section 6]} Since the semicircular Bridgeland walls are nested, we can order them by inclusion. If an ideal sheaf \( I_Z \) is destabilized along the wall \( W_c \), then \( I_Z \) is Bridgeland stable in the region bounded by \( W_c \) and \( s = 0 \). Let \( \sigma \prec \sigma' \) if all \( \sigma' \)-semistable ideal sheaves with Chern character \( \xi \) are \( \sigma \)-semistable. Consequently, Bridgeland stability induces a stratification of \( \mathbb{P}^2[\mathbb{n}] \),

\[
\mathbb{P}^2[\mathbb{n}] = \coprod_{\alpha} X^\alpha,
\]

where \( X^\alpha = \{ Z \in \mathbb{P}^2[\mathbb{n}] \mid I_Z \) is \( \alpha \)-semistable but \( \beta \)-unstable \( \forall \alpha \prec \beta \} \) and \( \alpha \) runs over a Bridgeland stability condition in each chamber. We have \( X^\alpha = \bigcup_{\beta \geq \alpha} X^\beta \) (see Section \[2\]). By \textsuperscript{[ABCH13 Sections 9, 10]} and \textsuperscript{[LZ]}, this stratification coincides with the stratification of \( \mathbb{P}^2[\mathbb{n}] \) according to the stable base loci of linear systems. Recall that the effective cone of a variety has a wall and chamber decomposition such that in each chamber the stable base locus of the divisors remain constant.

Similarly, there is a stratification induced by Castelnuovo-Mumford regularity:

\[
\mathbb{P}^2[\mathbb{n}] = \coprod_{r \in \mathbb{Z}} X^{r \text{-reg}},
\]

where \( X^{r \text{-reg}} \) is the collection of ideals whose Castelnuovo-Mumford regularity is \( r \). The regularity, being a cohomological invariant \textsuperscript{[Eis95 Proposition 20.16]}, is upper-semicontinuous, and we have \( X^{r \text{-reg}} = \coprod_{r' \geq r} X^{r' \text{-reg}} \).

This naturally raises the question of comparing the two stratifications. We will show that a general scheme destabilized at one of the \( \left\lfloor \frac{\mathbb{n}}{2} \right\rfloor \) largest Bridgeland walls has smaller regularity than the general scheme destabilized along the larger walls. Our main theorem will be proved in Section \[5\].

**Theorem.** Let \( p_i \) be the maximal ideal of the closed point \( p_i \in \mathbb{P}^2 \), \( i = 1, \ldots, s \). Let \( Z \) be the subscheme given by \( \bigcap_{i=1}^{s} p_i^{m_i} \) and let \( n \) be its length. Define

\[
h := \max \left\{ \sum_{j=1}^{t} m_{ij} \mid p_{i_1}, \ldots, p_{i_t} \text{ are collinear} \right\}.
\]

If \( n \leq 2h - 3 \), then \( Z \) is destabilized at the wall \( W_{\text{reg}(Z) - 1} \). In particular, general points destabilized at \( W_{k+1} \) have higher regularity than those destabilized at \( W_k \), \( \forall k \geq \frac{n}{2} - 1 \).

For zero-dimensional subschemes cut out by monomials, we have a more precise connection between regularity and Bridgeland stability:

**Proposition.** Let \( Z \) be a zero-dimensional monomial scheme in \( \mathbb{P}^2 \). If the ideal sheaf \( I_Z \) is destabilized at the wall \( W_{\mu(Z)} \) with center \( x = -\mu(Z) - \frac{3}{2} \), then

\[
\frac{3}{4} (\text{reg}(I_Z) - 1) \leq \mu(Z) \leq \text{reg}(I_Z) - 1.
\]

(1) The left equality holds if and only if \( \text{reg}(I_Z) + 1 = 2m \) is even and \( I_Z = \langle x^m, y^m \rangle \).
(2) The right equality holds if and only if $I_Z = \langle x^{a_1}, x^{a_2}y^{b_2}, \ldots, y^{b_r} \rangle$ with $\max_{1 \leq i \leq r-1}(a_i + b_{i+1} - 1) \leq \max(a_1, b_r)$.

In particular, for Borel fixed ideals, the regularity and the Bridgeland stability completely determine each other:

**Corollary.** Let $Z \subset \mathbb{P}^2$ be a zero-dimensional monomial scheme whose ideal is Borel-fixed (which holds if it is a generic initial ideal, for instance). Then the ideal sheaf $I_Z$ is destabilized at the wall $\mathcal{W}_{\text{reg}(I_Z)-1}$.

In general, the relation between regularity and the Bridgeland slope is not monotonic. Let $Z_1$ and $Z_2$ be two schemes of length $n$ destabilized along $\mathcal{W}_{\mu(Z_1)}$ and $\mathcal{W}_{\mu(Z_2)}$, respectively. It may happen that while $\text{reg}(Z_1) > \text{reg}(Z_2)$, we have $\mu(Z_1) < \mu(Z_2)$. We close the introduction with the following simple but illustrative example.

**Example 1.1.** Let $Z_1$ and $Z_2$ be the monomial scheme defined by $\langle x^4, y^4 \rangle$ and $\langle x^6, x^5y, x^4y^2, xy^3, y^4 \rangle$, respectively. Both are of length 16, and by the arguments of Section 3 we see that $\text{reg}(I_{Z_1}) = 7$, $\text{reg}(I_{Z_2}) = 6$ and $\mu(Z_1) = \frac{9}{2}$, $\mu(Z_2) = 5$.

We work over an algebraically closed field $\mathbb{K}$ of characteristic zero.

### 2. Preliminaries on Bridgeland stability conditions

We briefly review the basics of Bridgeland stability conditions on $\mathbb{P}^2$. We refer the reader to [ABCH13] and [CH14] for more details. Let $\mathcal{D}^b(\mathbb{P}^2)$ be the bounded derived category of coherent sheaves on $\mathbb{P}^2$, and $K(\mathbb{P}^2)$ be the $K$-group of $\mathcal{D}^b(\mathbb{P}^2)$.

**Definition 2.1.** A Bridgeland stability condition on $\mathbb{P}^2$ consists of a pair $(\mathcal{A}, \mathcal{Z})$, where $\mathcal{A}$ is the heart of a $t$-structure on $\mathcal{D}^b(\mathbb{P}^2)$ and $\mathcal{Z} : K(\mathbb{P}^2) \to \mathbb{C}$ is a homomorphism (called the central charge) satisfying

- if $0 \neq E \in \mathcal{A}$, $\mathcal{Z}(E)$ lies in the semiclosed upper half-plane $\{re^{i\pi\theta} \mid r > 0, 0 < \theta \leq 1\}$.
- $(\mathcal{A}, \mathcal{Z})$ has the Harder-Narasimhan property, which will be defined below.

**Definition 2.2.** Writing $\mathcal{Z} = -d + ir$, the slope $\mu(E)$ of $0 \neq E \in \mathcal{A}$ is defined by $\mu(E) = d(E)/r(E)$ if $r(E) \neq 0$ and $\mu(E) = \infty$ otherwise.

**Definition 2.3.** An object $E \in \mathcal{A}$ is called stable (resp. semistable) if for every proper subobject $F \subset E$ in $\mathcal{A}$, $\mu(F) < \mu(E)$ (resp. $\mu(F) \leq \mu(E)$).

**Definition 2.4.** The pair $(\mathcal{A}, \mathcal{Z})$ has the Harder-Narasimhan property if any non-zero object $E \in \mathcal{A}$ admits a finite filtration

$$0 \subset E_0 \subset E_1 \subset \cdots \subset E_n = E$$

such that each Harder-Narasimhan factor $F_i = E_i/E_{i-1}$ is semistable and $\mu(F_1) > \mu(F_2) > \cdots > \mu(F_n)$.

Let $L$ be the class of a line in $\mathbb{P}^2$.

**Definition 2.5.** Let $E$ be a coherent sheaf on $\mathbb{P}^2$. The Mumford slope of $E$ is defined to be $\deg(E)/\text{rank}(E)$, where $\deg(E) = ch_1(E) \cdot L$ and $\text{rank}(E) = ch_0(E) \cdot L^2$ are the ordinary degree and rank.

Let $\mu_{\text{min}}(E)$ (resp. $\mu_{\text{max}}(E)$) denote the minimum (resp. maximum) slope of a Harder-Narasimhan factor of a coherent sheaf $E$ with respect to the Mumford
slope. For \( s \in \mathbb{R} \), let \( Q_s \) and \( \mathcal{F}_s \) be the full subcategory of \( \text{Coh}(\mathbb{P}^2) \) defined by

- \( Q \in Q_s \) if \( Q \) is torsion or \( \mu_{\min}(Q) > s \).
- \( F \in \mathcal{F}_s \) if \( F \) is torsion-free and \( \mu_{\max}(F) \leq s \).

Each pair \( (\mathcal{F}_s, Q_s) \) is a torsion pair [Bri08, Lemma 6.1] and induces a \( t \)-structure via tilting on \( D^b(\mathbb{P}^2) \) with heart [HRS96]

\[ A_s = \{ E \in D^b(\mathbb{P}^2) \mid H^{-1}(E) \in \mathcal{F}_s, H^0(E) \in Q_s, \text{ and } H^1(E) = 0 \ otherwise \}. \]

**Theorem** ([Bri08]AB13, BM11). For each \( s \in \mathbb{R} \) and \( t > 0 \), define

\[ Z_{s,t}(E) = - \int_{\mathbb{P}^2} e^{-(s+it)L} \text{ch}(E). \]

Then the pair \( (A_s, Z_{s,t}) \) defines a Bridgeland stability condition on \( D^b(\mathbb{P}^2) \).

We thus obtain an upper half-plane \( H \) of Bridgeland stability conditions.

Fix a class \( \xi \) in the numerical Grothendieck group. If \( \xi \) has positive rank, define the slope and the discriminant by

\[ \mu(\xi) = \frac{\text{ch}_1(\xi)}{\text{rank}(\xi)}, \quad \Delta = \frac{1}{2} \mu(\xi)^2 - \frac{\text{ch}_2(\xi)}{\text{rank}(\xi)}. \]

For an ideal sheaf \( \mathcal{I}_n \) of \( n \) points, we have \( \mu = 0 \) and \( \Delta = n \). A sheaf \( E \) of positive rank is Gieseker semistable if for every proper subsheaf \( 0 \neq F \subset E \), \( \mu(F) \leq \mu(E) \) and in case of equality \( \Delta(F) \geq \Delta(E) \). The sheaf is called Gieseker stable if the second inequality is strict. The sheaf \( E \) is Gieseker semistable if and only if for some \( s \), \( E \) is \( Z_{s,t} \)-semistable for all \( t \gg 0 \) [ABCH13, Section 6]. Every ideal sheaf of points is Gieseker (in fact, slope) stable.

There exists a locally finite set of walls in the \((s,t)\)-half-plane depending on \( \xi \) such that the set of \( \sigma \)-semistable objects of class \( \xi \) does not change as the \( \sigma \) varies in a chamber [Bri08]BM11, BM14. These walls are called Bridgeland walls. For \( \mathbb{P}^2 \), the Bridgeland walls where a Gieseker semistable sheaf is destabilized consist of line \( s = \mu(\xi) \) and a finite number of nested semicircles with center \((c,0)\) with \( c < \mu \) [ABCH13, Section 6]. The largest semicircular wall is called the Gieseker wall, and the smallest semicircular wall is called the collapsing wall. If \( \xi = (1,0,-n) \), the Chern character of the ideal sheaf of a zero-dimensional subscheme of \( \mathbb{P}^2 \) of length \( n \), then the wall with center \((c,0)\) has radius \( \sqrt{c^2 - 2n} \). Throughout the paper \( W_\mu = W_\mu^n \) will denote the wall centered at \((\mu - \frac{3}{2},0)\). An ideal sheaf destabilized along \( W_\mu \) is Bridgeland stable for all Bridgeland stability conditions outside \( W_\mu \) and not semistable for any Bridgeland stability condition contained in \( W_\mu \). All Bridgeland walls for \( n \leq 9 \) were explicitly computed in [ABCH13, Section 10].

In Figure [1] we reproduce the example of \( n = 5 \). Along the Gieseker wall \( W_{-\frac{11}{2}} \) ideal sheaves of collinear points are destabilized. Along the wall \( W_{-\frac{9}{2}} \) ideal sheaves of schemes with a collinear subscheme of length four are destabilized. All ideal sheaves are destabilized along the collapsing wall \( W_{-\frac{7}{2}} \).
3. Monomial schemes

A monomial subscheme of $\mathbb{P}^2$ is a subscheme whose ideal is generated by monomials. For these schemes, the relation between Castelnuovo-Mumford regularity and Bridgeland stability is clear because the regularity is easy to compute and the Bridgeland stability is explicitly described by [CH14]. To reveal the relation, we need to study the combinatorics.

**Proposition 3.1.** Let $Z$ be a zero-dimensional monomial scheme in $\mathbb{P}^2$. If the ideal sheaf $I_Z$ is destabilized at the wall $W_{\mu(Z)}$ with center $x = -\mu(Z) - \frac{3}{2}$, then

$$\frac{3}{4} (\operatorname{reg}(I_Z) - 1) \leq \mu(Z) \leq \operatorname{reg}(I_Z) - 1.$$

1. The left equality holds if and only if $\operatorname{reg}(I_Z) + 1 = 2m$ is even and $I_Z = \langle x^m, y^m \rangle$.
2. The right equality holds if and only if $I_Z = \langle x^{a_1}, x^{a_2}y^{b_2}, \ldots, y^{b_r} \rangle$ satisfies $\max_{1 \leq i \leq r-1} (a_i + b_{i+1} - 1) \leq \max(a_1, b_r)$.

A zero-dimensional monomial subscheme $Z$ in $\mathbb{P}^2$, in a suitable affine coordinate system, has defining ideal $I_Z$ generated by a set of monomials

$$x^{a_1}, x^{a_2}y^{b_2}, \ldots, y^{b_r}$$

where $a_1 > \cdots > a_{r-1} > a_r = 0$ and $0 = b_1 < b_2 < \cdots < b_r$.

It is convenient to represent monomial subschemes by their block diagrams. The block diagram $D$ for $Z$ consists of $b_r$ left-aligned rows of consecutive boxes such that the $i$th row counting from the bottom has $a_j$ boxes if $b_j < i \leq b_{j+1}$. The lower left corner represents the monomial 1. The box to the right of (resp. above) $x^iy^j$ represents $x^{i+1}y^j$ (resp. $x^iy^{j+1}$). With this interpretation, the box diagram $D$ records the monomials in $\mathbb{K}[x, y]$ which are not in $I_Z$. Figure 2 shows an example.
We will always place the lower left corner of \( D \) at the origin and assume that the boxes in \( D \) are unit length.

**Proof of Proposition 3.1** We briefly recapitulate the computation of \( \mu(Z) \) in [CH14]. Index the rows of a box diagram \( D \) from bottom to top and the columns from left to right. Let \( h_j \) (resp. \( v_j \)) be the number of boxes in the \( j \)th row (resp. column). Let \( r(D) \) and \( c(D) \) be the number of rows and columns in \( D \). Define the \( k \)th horizontal slope \( \mu_k \) and the \( i \)th vertical slope \( \mu_i' \) by

\[
\mu_k = \frac{1}{k} \sum_{j=1}^{k} (h_j + j - 1) - 1, \quad \mu_i' = \frac{1}{i} \sum_{j=1}^{i} (v_j + j - 1) - 1.
\]

Then the slope \( \mu(Z) \) of \( Z \) is defined by

\[
\mu(Z) = \max_{1 \leq k \leq r(D), 1 \leq i \leq c(D)} \{ \mu_k, \mu_i' \}.
\]

By [CH14] Theorem 1.6], the ideal sheaf \( I_Z \) is destabilized at the wall \( W_{\mu(Z)} \) with center \( x = -\mu(Z) - \frac{3}{2} \).

On the other hand, the regularity of \( I_Z \) can be computed from its minimal free resolution given by

\[
0 \to \bigoplus_{i=1}^{r-1} \mathcal{O}(-a_i - b_{i+1}) \xrightarrow{M} \bigoplus_{i=1}^{r} \mathcal{O}(-a_i - b_i) \to I_Z \to 0,
\]

where \( M \) is the \( r \times (r-1) \) matrix with entries

\[
m_{i,i} = y^{b_{i+1} - b_i}, \quad m_{i+1,i} = -x^{a_i-a_{i+1}}, \quad \text{and} \quad m_{i,j} = 0 \text{ otherwise}.
\]

Since \( a_i + b_{i+1} - 1 \geq a_i + b_i \) for \( i = 1, \ldots, r-1 \) and \( a_{r-1} + b_r - 1 \geq a_r + b_r \), the Castelnuovo-Mumford regularity \( \text{reg}(I_Z) \) of \( I_Z \) is

\[
\text{reg}(I_Z) = \max_{1 \leq i \leq r-1} (a_i + b_{i+1} - 1).
\]

If we place the block diagram \( D \) in the \( a-b \) plane with its lower left corner at the origin and set every box to be a unit square, then the points \((a_i, b_{i+1})\) are the vertices of \( D \) contained in the first quadrant. Hence, the block diagrams representing ideals with regularity \( l \) are precisely those which lie below and touch the line \( a + b = l + 1 \).

Fix the regularity to equal \( l \). To maximize \( \mu(Z) \) subject to \( \text{reg}(Z) = l \), we need to maximize \( \mu_k \) and \( \mu_i' \) under the condition that the box diagram lies below and touches the line \( a + b = l + 1 \). Since the box diagram of \( I_Z = \langle x^l, x^{l-1}y, \ldots, y^l \rangle \) contains every positive integral lattice point under the line \( a + b = l + 1 \), it follows that \( Z \) gives the maximum \( \mu \)-value, which is \( l - 1 \). Note that \( \mu_k = l - 1 \) if and only if \( h_1 = l, h_2 = l - 1, \ldots, h_k = l - (k - 1) \). Hence, \( \mu(Z) = l - 1 \) precisely when either
Let $\text{Corollary 3.2.}$

A Borel-fixed ideal is of the form $\langle x^{a_1}, x^{a_2} y^{b_2}, \ldots, y^{b_r} \rangle$ if $I_Z$ satisfies $\max_{1 \leq i \leq r-1} (a_i + b_{i+1} - 1) \leq \max(a_1, b_r)$.

To minimize $\mu(Z)$ subject to $\text{reg}(Z) = l$, we use as few boxes as possible to minimize the slopes $\mu_k$ and $\mu'_k$. A box diagram that touches the line $a + b = l + 1$ at $(a', b')$ contains the box diagram of the ideal $\langle x^{a'}, y^{b'} \rangle$. It follows that the ideal of $Z$ should be of the form $\langle x^a, y^b \rangle$ with $a + b = l + 1$. Then

$$\max_{1 \leq k \leq r(D)} \{ \mu_k \} = \mu_b = a + b - \frac{1}{2} - 1$$

and similarly

$$\max_{1 \leq i \leq c(D)} \{ \mu'_i \} = \mu'_a = b + a - \frac{1}{2} - 1$$

so that

$$\mu(Z) = \max \left( a + b - \frac{1}{2} - 1, b + a - \frac{1}{2} - 1 \right).$$

Thus $\mu(Z)$ achieves the minimum when $a$ and $b$ are almost equal. If $l$ is even, then $(a, b) = \left( \frac{l}{2} + 1, \frac{l}{2} \right)$ gives $\mu(Z) = \frac{3l}{4} - \frac{3}{4}$. If $l$ is odd, then $(a, b) = \left( \frac{l+1}{2}, \frac{l+1}{2} \right)$ gives $\mu(Z) = \frac{3l}{4} - \frac{3}{4}$. Furthermore, if $n > \frac{(l+1)^2}{4}$, then either the horizontal slope $\mu'_{l+1}$ or the vertical slope $\mu'_{l+1}$ is strictly larger than $\frac{3l}{4} - \frac{3}{4}$. We conclude that $\frac{3l}{4} - \frac{3}{4} \leq \mu(Z)$ with equality only if $Z$ is the monomial ideal $\langle x^{\frac{l+1}{2}}, y^{\frac{l+1}{2}} \rangle$.

Recall that an ideal $I$ generated by monomials in $x$ and $y$ is Borel fixed if $x^i y^j \in I$ for some $j > 0$ implies $x^{i+1} y^{j-1} \in I$. Borel fixedness is one of the most important combinatorial properties in the study of monomial ideals. For instance, generic initial ideals with respect to a monomial order are Borel fixed. See Eis95 Theorem 15.20 for a detailed discussion. We obtain the following corollary.

**Corollary 3.2.** Let $Z \subset \mathbb{P}^2$ be a zero-dimensional monomial scheme whose ideal is Borel-fixed. Then the ideal sheaf $I_Z$ is destabilized at the wall $W_{\text{reg}(I_Z) - 1}$.

**Proof.** A Borel-fixed ideal is of the form $\langle x^a, x^{a-1} y^{b_{a-1}}, \ldots, y^{b_0} \rangle$ with $\lambda_0 > \cdots > \lambda_{a-1} > 0$. Then $i + \lambda_{i-1} - 1) \leq \lambda_0 = \max(a, \lambda_0)$ for $i = 1, \ldots, a$. The corollary follows from Proposition 3.1.2. □

Every possible Betti diagram of a zero-dimensional scheme in $\mathbb{P}^2$ occurs as the Betti diagram of a monomial scheme Eis05. Let $\binom{k}{2} < n \leq \binom{k+1}{2}$ and let $Z$ be a scheme of length $n$. Then the regularity of $Z$ can be any integer between $k$ and $n$. Given $k \leq l \leq n$, take a box diagram $D$ with $n$ boxes and at most $l$ rows such that $h_1 = l$ and $h_i \leq l + 1 - i$ for $2 \leq i \leq l$. Since $n \leq \left( \frac{l+1}{2} \right)$ such diagrams $D$ exist. Moreover, $\mu(Z) = l - 1 = \text{reg}(I_Z) - 1$, the maximum possible by Proposition 3.1.1.

We can also ask for the minimum possible $\mu(Z)$ given a scheme $Z$ of length $n$ and regularity $l$. If $0 < m \leq \frac{l}{2}$ and $m(l + 1 - m) \leq n < (m + 1)(l - m)$, then the tallest rectangle with upper right vertex on the line $x + y = l + 1$ is the $m \times (l + 1 - m)$ rectangle. Hence, $\mu(Z) \geq \text{reg}(I_Z) - \frac{1}{2} - \frac{n}{2}$. Equality occurs, for instance, when $n = m(l+1-m)$. In case, $l$ is even (resp. odd) and $n \geq \frac{l}{2}(\frac{l}{2} + 1)$ (resp. $n \geq \left( \frac{l+1}{2} \right)$), then $\mu(Z) \geq \frac{3}{4}\text{reg}(I_Z) - \frac{3}{4}$. In particular, we conclude that

$$1 \leq \text{reg}(I_Z) - \mu(Z) \leq \frac{\sqrt{n} + 1}{2}.$$
Equality is attained on the right hand side when \( \text{reg}(I_Z) \) is odd and \( n = \frac{(\text{reg}(I_Z)+1)^2}{4} \). We summarize this in the following proposition.

**Proposition 3.3.** Let \( Z \) be a monomial scheme of length \( n \) and regularity \( l \). If \( 0 < m \leq \frac{l}{2} \) and \( m(l + 1 - m) \leq n < (m + 1)(l - m) \), then

\[
1 \leq \text{reg}(I_Z) - \mu(Z) \leq \frac{m}{2} + \frac{1}{2}.
\]

In general,

\[
1 \leq \text{reg}(I_Z) - \mu(Z) \leq \frac{\sqrt{n} + 1}{2}.
\]

4. General points

In this section, we discuss the relation between Bridgeland stability and regularity for general points on \( \mathbb{P}^2 \).

Let \( \binom{r}{s} < n \leq \binom{r+1}{s} \). Then, for a dense open set \( U \in \mathbb{P}^{2[n]} \), the minimal free resolution of \( I_Z \) is the Gaeta resolution

\[
0 \to \mathcal{O}^{\oplus a}(-r - 1) \oplus \mathcal{O}^{\oplus \max(0,b)}(-r) \to \mathcal{O}^{\oplus \max(0,b)}(-r) \oplus \mathcal{O}^{\oplus c}(-r + 1) \to I_Z \to 0,
\]

where \( a = n - \binom{r}{s} > 0 \), \( c = \binom{r+1}{s} - n \geq 0 \) and \( b = c - a + 1 \) [Eis05]. The regularity of \( I_Z \) is \( r \). Since regularity is upper-semicontinuous and \( \mathbb{P}^{2[n]} \) is irreducible, there exists an open set \( U_1 \) containing \( U \) such that \( \text{reg}(I_Z) = r \) for \( Z \in U_1 \).

On the other hand, there exists an open dense set \( U_2 \in \mathbb{P}^{2[n]} \) such that for \( Z \in U_2 \) the ideal sheaf \( I_Z \) is destabilized at the collapsing wall \( W_{\mu_n} \) with center \( (-\mu_n - \frac{3}{2}, 0) \).

By a general point of \( \mathbb{P}^{2[n]} \), we will mean a point \( Z \in U_1 \cap U_2 \). For such \( Z \), there exists a precise relation between the regularity \( k \) and the Bridgeland slope \( \mu_n \).

Huizenga computed \( \mu_n \) for all \( n \) [Hui]. The slope \( \mu_n \) is the smallest positive slope of a stable vector bundle on the parabola \( \mu^2 + 3\mu + 2 - 2n = 2\Delta \), where \( \mu \) is the slope and \( \Delta \) is the discriminant. The computation of \( \mu_n \), while easy for any given \( n \), depends on a fractal curve. Consequently, it is hard to write a closed formula.

Luckily, there are good bounds for \( \mu_n \). Let

\[
S = \left\{ \binom{0}{1}, \binom{1}{2}, \binom{3}{5}, \binom{8}{13}, \ldots \right\} \cup \left\{ \alpha > \phi^{-1} = \frac{\sqrt{5} - 1}{2} \right\},
\]

consisting of consecutive ratios of Fibonacci numbers and numbers larger than the inverse of the golden ratio. Let \( n = \binom{k}{2} + s \) with \( 0 \leq s < k \). By [ABCH13] Theorem 4.5, we have

\[
\mu_n = \begin{cases} 
  k - 2 + \frac{s}{k-1} & \text{if } \frac{s}{k-1} \in S, \\
  k - 1 - \frac{s}{k+1} & \text{if } 1 - \frac{s+1}{k+1} \in S.
\end{cases}
\]

Furthermore, by [ABCH13] Lemma 4.1, Corollary 4.9], the inequalities

\[
\mu_{n-1} \leq \mu_n \leq \begin{cases} 
  k - 2 + \frac{s}{k-1} & \text{if } \frac{s}{k-1} \geq \frac{1}{2}, \\
  k - 1 - \frac{s}{k+1} & \text{if } \frac{s}{k-1} \leq \frac{1}{2}
\end{cases}
\]

hold. When \( k \) is odd and \( s = \frac{k-1}{2} \), then \( \frac{s}{k-1} = \frac{1}{2} \in S \) and \( \mu_n = k - \frac{3}{2} \). When \( k \) is even and \( n = \binom{k}{2} + \frac{k}{2} + 1 \), then the positive root \( x_p \) of \( \frac{1}{2}(\mu^2 + 3\mu + 2) - n = \frac{1}{2} \) satisfies \( x_p > k - \frac{3}{2} \). By [Hui] Theorem 7.2, we conclude that \( \mu_n > k - \frac{3}{2} \). Combining these inequalities we deduce the following proposition.
Proposition 4.1. Let $Z$ be a general point of $\mathbb{P}^2[n]$. Let $\mathcal{W}_{\mu_n}$ be the collapsing wall.

1. If $n = \binom{k}{2}$, then $\mu_n = \text{reg}(\mathcal{I}_Z) - 1$.
2. If $n = \binom{k}{2} + s$ with $\frac{1}{2} \leq \frac{s}{k-1} > 0$, then
   \[ \text{reg}(\mathcal{I}_Z) - 1 - \frac{\max(k-s, \lceil \phi^{-1}(k+1) \rceil)}{k+1} \leq \mu_n \leq \text{reg}(\mathcal{I}_Z) - 1 - \frac{k-s}{k+1} \]
   and the right inequality is an equality if $1 - \frac{s+1}{k+1} \in S$.
3. If $n = \binom{k}{2} + s$ with $\frac{s}{k-1} \geq \frac{1}{2}$, then
   \[ \text{reg}(\mathcal{I}_Z) - \frac{3}{2} \leq \mu_n \leq \text{reg}(\mathcal{I}_Z) - 2 + \frac{s}{k-1} \]
   and the right inequality is an equality if $\frac{s}{k-1} \in S$.

In particular, $\text{reg}(\mathcal{I}_Z) - 2 < \mu_n \leq \text{reg}(\mathcal{I}_Z) - 1$ for a general $Z$.

We point out that the sets $U_1 - U_2$ and $U_2 - U_1$ are both nonempty in general.

Example 4.2. The minimum regularity for a scheme $Z$ of length 7 is 4 and $\mu_7 = \frac{11}{5}$ [Hui, Table 1]. Consider the monomial scheme generated with defining ideal $\langle x^4, xy, y^4 \rangle$. The regularity of this scheme is 4, but it is destabilized along the wall $\mathcal{W}_3$. Hence, this monomial scheme is a point of $U_1$ which is not in $U_2$.

Example 4.3. The minimum regularity for a scheme $Z$ of length 9 is 4. For a complete intersection scheme of type $(3, 3)$, the minimal resolution is
\[ 0 \to \mathcal{O}(-6) \to \mathcal{O}(-3) \oplus \mathcal{O}(-3) \to \mathcal{I}_Z \to 0. \]
Hence, the regularity is 5. On the other hand, the general scheme and a complete intersection scheme both have $\mu = 3$ [ABCH13, CH14, Theorem 5.1]. Hence, the complete intersection scheme is in $U_2$ but not in $U_1$.

5. Outer walls of the Bridgeland manifold

In general, it is hard to test whether a specific ideal sheaf $\mathcal{I}_Z$ is destabilized along a given wall $\mathcal{W}_\mu$. However, for the largest $\left\lfloor \frac{n}{2} \right\rfloor$ semicircular Bridgeland walls, one can give a concrete characterization of the ideal sheaves destabilized along the wall. This characterization allows us to compute the regularity.

Let $Y_\mu^n$ denote the locally closed subset of $\mathbb{P}^2[n]$ parameterizing subschemes $Z$ destabilized along $\mathcal{W}_\mu$. By the one-to-one correspondence between the Bridgeland walls and Mori walls [ABCH13], we may rephrase [ABCH13 Proposition 4.16] as follows.

Proposition 5.1. Let $n \leq k(k+3)/2$. Let $\mathcal{W}_k$ be the wall with center $x = -k - \frac{3}{2}$.

(a) If $n \leq 2k + 1$, then $Y_\mu^n$ parameterizes $Z$ that have a linear subscheme of length $k + 2$ but no linear subscheme of length greater than $k + 2$.
(b) If $n = 2k + 2$, then $Y_\mu^n$ parameterizes $Z$ that are contained in a conic or have a linear subscheme of length $k + 2$ but do not have a linear subscheme of length greater than $k + 2$.

Fatabbi’s theorem [Fat94] allows us to say more about the regularity of the schemes destabilized along $\mathcal{W}_k$.
Proposition 5.2 (Fat points). Let $p_i$ be the maximal ideals of distinct closed points $p_i \in \mathbb{P}^1$, $i = 1, \ldots, s$. Let $Z$ be the subscheme given by $\bigcap_{i=1}^s p_i^{m_{ij}}$ and suppose that $Z$ is of length $n$. Define

$$h := \max \left\{ \sum_{j=1}^t m_{ij} \mid p_{i_1}, \ldots, p_{i_t} \text{ are collinear} \right\}.$$  

If $n \leq 2h - 3$, then $Z$ is destabilized at the wall $W_{\text{reg}(Z)} - 1$. In particular, a general member of $Y^n_{k+1}$ has a higher regularity than a general member of $Y^n_k$, $\forall k \geq \frac{n}{2} - 1$.

Proof. The assumption $n \leq 2h - 3$ allows us to apply [Fat94, Theorem 3.3] and conclude that the regularity of $Z$ equals $h$. We shall prove that $Z$ has no linear subschemes of length $h + 1$. Let $L$ be a linear subscheme of $Z$ supported on $p_{i_1}, \ldots, p_{i_t}$. Let $f$ be a linear form vanishing on $p_{i_1}, \ldots, p_{i_t}$. Then $p_{ij} = \langle f, g_{ij} \rangle$ for some linear form $g_{ij}$ and $f$ and $p_{ij}^{m_{ij}}$, $j = 1, \ldots, t$, are contained in the ideal $I_L$ of $L$.

For the length of $L$ to be as large as possible, we take the smallest possible ideal that contains $f + \sum_{j=1}^t p_{ij}$. Since $p_{ij}^{m_{ij}} = \langle f^{m_{ij}}, f^{m_{ij}-1}g_{ij}, \ldots, g_{ij}^{m_{ij}} \rangle$, any ideal containing $f + \sum_{j=1}^t p_{ij}$ must also contain $g_{ij}^{m_{ij}}$. It follows that $\langle f, g_{ij}^{m_{ij}} \rangle \cap \cdots \cap \langle f, g_{ij}^{m_{ij}} \rangle$ defines a linear subscheme of $Z$ of maximal length $\sum_{j=1}^t m_{ij}$ supported on the cycle $\sum_{j=1}^t m_{ij}p_{ij}$. Since the regularity $h$ is the maximum that the degree $\sum_{j=1}^t m_{ij}$ can achieve, it is the maximum length of a linear subscheme of $Z$. Now, since $n \leq 2(h - 2) + 1$ by assumption, we may apply Proposition 5.1 and obtain the first assertion.

General points $Z$ of $Y^n_k$, $k \geq \frac{n}{2} - 1$, have no multiplicities, i.e. $m_{ij} = 1 \forall i$, have $k + 2$ collinear points, and the rest are in general position. This corresponds to the case $h = k + 2 \geq \frac{n}{2} + 1 > \left[ \frac{n}{2} \right]$, so Fatabbi’s theorem applies and $\text{reg}(I_Z) = h = k + 2$.

We emphasize again that, as we have noted in the introduction, the relation between regularity and the Bridgeland slope in general is not monotonic (Example 1.1).

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