

## A “QUANTUM” RAMSEY THEOREM FOR OPERATOR SYSTEMS

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ABSTRACT. Let  $\mathcal{V}$  be a linear subspace of  $M_n(\mathbb{C})$  which contains the identity matrix and is stable under the formation of Hermitian adjoints. We prove that if  $n$  is sufficiently large, then there exists a rank  $k$  orthogonal projection  $P$  such that  $\dim(P\mathcal{V}P) = 1$  or  $k^2$ .

### 1. BACKGROUND

An *operator system* in finite dimensions is a linear subspace  $\mathcal{V}$  of  $M_n(\mathbb{C})$  with the properties

- $I_n \in \mathcal{V}$ ,
- $A \in \mathcal{V} \Rightarrow A^* \in \mathcal{V}$ ,

where  $I_n$  is the  $n \times n$  identity matrix and  $A^*$  is the Hermitian adjoint of  $A$ . In this paper the scalar field will be complex and we will write  $M_n = M_n(\mathbb{C})$ .

Operator systems play a role in the theory of quantum error correction. In classical information theory, the “confusability graph” is a bookkeeping device which keeps track of possible ambiguity that can result when a message is transmitted through a noisy channel. It is defined by taking as vertices all possible source messages, and placing an edge between two messages if they are sufficiently similar that data corruption could lead to them being indistinguishable on reception. Once the confusability graph is known, one is able to overcome the problem of information loss by using an independent subset of the confusability graph, which is known as a “code”. If it is agreed that only code messages will be sent, then we can be sure that the intended message is recoverable.

When information is stored in quantum mechanical systems, the problem of error correction changes radically. The basic theory of quantum error correction was laid down in [3]. In [2] it was suggested that in this setting the role of the confusability graph is played by an operator system, and it was shown that for every operator system a “quantum Lovász number” could be defined, in analogy to the classical Lovász number of a graph. This is an important parameter in classical information theory. See also [5] for much more along these lines.

The interpretation of operator systems as “quantum graphs” was also proposed in [8], based on the more general idea of regarding linear subspaces of  $M_n$  as “quantum relations”, and taking the conditions  $I_n \in \mathcal{V}$  and  $A \in \mathcal{V} \Rightarrow A^* \in \mathcal{V}$  to respectively

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express reflexivity and symmetry conditions. The idea is that the edge structure of a classical graph can be encoded in an obvious way as a reflexive, symmetric relation on a set. This point of view was explicitly connected to the quantum error correction literature in [9].

Ramsey's theorem states that for any  $k$  there exists  $n$  such that every graph with at least  $n$  vertices contains either a  $k$ -clique or a  $k$ -anticlique, i.e., a set of  $k$  vertices among which either all edges are present or no edges are present. Simone Severini asked the author whether there is a "quantum" version of this theorem for operator systems. The natural notions of  $k$ -clique and  $k$ -anticlique are the following.

**Definition 1.1.** Let  $\mathcal{V} \subseteq M_n$  be an operator system. A *quantum  $k$ -clique* of  $\mathcal{V}$  is an orthogonal projection  $P \in M_n$  (i.e., a matrix satisfying  $P = P^2 = P^*$ ) whose rank is  $k$ , such that  $P\mathcal{V}P = \{PAP : A \in \mathcal{V}\}$  is maximal; that is, such that  $P\mathcal{V}P = PM_nP \cong M_k$ , or equivalently,  $\dim(P\mathcal{V}P) = k^2$ . A *quantum  $k$ -anticlique* of  $\mathcal{V}$  is a rank  $k$  projection  $P$  such that  $P\mathcal{V}P$  is minimal; that is, such that  $P\mathcal{V}P = \mathbb{C} \cdot P \cong M_1$ , or equivalently,  $\dim(P\mathcal{V}P) = 1$ .

The definition of quantum  $k$ -anticlique is supported by the fact that in quantum error correction a code is taken to be the range of a projection satisfying just this condition,  $P\mathcal{V}P = \mathbb{C} \cdot P$  [3]. As mentioned earlier, classical codes are taken to be independent sets, which is to say, anticliques. See also Section 4 of [9], where intuition for why  $P\mathcal{V}P$  is correctly thought of as a "restriction" of  $\mathcal{V}$  is given.

The main result of this paper is a quantum Ramsey theorem which states that for every  $k$  there exists  $n$  such that every operator system in  $M_n$  (or indeed, in  $M_{n'}$  for any  $n' \geq n$ ) has either a quantum  $k$ -clique or a quantum  $k$ -anticlique. This answers Severini's question positively. The quantum Ramsey theorem is not merely analogous to the classical Ramsey theorem; using the bimodule formalism of [8], we can formulate a common generalization of the two results. This will be done in the final section of the paper.

## 2. EXAMPLES

If  $G = (V, \mathcal{E})$  is any finite simple graph, without loss of generality suppose  $V = \{1, \dots, n\}$  and define  $\mathcal{V}_G$  to be the operator system

$$\mathcal{V}_G = \text{span}\{E_{ij} : i = j \text{ or } \{i, j\} \in \mathcal{E}\} \subseteq M_n.$$

Here we use the notation  $E_{ij}$  for the  $n \times n$  matrix with a 1 in the  $(i, j)$  entry and 0's elsewhere. Also, let  $(e_i)$  be the standard basis of  $\mathbb{C}^n$ , so that  $E_{ij} = e_i e_j^*$ .

The inclusion of the diagonal  $E_{ii}$  matrices in  $\mathcal{V}_G$  corresponds to including a loop at each vertex in  $G$ . In the error correction setting this is natural: we place an edge between any two messages that might be indistinguishable on reception, and this is certainly true of any message and itself. Once we adopt the convention that every graph has a loop at each vertex, an anticlique should no longer be a subset  $S \subseteq V$  which contains no edges, it should be a subset which contains no edges except loops. Such a set corresponds to the projection  $P_S$  onto  $\text{span}\{e_i : i \in S\}$ , which has the property that  $P_S \mathcal{V}_G P_S = \text{span}\{E_{ii} : i \in S\}$ . Of course this is very different from a quantum anticlique where  $P\mathcal{V}P$  is one-dimensional.

To illustrate the dissimilarity between classical and quantum cliques and anticliques, consider the *diagonal operator system*  $D_n \subseteq M_n$  consisting of the diagonal

$n \times n$  complex matrices. In the notation used above, this is just the operator system  $\mathcal{V}_G$  corresponding to the empty graph on  $n$  vertices. It might at first appear to falsify the desired quantum Ramsey theorem, because of the following fact.

**Proposition 2.1.**  *$D_n$  has no quantum  $k$ -anticlique for  $k \geq 2$ .*

*Proof.* Let  $P \in M_n$  be a projection of rank  $k \geq 2$ . Since  $\text{rank}(E_{ii}) = 1$  for all  $i$ , it follows that  $\text{rank}(PE_{ii}P) = 0$  or  $1$  for each  $i$ . If  $PE_{ii}P = 0$  for all  $i$ , then  $P = \sum_{i=1}^n PE_{ii}P = 0$ , a contradiction. Thus we must have  $\text{rank}(PE_{ii}P) = 1$  for some  $i$ , but then  $PE_{ii}P$  cannot belong to  $\mathbb{C} \cdot P = \{aP : a \in \mathbb{C}\}$ , since every matrix in this set has rank  $0$  or  $k$ . So  $PD_nP \neq \mathbb{C} \cdot P$ .  $\square$

Since every operator system of the form  $\mathcal{V}_G$  contains the diagonal matrices, none of these operator systems has nontrivial quantum anticliques. The surprising thing is that for  $n$  sufficiently large, they all have quantum  $k$ -cliques. This follows from the next result.

**Proposition 2.2.** *If  $n \geq k^2 + k - 1$ , then  $D_n$  has a quantum  $k$ -clique.*

*Proof.* Without loss of generality let  $n = k^2 + k - 1$ . Start by considering  $M_k$  acting on  $\mathbb{C}^k$ . Find  $k^2$  vectors  $v_1, \dots, v_{k^2}$  in  $\mathbb{C}^k$  such that the rank 1 matrices  $v_i v_i^*$  are linearly independent. (For example, we could take the  $k$  standard basis vectors  $e_i$  plus the  $\frac{k^2-k}{2}$  vectors  $e_i + e_j$  for  $i \neq j$  plus the  $\frac{k^2-k}{2}$  vectors  $e_i + i e_j$  for  $i \neq j$ . The corresponding rank 1 matrices span  $M_k$  and thus they must be independent since  $\dim(M_k) = k^2$ .) Making the identification  $\mathbb{C}^n \cong \mathbb{C}^k \oplus \mathbb{C}^{k^2-1}$ , we can extend the  $v_i$  to orthogonal vectors  $w_i \in \mathbb{C}^n$  as follows: take  $w_1 = v_1 \oplus (1, 0, \dots, 0)$ ,  $w_2 = v_2 \oplus (a_1, 1, 0, \dots, 0)$ ,  $w_3 = v_3 \oplus (b_1, b_2, 1, 0, \dots, 0)$ , etc., with  $a_1, b_1, b_2, \dots$  successively chosen so that  $\langle w_i, w_j \rangle = 0$  for  $i \neq j$ . Since  $w_{k^2}$  does not require a final “1” entry, we need  $k^2 - 1$  extra dimensions to accomplish this. Now let  $P$  be the rank  $k$  projection of  $\mathbb{C}^n$  onto  $\mathbb{C}^k$  and let  $D_n$  be the diagonal operator system relative to any orthonormal basis of  $\mathbb{C}^n$  that contains the vectors  $\frac{w_i}{\|w_i\|}$  for  $1 \leq i \leq k^2$ . Then  $PD_nP$  contains  $Pw_i w_i^* P = v_i v_i^*$  for all  $i$ , so  $\dim(PD_nP) = k^2$ .  $\square$

A stronger version of this result will be proven in Lemma 4.3. The value  $n = k^2 + k - 1$  may not be optimal, but note that in order for  $D_n$  to have a quantum  $k$ -clique  $n$  must be at least  $k^2$ , since  $\dim(D_n) = n$  and we need  $\dim(PD_nP) = k^2$ .

Next, we show that operator systems of arbitrarily large dimension may lack quantum 3-cliques.

**Proposition 2.3.** *Let  $\mathcal{V}_n = \text{span}\{I_n, E_{11}, E_{12}, \dots, E_{1n}, E_{21}, \dots, E_{n1}\} \subseteq M_n$ . Then  $\mathcal{V}_n$  has no quantum 3-cliques.*

*Proof.* Let  $P \in M_n$  be any projection. If  $Pe_1 = 0$ , then  $PE_{1i}P = PE_{i1}P = 0$  for all  $i$ , so  $P$  is a quantum anticlique. Otherwise let  $k = \text{rank}(P)$  and let  $f_1, \dots, f_k$  be an orthonormal basis of  $\text{rank}(P)$  with  $f_1 = \frac{Pe_1}{\|Pe_1\|}$ . Then  $PE_{1i}P = Pe_1 e_i^* P = f_1 v_i^*$  where  $v_i = \|Pe_1\| Pe_i$ . The span of these matrices  $f_1 v_i^*$  is precisely  $\text{span}\{f_1 f_i^*\}$ , since the projections of the  $e_i$  span  $\text{rank}(P)$ . Similarly, the span of the matrices  $PE_{i1}P$  is precisely  $\text{span}\{f_i f_1^*\}$ . So  $P\mathcal{V}_n P$  is just  $\mathcal{V}_k \subseteq M_k \cong PM_n P$ , relative to the  $(f_i)$  basis. If  $k \geq 3$ , then  $\dim(\mathcal{V}_k) = 2k < k^2$ , so  $P$  cannot be a quantum clique.  $\square$

### 3. QUANTUM 2-CLIQUE

In contrast to Proposition 2.3, we will show in this section that any operator system whose dimension is at least four must have a quantum 2-clique. This result is clearly sharp. It is somewhat analogous to the trivial classical fact that any graph that contains at least one edge must have a 2-clique.

Define the Hilbert-Schmidt inner product of  $A, B \in M_n$  to be  $\text{Tr}(AB^*)$ . Denote the set of Hermitian  $n \times n$  matrices by  $M_n^h$ . Observe that any operator system is spanned by its Hermitian part since any matrix  $A$  satisfies  $A = \text{Re}(A) + i\text{Im}(A)$  where  $\text{Re}(A) = \frac{1}{2}(A + A^*)$  and  $\text{Im}(A) = \frac{1}{2i}(A - A^*)$ .

**Lemma 3.1.** *Let  $\mathcal{V} \subseteq M_n$  be an operator system and suppose  $\dim(\mathcal{V}) \leq 3$ . Then its Hilbert-Schmidt orthocomplement is spanned by rank 2 Hermitian matrices.*

*Proof.* Work in  $M_n^h$ . Let  $\mathcal{V}_0 = \mathcal{V} \cap M_n^h$  and let  $\mathcal{W}_0$  be the real span of the Hermitian matrices in  $\mathcal{V}_0^\perp$  whose rank is 2. We will show that  $\mathcal{W}_0 = \mathcal{V}_0^\perp$  (in  $M_n^h$ ); taking complex spans then yields the desired result.

Suppose to the contrary that there exists a nonzero Hermitian matrix  $B \in \mathcal{V}_0^\perp$  which is orthogonal to  $\mathcal{W}_0$ . Say  $\mathcal{V}_0 = \text{span}\{I_n, A_1, A_2\}$ , where  $A_1$  and  $A_2$  are not necessarily distinct from  $I_n$ . Since  $B \in \mathcal{V}_0^\perp$ , we have  $\text{Tr}(I_n B) = \text{Tr}(A_1 B) = \text{Tr}(A_2 B) = 0$ , but  $\text{Tr}(B^2) \neq 0$ . We will show that there is a rank 2 Hermitian matrix  $C$  whose inner products against  $I_n, A_1, A_2$ , and  $B$  are the same as their inner products against  $B$ . This will be a matrix in  $\mathcal{W}_0$  which is not orthogonal to  $B$ , a contradiction.

Since  $B$  is Hermitian, we can choose an orthonormal basis  $(f_i)$  of  $\mathbb{C}^n$  with respect to which it is diagonal, say  $B = \text{diag}(b_1, \dots, b_n)$ . We may assume  $b_1, \dots, b_j \geq 0$  and  $b_{j+1}, \dots, b_n < 0$ . Let

$$B^+ = \text{diag}(b_1, \dots, b_j, 0, \dots, 0)$$

and

$$B^- = \text{diag}(0, \dots, 0, -b_{j+1}, \dots, -b_n)$$

be the positive and negative parts of  $B$ , so that  $B = B^+ - B^-$ . Let  $\alpha = \text{Tr}(B^+) = \text{Tr}(B^-)$  (they are equal since  $\text{Tr}(B) = \text{Tr}(I_n B) = 0$ ). Then  $\frac{1}{\alpha}B^+$  is a convex combination of the rank 1 matrices  $f_1 f_1^*, \dots, f_j f_j^*$ ; consequently, the linear functional  $A \mapsto \frac{1}{\alpha}\text{Tr}(AB^+)$  is a convex combination of the linear functionals  $A \mapsto \langle A f_i, f_i \rangle$  for  $1 \leq i \leq j$ . By the convexity of the joint numerical range of three Hermitian matrices [1], there exists a unit vector  $v \in \mathbb{C}^n$  such that  $\frac{1}{\alpha}\text{Tr}(AB^+) = \langle Av, v \rangle$  for  $A = A_1, A_2$ , and  $B$ . Similarly, there exists a unit vector  $w$  such that  $\frac{1}{\alpha}\text{Tr}(AB^-) = \langle Aw, w \rangle$  for  $A = A_1, A_2$ , and  $B$ . Then  $C = \alpha(vv^* - ww^*)$  is a rank 2 Hermitian matrix whose inner products against  $I_n, A_1, A_2$ , and  $B$  are the same as their inner products against  $B$ . So  $C$  has the desired properties.  $\square$

**Lemma 3.2.** *Let  $\mathcal{V} \subseteq M_3$  be an operator system and suppose  $\dim(\mathcal{V}) = 4$ . Then  $\mathcal{V}$  has a quantum 2-clique.*

*Proof.* The proof is computational. Say  $\mathcal{V} = \text{span}\{A_0, A_1, A_2, A_3\}$  where  $A_0 = I_3$  and the other  $A_i$  are Hermitian. It will suffice to find two vectors  $v, w \in \mathbb{C}^3$  such that the four vectors  $[\langle A_i v, v \rangle, \langle A_i v, w \rangle, \langle A_i w, v \rangle, \langle A_i w, w \rangle] \in \mathbb{C}^4$  for  $0 \leq i \leq 3$  are independent. That is, we need the  $4 \times 4$  matrix whose rows are these vectors to have nonzero determinant. Then letting  $P$  be the orthogonal projection onto  $\text{span}\{v, w\}$  will verify the lemma.

We can simplify by putting the  $A_i$  in a special form. First, by choosing a basis of eigenvectors, we can assume  $A_1$  is diagonal. By subtracting a suitable multiple of  $A_0$  from  $A_1$ , multiplying by a nonzero scalar, and possibly reordering the basis vectors, we can arrange that  $A_1$  has the form  $\text{diag}(0, 1, a)$ . (Note that  $\dim(\mathcal{V}) = 4$  implies that  $A_1$  cannot be a scalar multiple of  $A_0$ .) These operations do not affect  $\text{span}\{A_0, A_1, A_2, A_3\}$ . Then, by subtracting suitable linear combinations of  $A_0$  and  $A_1$ , we can arrange that  $A_2$  and  $A_3$  have the forms

$$\begin{bmatrix} 0 & a_{12} & a_{13} \\ \bar{a}_{12} & 0 & a_{23} \\ \bar{a}_{13} & \bar{a}_{23} & a_{33} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & b_{12} & b_{13} \\ \bar{b}_{12} & 0 & b_{23} \\ \bar{b}_{13} & \bar{b}_{23} & b_{33} \end{bmatrix}.$$

Let

$$v = \begin{bmatrix} 1 \\ \alpha \\ 0 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} 1 \\ 0 \\ \beta \end{bmatrix};$$

then evaluate the determinant of the  $4 \times 4$  matrix described above and expand it as a polynomial in  $\alpha, \beta, \bar{\alpha}$ , and  $\bar{\beta}$ . We just need this determinant to be nonzero for some values of  $\alpha$  and  $\beta$ ; if this fails, then the polynomial coefficients must all be zero, and direct computation shows that this forces one of  $A_2$  and  $A_3$  to be a scalar multiple of the other. We omit the tedious but straightforward details.<sup>1</sup>  $\square$

**Theorem 3.3.** *Let  $\mathcal{V} \subseteq M_n$  be an operator system and suppose  $\dim(\mathcal{V}) \geq 4$ . Then  $\mathcal{V}$  has a quantum 2-clique.*

*Proof.* Without loss of generality we can suppose that  $\dim(\mathcal{V}) = 4$ . Say  $\mathcal{V} = \text{span}\{I_n, A_1, A_2, A_3\}$  where the  $A_i$  are Hermitian and linearly independent.

We first claim that there is a projection  $P$  of rank at most 3 such that  $PI_nP, PA_1P,$  and  $PA_2P$  are linearly independent. If  $A_1$  and  $A_2$  are jointly diagonalizable, then we can find three common eigenvectors  $v_1, v_2,$  and  $v_3$  such that the vectors  $(1, 1, 1), (\lambda_1, \lambda_2, \lambda_3), (\mu_1, \mu_2, \mu_3) \in \mathbb{C}^3$  are linearly independent, where  $\lambda_i$  and  $\mu_i$  are the eigenvalues belonging to  $v_i$  for  $A_1$  and  $A_2$ , respectively. Then the projection onto  $\text{span}\{v_1, v_2, v_3\}$  verifies the claim. If  $A_1$  and  $A_2$  are not jointly diagonalizable, then we can find two eigenvectors  $v_1$  and  $v_2$  of  $A_1$  such that  $\langle A_2v_1, v_2 \rangle \neq 0$ . Letting  $v_3$  be a third eigenvector of  $A_1$  with the property that the eigenvalues of  $A_1$  belonging to  $v_1, v_2,$  and  $v_3$  are not all equal, we can again use the projection onto  $\text{span}\{v_1, v_2, v_3\}$ . This establishes the claim.

Now let  $P$  be as in the claim and find  $B \in M_n$  such that  $PI_nP, PA_1P, PA_2P,$  and  $PBP$  are linearly independent. By Lemma 3.2 we can then find a rank 2 projection  $Q \leq P$  such that  $QI_nQ, QA_1Q, QA_2Q,$  and  $QBQ$  are linearly independent.

If  $QI_nQ, QA_1Q, QA_2Q,$  and  $QA_3Q$  are linearly independent, then we are done. Otherwise, let  $\alpha, \beta,$  and  $\gamma$  be the unique scalars such that  $QA_3Q = \alpha QI_nQ + \beta QA_1Q + \gamma QA_2Q$ . By Lemma 3.1 we can find a rank 2 Hermitian matrix  $C$  such that  $\text{Tr}(I_nC) = \text{Tr}(A_1C) = \text{Tr}(A_2C) = 0$  but  $\text{Tr}(A_3C) \neq 0$ . Then  $C = vv^* - ww^*$  for some orthogonal vectors  $v$  and  $w$ . Thus,  $\langle Av, v \rangle = \langle Aw, w \rangle$  for  $A = I_n, A_1,$  and  $A_2$ , but not for  $A = A_3$ . It follows that the two conditions

$$\langle A_3v, v \rangle = \alpha \langle I_nv, v \rangle + \beta \langle A_1v, v \rangle + \gamma \langle A_2v, v \rangle$$

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<sup>1</sup>First observe that if  $ay = bx$  and  $b\bar{y} = \bar{b}y$ , then  $a\bar{y} = x\bar{b}$ . The monomials  $\alpha^2\bar{\beta}^2, \alpha^2\bar{\alpha}^2\beta\bar{\beta}, \alpha\bar{\alpha}\beta^2\bar{\beta}^2, \beta^2\bar{\beta}^2$  suffice to handle the  $b\bar{y} = \bar{b}y$  terms and the monomials  $\alpha^2\bar{\alpha}\bar{\beta}, \alpha^2\bar{\alpha}^2\beta, \alpha^2\bar{\alpha}\beta\bar{\beta}, \alpha\beta^2\bar{\beta}^2, \alpha^2\bar{\alpha}\beta^2\bar{\beta}^2, \alpha\bar{\alpha}\beta^2\bar{\beta}, \alpha^2\bar{\alpha}\beta\bar{\beta}^2$  suffice to handle the mixed terms.

and

$$\langle A_3w, w \rangle = \alpha \langle I_n w, w \rangle + \beta \langle A_1 w, w \rangle + \gamma \langle A_2 w, w \rangle$$

cannot both hold. Without loss of generality suppose the first fails. Then letting  $Q'$  be the projection onto  $\text{span}(\text{ran}(Q) \cup \{v\})$ , we cannot have  $Q'A_3Q' = \alpha Q'I_nQ' + \beta Q'A_1Q' + \gamma Q'A_2Q'$ . Thus  $\text{rank}(Q') = 3$  and  $\dim(Q'\mathcal{V}Q') = 4$ . The theorem now follows by applying Lemma 3.2 to  $Q'\mathcal{V}Q'$ .  $\square$

Theorem 3.3 does not generalize to arbitrary four-dimensional subspaces of  $M_n$ . For instance, let  $\mathcal{V} = \text{span}\{E_{11}, E_{12}, E_{13}, E_{14}\} \subset M_4$ ; by reasoning similar to that in the proof of Proposition 2.3, if  $P$  is any rank 2 projection in  $M_4$ , then  $\dim(P\mathcal{V}P) \leq 2$ .

#### 4. THE MAIN THEOREM

The proof of our main theorem proceeds through a series of lemmas.

**Lemma 4.1.** *Suppose the operator system  $\mathcal{V}$  is contained in the diagonal operator system  $D_n$ . If  $\dim(\mathcal{V}) \geq k^2 + k - 1$ , then  $\mathcal{V}$  has a quantum  $k$ -clique. If  $\dim(\mathcal{V}) \leq \frac{n-k}{k-1}$ , then  $\mathcal{V}$  has a quantum  $k$ -anticlique. If  $n \geq k^3 - k + 1$ , then  $\mathcal{V}$  has either a quantum  $k$ -clique or a quantum  $k$ -anticlique.*

*Proof.* If  $\dim(\mathcal{V}) \geq k^2 + k - 1 = m$ , then we can find a set of indices  $S \subseteq \{1, \dots, n\}$  of cardinality  $m$  such that  $\dim(P\mathcal{V}P) = m$  where  $P$  is the orthogonal projection onto  $\text{span}\{e_i : i \in S\}$ . Then  $P\mathcal{V}P \cong D_m \subseteq M_m \cong PM_nP$  and Proposition 2.2 yields that  $P\mathcal{V}P$ , and hence also  $\mathcal{V}$ , has a quantum  $k$ -clique. If  $\dim(\mathcal{V}) \leq \frac{n-k}{k-1}$ , then a result of Tverberg [6, 7] can be used to extract a quantum  $k$ -anticlique; this is essentially Theorem 4 of [4]. Thus if  $k^2 + k - 1 \leq \frac{n-k}{k-1}$ , then one of the two cases must obtain, i.e.,  $\mathcal{V}$  must have either a quantum  $k$ -clique or a quantum  $k$ -anticlique. A little algebra shows that this inequality is equivalent to  $n \geq k^3 - k + 1$ .  $\square$

**Lemma 4.2.** *Let  $v_1, \dots, v_r$  be vectors in  $\mathbb{C}^s$ . Then there are vectors  $w_1, \dots, w_r \in \mathbb{C}^{r-1}$  such that the vectors  $v_i \oplus w_i \in \mathbb{C}^{s+r-1}$  are pairwise orthogonal and all have the same norm.*

*Proof.* Let  $G$  be the Gramian matrix of the vectors  $v_i$  and let  $\|G\|$  be its operator norm. Then  $\text{rank}(\|G\|I_r - G) \leq r - 1$ , so we can find vectors  $w_i \in \mathbb{C}^{r-1}$  whose Gramian matrix is  $\|G\|I_r - G$ . The Gramian matrix of the vectors  $v_i \oplus w_i$  is then  $\|G\|I_r$ , as desired.  $\square$

Then next lemma improves Proposition 2.2.

**Lemma 4.3.** *Let  $n = k^2 + k - 1$  and suppose  $A_1, \dots, A_{k^2}$  are Hermitian matrices in  $M_n$  such that for each  $i$  we have  $\langle A_i e_i, e_i \rangle = 1$ , and also  $\langle A_i e_r, e_s \rangle = 0$  whenever  $\max\{r, s\} > i$ . Then  $\mathcal{V} = \text{span}\{I, A_1, \dots, A_{k^2}\}$  has a quantum  $k$ -clique.*

*Proof.* Let  $A_i$  have matrix entries  $(a_{r,s}^i)$ . The goal is to find vectors  $v_1, \dots, v_{k^2} \in \mathbb{C}^k$  such that the matrices

$$A'_i = \sum_{1 \leq r, s \leq k^2} a_{r,s}^i v_r v_s^* \in M_k$$

are linearly independent. Once we have done this, find vectors  $w_i \in \mathbb{C}^{k^2-1}$  as in Lemma 4.2 and let  $f_i = \frac{1}{N}(v_i \oplus w_i) \in \mathbb{C}^n \cong \mathbb{C}^k \oplus \mathbb{C}^{k^2-1}$  where  $N$  is the common norm of the  $v_i \oplus w_i$ . Then the  $f_i$  form an orthonormal set in  $\mathbb{C}^n$ , so they can be extended to an orthonormal basis, and the operators whose matrices for this basis

are the  $A_i$  compress to the matrices  $\frac{1}{N^2}A'_i$  on the initial  $\mathbb{C}^k$ , which are linearly independent. So  $PVP$  contains  $k^2$  linearly independent matrices, where  $P$  is the orthogonal projection onto  $\mathbb{C}^k$ , showing that  $\mathcal{V}$  has a quantum  $k$ -clique.

The vectors  $v_i$  are constructed inductively. Once  $v_1, \dots, v_i$  are chosen so that  $A'_1, \dots, A'_i$  are independent, future choices of the  $v$ 's cannot change this since  $A_1, \dots, A_i$  all live on the initial  $i \times i$  block. We can let  $v_1$  be any nonzero vector in  $\mathbb{C}^k$ , since  $A_1 = e_1 e_1^*$ , so that  $A'_1 = v_1 v_1^*$  and this only has to be nonzero. Now suppose  $v_1, \dots, v_{i-1}$  have been chosen and we need to select  $v_i$  so that  $A'_i$  is independent of  $A'_1, \dots, A'_{i-1}$ . After choosing  $v_i$  we will have  $A'_i = \sum_{1 \leq r, s \leq i} a_{rs}^i v_r v_s^*$ . Let  $B$  be this sum restricted to  $1 \leq r, s \leq i-1$ . That part is already determined since  $v_i$  does not appear. Also let

$$u = a_{1i}^i v_1 + \dots + a_{(i-1)i}^i v_{i-1};$$

then we will have

$$A'_i = B + uv_i^* + v_i u^* + v_i v_i^*$$

(using the assumption that  $a_{ii}^i = 1$ ). That is,

$$A'_i = (B - uu^*) + (u + v_i)(u + v_i)^* = B' + \tilde{u}\tilde{u}^*$$

where  $B' = B - uu^*$  and  $\tilde{u} = u + v_i$  is arbitrary, and the question is whether  $\tilde{u}$  can be chosen to make this matrix independent of  $A'_1, \dots, A'_{i-1}$ . But the possible choices of  $A'_i$  span  $M_k$  (there is no matrix which is Hilbert-Schmidt orthogonal to  $B' + \tilde{u}\tilde{u}^*$  for all  $\tilde{u}$ ) so there must be a choice of  $\tilde{u}$  which makes  $A'_i$  independent of  $A'_1, \dots, A'_{i-1}$ , as desired.  $\square$

Next we prove a technical variation on Lemma 4.3.

**Lemma 4.4.** *Let  $n = k^4 + k^3 + k - 1$  and let  $\mathcal{V}$  be an operator system contained in  $M_n$ . Suppose  $\mathcal{V}$  contains matrices  $A_1, \dots, A_{k^4+k^3}$  such that for each  $i$  we have  $\langle A_i e_i, e_{i+1} \rangle \neq 0$ , and also  $\langle A_i e_r, e_s \rangle = 0$  whenever  $\max\{r, s\} > i + 1$  and  $r \neq s$ . Then  $\mathcal{V}$  has a quantum  $k$ -clique.*

*Proof.* Let  $A_i$  have matrix entries  $(a_{rs}^i)$ . Observe that for each  $i$  the compression of  $A_i$  to  $\text{span}\{e_{i+2}, \dots, e_n\}$  is diagonal. For each  $r > i + 1$ , let the  $r$ -tail of  $A_i$  be the vector  $(a_{rr}^i, \dots, a_{nn}^i)$ . Suppose there exist indices  $i_1, \dots, i_{k^2+k-1}$  such that the  $r$ -tails of the  $A_{i_j}$ ,  $1 \leq j \leq k^2 + k - 1$ , are linearly independent, where  $r = \max_j \{i_j + 2\}$ . Then the compression of  $\mathcal{V}$  to  $\text{span}\{e_r, \dots, e_n\}$  contains  $k^2 + k - 1$  linearly independent diagonal matrices, so it has a quantum  $k$ -clique by the first assertion of Lemma 4.1. Thus, we may assume that for any  $k^2 + k - 1$  distinct indices  $i_j$  the matrices  $A_{i_j}$  have linearly dependent  $r$ -tails.

We construct an orthonormal sequence of vectors  $v_i$  and a sequence of Hermitian matrices  $B_i \in \mathcal{V}$ ,  $1 \leq i \leq k^2$ , such that each  $v_i$  is orthogonal to each  $e_{k^4+k^3+j}$  for  $1 \leq j \leq k - 1$ , and the compressions of the  $B_i$  to  $\text{span}\{v_1, \dots, v_{k^2}, e_{k^4+k^3+1}, \dots, e_{k^4+k^3+k-1}\}$  satisfy the hypotheses of Lemma 4.3. This will ensure the existence of a quantum  $k$ -clique.

The first  $k^2 + k - 1$  matrices  $A_1, \dots, A_{k^2+k-1}$  have linearly dependent  $r$ -tails for  $r = k^2 + k + 1$ . Thus there is a nontrivial linear combination  $B'_1 = \sum_{i=1}^{k^2+k-1} \alpha_i A_i$  whose  $r$ -tail is the zero vector. Letting  $j$  be the largest index such that  $\alpha_j$  is nonzero, we have  $\langle B'_1 e_j, e_{j+1} \rangle \neq 0$  because  $\langle A_j e_j, e_{j+1} \rangle \neq 0$  but  $\langle A_i e_j, e_{j+1} \rangle = 0$  for  $i < j$ . Thus the compression of  $B'_1$  to  $\text{span}\{e_1, \dots, e_{k^2+k}\}$  is nonzero, so there exists a unit vector  $v_1$  in this span such that  $\langle B'_1 v_1, v_1 \rangle \neq 0$ . Then let  $B_1$  be a scalar

multiple of either the real or imaginary part of  $B'_1$  which satisfies  $\langle B_1 v_1, v_1 \rangle = 1$ . Note that  $\langle B_1 e_r, e_s \rangle = 0$  for any  $r, s$  with  $\max\{r, s\} > k^2 + k$ . Apply the same reasoning to the next block of  $k^2 + k - 1$  matrices  $A_{k^2+k+1}, \dots, A_{2k^2+2k-1}$  to find  $v_2$  and  $B_2$ , and proceed inductively. After  $k^2$  steps,  $k^2(k^2 + k) = k^4 + k^3$  indices will have been used up and  $k - 1$  (namely,  $e_{k^4+k^3+1}, \dots, e_{k^4+k^3+k-1}$ ) will remain, as needed.  $\square$

**Theorem 4.5.** *For any  $n \geq 8k^{11}$ , every operator system in  $M_n$  has either a quantum  $k$ -clique or a quantum  $k$ -anticlique.*

*Proof.* By first compressing  $M_n$  to  $M_{8k^{11}}$  we can reduce to the case that  $n = 8k^{11}$ . Fix this value and let  $\mathcal{V}$  be an operator system in  $M_n$ . Find a unit vector  $v_1 \in \mathbb{C}^n$ , if one exists, such that the dimension of  $\mathcal{V}v_1 = \{Av_1 : A \in \mathcal{V}\}$  is less than  $8k^8$ . Then find a unit vector  $v_2 \in (\mathcal{V}v_1)^\perp$ , if one exists, such that the dimension of  $(\mathcal{V}v_1)^\perp \cap (\mathcal{V}v_2)$  is less than  $8k^8$ . Proceed in this fashion, at the  $r$ th step trying to find a unit vector  $v_r$  in

$$(\mathcal{V}v_1)^\perp \cap \dots \cap (\mathcal{V}v_{r-1})^\perp$$

such that the dimension of

$$(\mathcal{V}v_1)^\perp \cap \dots \cap (\mathcal{V}v_{r-1})^\perp \cap (\mathcal{V}v_r)$$

is less than  $8k^8$ . If this construction lasts for  $k^3$  steps, then the compression of  $\mathcal{V}$  to  $\text{span}\{v_1, \dots, v_{k^3}\}$  (putting  $\mathcal{V}$  in  $M_{k^3}$ ) is contained in  $D_{k^3}$ , so this compression, and hence also  $\mathcal{V}$ , has either a quantum  $k$ -clique or a quantum  $k$ -anticlique by Lemma 4.1.

Otherwise, the construction fails at some stage  $d$ . This means that the compression  $\mathcal{W}$  of  $\mathcal{V}$  to  $F = (\mathcal{V}v_1)^\perp \cap \dots \cap (\mathcal{V}v_d)^\perp$  has the property that the dimension of  $\mathcal{W}v$  is at least  $8k^8$ , for every unit vector  $v \in F$ .

Work in  $F$ . Choose any nonzero vector  $w_1 \in F$  and find  $A_1 \in \mathcal{W}$  such that  $w_2 = A_1 w_1$  is nonzero and orthogonal to  $w_1$ . Then find  $A_2 \in \mathcal{W}$  such that  $w_3 = A_2 w_2$  is nonzero and orthogonal to  $\text{span}\{w_1, w_2, A_1 w_1, A_1^* w_1, A_1 w_2, A_1^* w_2\}$ . Continue in this way, at the  $r$ th step finding  $A_r \in \mathcal{W}$  such that  $w_{r+1} = A_r w_r$  is nonzero and orthogonal to  $\text{span}\{w_j, A_i w_j, A_i^* w_j : i < r \text{ and } j \leq r\}$ . The dimension of this span is at most  $2r^2 - r$ , so as long as  $r \leq 2k^4$  its dimension is less than  $8k^8$  and a vector  $w_{r+1}$  can be found. Compressing to the span of the  $w_i$  then puts us in the situation of Lemma 4.4 with  $n = 2k^4$ , which is more than enough. So there exists a quantum  $k$ -clique by that lemma.  $\square$

The constants in the proof could easily be improved, but only marginally. Very likely the problem of determining optimal bounds on quantum Ramsey numbers is open-ended, just as in the classical case.

### 5. A GENERALIZATION

In this section we will present a result which simultaneously generalizes the classical and quantum Ramsey theorems. This is less interesting than it sounds because the proof involves little more than a reduction to these two special cases. Perhaps the statement of the theorem is more significant than its proof.

At the beginning of Section 2 we showed how any simple graph  $G$  on the vertex set  $\{1, \dots, n\}$  gives rise to an operator system  $\mathcal{V}_G \subseteq M_n$ . This operator system has the special property that it is a bimodule over  $D_n$ , i.e., it is stable under left and right multiplication by diagonal matrices. Conversely, it is not hard to see that any



operator system in  $M_n$  which is also a  $D_n$ - $D_n$ -bimodule must have the form  $\mathcal{V}_G$  for some  $G$  [8, Propositions 2.2 and 2.5]. The general definition therefore goes as follows:

**Definition 5.1** ([8, Definition 2.6 (d)]). Let  $\mathcal{M}$  be a unital  $*$ -subalgebra of  $M_n$ . A quantum graph on  $\mathcal{M}$  is an operator system  $\mathcal{V} \subseteq M_n$  which satisfies  $\mathcal{M}'\mathcal{V}\mathcal{M}' = \mathcal{V}$ .

Here  $\mathcal{M}' = \{A \in M_n : AB = BA \text{ for all } B \in \mathcal{M}\}$  is the commutant of  $\mathcal{M}$ . This definition is actually representation-independent: if  $\mathcal{M}$  and  $\mathcal{N}$  are  $*$ -isomorphic unital  $*$ -subalgebras of two matrix algebras (possibly of different sizes), then the quantum graphs on  $\mathcal{M}$  naturally correspond to the quantum graphs on  $\mathcal{N}$  [8, Theorem 2.7]. More properly, one could say that the pair  $(\mathcal{M}, \mathcal{V})$  is the quantum graph, just as a classical graph is a pair  $(V, \mathcal{E})$ .

If  $\mathcal{M} = M_n$ , then its commutant is  $\mathbb{C} \cdot I_n$  and the bimodule condition in Definition 5.1 is vacuous: any operator system in  $M_n$  is a quantum graph on  $M_n$ . On the other hand, the commutant of  $\mathcal{M} = D_n$  is itself, so that by the comment made above, the quantum graphs on  $D_n$  — the operator systems which are  $D_n$ - $D_n$ -bimodules — correspond to simple graphs on the vertex set  $\{1, \dots, n\}$ . In this correspondence, subsets of  $\{1, \dots, n\}$  give rise to orthogonal projections  $P \in D_n$ , and the  $k$ -cliques and  $k$ -anticliques of the graph are realized in the matrix picture as rank  $k$  orthogonal projections  $P \in D_n$  which satisfy  $P\mathcal{V}_G P = PM_n P$  or  $PD_n P$ , respectively. This suggests the following definition.

**Definition 5.2.** Let  $\mathcal{M}$  be a unital  $*$ -subalgebra of  $M_n$  and let  $\mathcal{V} \subseteq M_n$  be a quantum graph on  $\mathcal{M}$ . A rank  $k$  projection  $P \in \mathcal{M}$  is a quantum  $k$ -clique if it satisfies  $P\mathcal{V}P = PM_n P$  and a quantum  $k$ -anticlique if it satisfies  $P\mathcal{V}P = P\mathcal{M}'P$ .

Since every operator system contains the identity matrix, if  $\mathcal{V}$  is a quantum graph on  $\mathcal{M}$ , then  $\mathcal{M}' \subseteq \mathcal{V}$ . So  $P\mathcal{V}P = P\mathcal{M}'P$  is the minimal possibility, as  $P\mathcal{V}P = PM_n P$  is the maximal possibility. Note the crucial requirement that  $P$  must belong to  $\mathcal{M}$ .

If  $\mathcal{M} = M_n$ , then  $\mathcal{M}' = \mathbb{C} \cdot I_n$  and the preceding definition duplicates the notions of quantum  $k$ -clique and quantum  $k$ -anticlique used earlier in the paper, whereas if  $\mathcal{M} = D_n$  it effectively reproduces the classical notions of  $k$ -clique and  $k$ -anticlique in a finite simple graph. In the classical setting fewer operator systems count as graphs, but one also has less freedom in the choice of  $P$  when seeking cliques or anticliques.

We require only the following simple lemma.

**Lemma 5.3.** Let  $\mathcal{V} \subseteq M_{nd} \cong M_n \otimes M_d$  be a quantum graph on  $M_n \otimes I_d$ . If  $nd \geq 8k^{11}$ , then there is a projection in  $M_n \otimes I_d$  whose rank is at least  $k$ , and which is either a quantum clique or a quantum anticlique of  $\mathcal{V}$ .

*Proof.* Since  $\mathcal{V}$  is a bimodule over  $(M_n \otimes I_d)' = I_n \otimes M_d$ , it has the form  $\mathcal{V} = \mathcal{W} \otimes M_d$  for some operator system  $\mathcal{W} \subseteq M_n$ . If  $d = 1$ , then the desired result was proven in Theorem 4.5, and if  $d \geq k$ , then any projection of the form  $P \otimes I_d$ , where  $P$  is a rank 1 projection in  $M_n$ , will have rank at least  $k$  and be both a quantum clique and a quantum anticlique. So assume  $2 \leq d < k$ .

Now if  $d \geq 3$ , then  $d^{10/11} > 2$ , so  $d < k(d^{10/11} - 1)$ . Thus  $1 < \frac{k}{d}(d^{10/11} - 1)$ , i.e.,  $\frac{k}{d} + 1 < \frac{k}{d} \cdot d^{10/11} = \frac{k}{d^{1/11}}$ , which implies  $(\frac{k}{d} + 1)^{11} < \frac{k^{11}}{d}$ . So finally

$$n \geq \frac{8k^{11}}{d} > 8 \left(\frac{k}{d} + 1\right)^{11} > 8 \left\lceil \frac{k}{d} \right\rceil^{11}.$$

If  $d = 2$ , then  $d < 3(d^{10/11} - 1)$ , so the same reasoning leads to the same inequality  $n \geq 8\lceil \frac{k}{d} \rceil^{11}$  provided  $k \geq 3$ . So we conclude that in all cases  $n \geq 8\lceil \frac{k}{d} \rceil^{11}$ . By Theorem 4.5,  $\mathcal{W}$  has a quantum  $\lceil \frac{k}{d} \rceil$ -clique or a quantum  $\lceil \frac{k}{d} \rceil$ -anticlique  $Q \in M_n$ . Then  $Q \otimes I_d$  is correspondingly either a quantum  $\lceil \frac{k}{d} \rceil \cdot d$ -clique or a quantum  $\lceil \frac{k}{d} \rceil \cdot d$ -anticlique of  $\mathcal{V}$ , which is enough.  $\square$

Note that we cannot promise a quantum  $k$ -clique or -anticlique, only a  $\geq k$ -clique or -anticlique, since the rank of any projection in  $M_n \otimes I_d$  is a multiple of  $d$ .

**Theorem 5.4.** *For every  $k$  there exists  $n$  such that for any  $n' \geq n$ , if  $\mathcal{M}$  is a unital  $*$ -subalgebra of  $M_{n'}$  and  $\mathcal{V} \subseteq M_{n'}$  is an operator system satisfying  $\mathcal{M}'\mathcal{V}\mathcal{M}' = \mathcal{V}$ , then there is a projection  $P \in \mathcal{M}$  whose rank is at least  $k$  and such that  $PVP = PM_nP$  or  $PM'P$ .*

*Proof.* Let  $R(k, k)$  be the classical Ramsey number and fix  $n \geq 8k^{11} \cdot R(k, k)$ . Now  $\mathcal{M}$  has the form  $(M_{n_1} \otimes I_{d_1}) \oplus \cdots \oplus (M_{n_r} \otimes I_{d_r})$  for some pair of sequences  $(n_1, \dots, n_r)$  and  $(d_1, \dots, d_r)$  such that  $n_1d_1 + \cdots + n_rd_r = n$ . Thus if  $r \leq R(k, k)$ , then for some  $i$  we must have  $n_id_i \geq 8k^{11}$ , and compressing to that block then yields the desired conclusion by appealing to the lemma. Otherwise, if  $r > R(k, k)$ , then choose a sequence of rank 1 projections  $Q_i \in M_{n_i}$  and work in  $QM_nQ$  where  $Q = (Q_1 \otimes I_{d_1}) \oplus \cdots \oplus (Q_r \otimes I_{d_r})$ . Then  $QM_nQ \cong M_{d_1+\dots+d_r}$ ,  $QM_nQ \cong \mathbb{C} \cdot I_{d_1} \oplus \cdots \oplus \mathbb{C} \cdot I_{d_r} \cong D_r$ , and  $Q\mathcal{V}Q$  is a bimodule over the commutant of  $QM_nQ$  in  $QM_nQ$ , i.e., the  $*$ -algebra  $M_{d_1} \oplus \cdots \oplus M_{d_r}$ . It follows that there is a graph  $G = (V, \mathcal{E})$  on the vertex set  $V = \{1, \dots, r\}$  such that  $Q\mathcal{V}Q$  has the form

$$Q\mathcal{V}Q = \sum E_{ij} \otimes M_{d_id_j},$$

taking the sum over the set of pairs  $\{(i, j) : i = j \text{ or } \{i, j\} \in \mathcal{E}\}$  [8, Theorem 2.7]. Since  $r > R(k, k)$ , there exists either a  $k$ -clique or a  $k$ -anticlique in  $G$ , and this gives rise to a diagonal projection in  $QM_nQ$  whose rank is at least  $k$  and which is either a quantum  $k$ -clique or a quantum  $k$ -anticlique of  $\mathcal{V}$ .  $\square$

Again, when  $\mathcal{M} = M_n$  Theorem 5.4 recovers the quantum Ramsey theorem and when  $\mathcal{M} = D_n$  it recovers the classical Ramsey theorem (though in both cases with worse constants).

Theorem 5.4 could also be proven by mimicking the proof of Theorem 4.5. However, in order to accommodate the requirement that  $P$  belong to  $\mathcal{M}$  we need to modify the last part of the proof so as to be sure that each  $w_r$  belongs to  $\mathcal{W}w_1 \cap \cdots \cap \mathcal{W}w_{r-1}$ . This means that instead of needing  $\mathcal{W}v$  to have sufficiently large dimension for each  $v$ , we need it to have sufficiently small codimension. Ensuring that this must be the case if the construction in the first part of the proof fails then requires that construction to take place in a space whose dimension is exponential in  $k$ . This explains the dramatic difference between classical and quantum Ramsey numbers (the first grow exponentially, the second polynomially).

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