

## AN IRREDUCIBILITY CRITERION FOR POWER SERIES

GUILLAUME ROND AND BERND SCHOBER

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ABSTRACT. We prove an irreducibility criterion for polynomials with power series coefficients generalizing previous results given by García Barroso and González-Pérez.

### 1. INTRODUCTION

The aim of this paper is to provide a simpler approach to an irreducibility criterion for polynomials with power series coefficients (see Theorem 2.4). The first version of the criterion has been given in [GBGP] and then has been generalized in [ACLM1]. In this paper we give a more natural and elementary proof of a general version of this criterion. In particular, our statement holds over any field while the previous ones were only proven for algebraically closed fields of characteristic zero. Moreover, the only hypothesis that we need is that the projection of the Newton polyhedron has exactly one vertex while the previous known versions were involving additional technical conditions, i.e., for  $\nu$ -quasi-ordinary polynomials (Let us mention that the criterion given in [ACLM1] is stated for  $\nu$ -quasi-ordinary polynomials while the provided proof apparently holds more generally for polynomials over algebraically closed fields of characteristic zero for which the projection of the Newton polyhedron has exactly one vertex).

Let us recall that the proof given in [GBGP] uses toric geometry and Zariski's Main Theorem while the one provided in [ACLM1] is based on a generalization of Newton's method for plane curves. Our proof is essentially based on the following well-known version of Hensel's Lemma (see for instance [EGA] (18.5.13)):

**Proposition 1.1** (Hensel's Lemma). *Let  $(R, \mathfrak{m})$  be a Henselian local ring and  $P = P(Z) \in R[Z]$  a monic polynomial. Suppose  $P$  is the product of two monic coprime polynomials  $P_1$  and  $P_2$  modulo  $\mathfrak{m}R[Z]$ ,  $P \equiv P_1 \cdot P_2 \pmod{\mathfrak{m}R[Z]}$ . Then  $P(Z)$  is the product of two coprime monic polynomials in  $R[Z]$  whose reductions modulo  $\mathfrak{m}R[Z]$  are equal to  $P_1$ , respectively,  $P_2$ .*

We begin by giving some definitions and our main result (Theorem 2.4). In a second part we give an example showing that our main result cannot be extended in a more general setting.

Finally, let us mention that this irreducibility criterion is very useful in the study of quasi-ordinary hypersurfaces (see [ACLM2] or [MS]). Different irreducibility criteria for quasi-ordinary power series were also shown in [A], [GBG] and [GV].

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2. AN IRREDUCIBILITY CRITERION

We denote by  $\mathbb{k}[[x]]$  the ring of formal power series in  $n$  variables  $x := (x_1, \dots, x_n)$  over a field  $\mathbb{k}$ . For any vector  $\beta \in \mathbb{Z}^n$  we set

$$x^\beta := x_1^{\beta_1} \cdots x_n^{\beta_n}$$

and for any positive integer  $q$ ,

$$x^q := x_1^q \cdots x_n^q.$$

Let  $P(Z) \in \mathbb{k}[[x]][Z]$  be a monic Weierstraß polynomial with coefficients in  $\mathbb{k}[[x]]$ . Let us write

$$P(Z) = Z^d + \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n, j < d} c_{\alpha,j} x^\alpha Z^j.$$

Recall that the *Newton polyhedron* of  $P = P(Z)$ , denoted by  $\text{NP}(P)$ , is the convex hull of  $\{(0, d), (\alpha, j) \mid c_{\alpha,j} \neq 0\} + \mathbb{R}_{\geq 0}^{n+1}$ . In this note we assume that  $P(Z) \neq Z^d$ .

The *associated polyhedron* of  $P$ , denoted by  $\Delta_P$ , is the convex hull of

$$\left\{ \frac{d\alpha}{d-j} \mid c_{\alpha,j} \neq 0 \right\} + \mathbb{R}_{\geq 0}^n.$$

Note that  $\Delta_P$  is the projection of  $\text{NP}(P)$  from the point  $(0, \dots, 0, d)$  on the subspace given by the first  $n$  coordinates.

**Definition 2.1.** Let  $\omega \in \mathbb{R}_{>0}^n$ . For a nonzero element  $b = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} b_\alpha x^\alpha$  of  $\mathbb{k}[[x]]$  we set

$$\nu_\omega(b) := \min\{\alpha \cdot \omega = \sum_{i=1}^n \alpha_i \omega_i \mid b_\alpha \neq 0\} \in \mathbb{R}_{\geq 0}$$

$$\text{and } \text{In}_\omega(b) := \sum_{\alpha \mid \alpha \cdot \omega = \nu_\omega(b)} b_\alpha x^\alpha.$$

For such a  $\omega$  and  $P(Z) \in \mathbb{k}[[x]][Z]$  as before we define  $\omega_{n+1} \in \mathbb{R}_{>0}$  by

$$\omega_{n+1} := \frac{\min\{v \cdot \omega \mid v \in \Delta_P\}}{d} \in \mathbb{R}_{>0}.$$

Then we set  $\omega' := (\omega, \omega_{n+1})$  and we define

$$\nu_{\omega'}(P) := \min\{\alpha \cdot \omega + j\omega_{n+1} \mid c_{\alpha,j} \neq 0\} = d\omega_{n+1}$$

$$\text{and } \text{In}_{\omega'}(P) := Z^d + \sum_{(\alpha,j) \mid (\alpha,j) \cdot \omega' = \nu_{\omega'}(P)} c_{\alpha,j} x^\alpha Z^j.$$

This former polynomial is weighted homogeneous for the weights  $\omega_1, \dots, \omega_n, \omega_{n+1}$ .

**Definition 2.2.** Let  $P(Z) \in \mathbb{k}[[x]][Z]$  be a monic polynomial of degree  $d$  in  $Z$ . The polynomial  $P$  has an *orthant associated polyhedron* if  $\Delta_P = d\gamma + \mathbb{R}_{\geq 0}^n$  for some  $\gamma \in \mathbb{Q}_{\geq 0}^n$ .

In this case  $\text{In}_{\omega'}(P)$  does not depend on  $\omega$  and we denote it by  $P_\Gamma$ , i.e.,

$$P_\Gamma(x, Z) := Z^d + \sum_{(\alpha,j) \mid \frac{\alpha}{d-j} = \gamma} c_{\alpha,j} x^\alpha Z^j,$$

where  $\Gamma$  is the compact edge of the Newton polyhedron of  $P$  containing the point  $(0, d)$ .

In this case we define

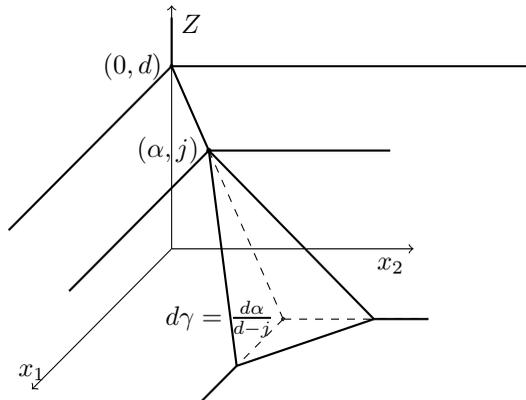
$$\overline{P}(Z) := P_\Gamma(1, Z) = Z^d + \sum_{(\alpha, j) | \frac{\alpha}{d-j} = \gamma} c_{\alpha, j} Z^j \in \mathbb{k}[Z].$$

If we write  $\gamma = \frac{\beta}{q}$ , where  $\beta \in \mathbb{Z}_{\geq 0}^n$ ,  $q \in \{1, \dots, d\}$  and  $\gcd(\beta_1, \dots, \beta_n, q) = 1$ , we have that

$$x^{d\beta} \overline{P}(Z) = P_\Gamma(x_1^q, \dots, x_n^q, x^\beta Z).$$

*Remark 2.3.* A polynomial  $P(Z) \in \mathbb{k}[[x]][Z]$  is called  $\nu$ -quasi-ordinary if it has an orthant associated polyhedron and  $P_\Gamma(x, Z)$  is not the power of a polynomial of degree one in  $Z$ . This definition has been introduced by H. Hironaka in [H] and is more standard, but more restrictive. Over a characteristic zero field if  $P(Z)$  has an orthant associated polyhedron and the coefficient of  $Z^{d-1}$  is zero, then  $P(Z)$  is  $\nu$ -quasi-ordinary.

Here is a picture of the Newton polyhedron of a polynomial having an orthant associated polyhedron with  $n = 2$  (thick lines represent the edges of the Newton polyhedron):



**Theorem 2.4.** Let  $P(Z) \in \mathbb{k}[[x]][Z]$  be a monic Weierstraß polynomial. Assume that  $P(Z)$  has an orthant associated polyhedron and that  $P_\Gamma(x, Z) \in \mathbb{k}[x, Z]$  is the product of two coprime monic polynomials  $P_1, P_2 \in \mathbb{k}[x, Z]$ , respectively, of degree  $d_1$  and  $d_2$ . Then there exist two monic polynomials  $S_1, S_2 \in \mathbb{k}[[x]][Z]$ , respectively, of degrees  $d_1$  and  $d_2$  in  $Z$  such that

- i)  $P = S_1 S_2$ ,
- ii) there is at least one  $i \in \{1, 2\}$  such that  $S_i$  has an orthant associated polyhedron and if  $\Gamma_i$  denotes the compact face of  $\text{NP}(S_i)$  containing the points  $(0, d_i)$ , then  $S_{i\Gamma_i} = P_i$  and  $\Gamma_i$  is parallel to  $\Gamma$ .

*Proof.* By assumption, we have for  $P_1 = P_1(x, Z)$  and  $P_2 = P_2(x, Z)$ ,

$$P_\Gamma(x, Z) = P_1(x, Z) \cdot P_2(x, Z)$$

and  $d = d_1 + d_2$ . Since  $Z$  cannot divide  $P_1$  and  $P_2$  simultaneously we may assume that  $P_1$  and  $Z$  are coprime. Let us write  $\overline{P}_i(Z) := P_i(1, Z)$  for  $i = 1, 2$ . Thus we have that

$$\overline{P}(Z) = \overline{P}_1(Z) \cdot \overline{P}_2(Z).$$

Moreover,  $Z$  and  $\bar{P}_1$  are coprime. We define the monomial map  $\sigma : \mathbb{k}[[x]][Z] \rightarrow \mathbb{k}[[x]][Z]$  by

$$\sigma(x_1) = x_1^q, \dots, \sigma(x_n) = x_n^q, \quad \sigma(Z) = x^\beta Z.$$

Let  $M = M(x, Z) := cx^\alpha Z^j$  be a monomial of  $P(x, Z)$ . We have that

$$\sigma(M) = cx^{q\alpha+j\beta} Z^j$$

and  $q\alpha + j\beta \succeq d\beta$  since  $\frac{\alpha}{d-j} \succeq \frac{\beta}{q}$  if  $j < d$ , where  $\succeq$  denotes the product order on  $\mathbb{R}_{\geq 0}^n$ . Thus we have

$$(1) \quad \sigma(P) = P(x_1^q, \dots, x_n^q, x^\beta Z) = x^{d\beta} (\bar{P}(Z) + Q(x, Z))$$

for some  $Q(x, Z) \in (x)\mathbb{k}[[x]][Z]$ . In particular,  $\bar{P}(Z) + Q(x, Z) = \bar{P}_1(Z)\bar{P}_2(Z)$  modulo  $(x)$ . Thus, by Hensel’s Lemma,

$$\bar{P}(Z) + Q(x, Z) = \tilde{P}_1(x, Z) \cdot \tilde{P}_2(x, Z),$$

for some monic polynomials  $\tilde{P}_1(x, Z)$  and  $\tilde{P}_2(x, Z) \in \mathbb{k}[[x]][Z]$  equal, respectively, to  $\bar{P}_1(Z)$  and  $\bar{P}_2(Z)$  modulo  $(x)$ . We obtain that

$$\sigma(P) = \left(x^{d_1\beta} \tilde{P}_1(x, Z)\right) \cdot \left(x^{d_2\beta} \tilde{P}_2(x, Z)\right).$$

Moreover, for every  $\omega' \in \mathbb{R}_{>0}^{n+1}$  given as in Definition 2.1 for the polynomial  $P(Z)$  we have that

$$\text{In}_{\omega'} \left(x^{d_i\beta} \tilde{P}_i(x, Z)\right) = \sigma(P_i) \text{ for } i = 1, 2.$$

But we have that

$$x^{d_i\beta} \tilde{P}_i(x, Z) = R_i(x, x^\beta Z)$$

for some monic polynomials  $R_i(x, Z) \in \mathbb{k}[[x]][Z]$  of degree  $d_i$ . Thus,

$$P(x^q, Z) = R_1(x, Z) \cdot R_2(x, Z)$$

and  $\text{In}_{\omega'}(R_i)(x, Z) = P_i(x^q, Z) \in \mathbb{k}[x^q, Z]$  for  $i = 1, 2$ . Since  $P_\Gamma = \text{In}_{\omega'}(P)$  for every  $\omega' \in \mathbb{R}_{>0}^{n+1}$  given as in Definition 2.1 for the polynomial  $P(Z)$ , we can apply Lemma 2.6 for  $P_0 = P(x^q, Z)$  to see that  $R_1(x, Z), R_2(x, Z) \in \mathbb{k}[x^q][Z]$  so they can be written as  $R_i(x, Z) = S_i(x^q, Z)$  for  $i = 1, 2$ . This means that  $\sigma(P) = \sigma(S_1)\sigma(S_2)$ . Thus,  $P = S_1 S_2$ .

Since  $\text{In}_{\omega'}(\tilde{P}_1)$  does not depend on  $\omega' \in \mathbb{R}_{>0}^{n+1}$  as above,  $\text{In}_{\omega'}(S_1)(x^q, Z)$  is also independent of  $\omega'$ . Moreover,  $\text{In}_{\omega'}(\tilde{P}_1)$  has at least two nonzero monomials since  $Z$  does not divide  $\bar{P}_1$ . Therefore,  $\text{In}_{\omega'}(S_1)(x^q, Z)$  has also at least two nonzero monomials. This shows that  $S_1$  has an orthant associated polyhedron and  $\Gamma_1$  is parallel to  $\Gamma$ .  $\square$

*Remark 2.5.* The key point in the proof of this theorem is the fact that equation (1) is satisfied when  $P$  has an orthant associated polyhedron.

**Lemma 2.6.** *Let  $P_0 \in \mathbb{k}[[x^q]][Z]$  be a monic Weierstraß polynomial, where  $q \in \mathbb{Z}_{>0}$ , and let us assume that  $P_0 = R_1 R_2$ , where  $R_1$  and  $R_2$  are monic polynomials of  $\mathbb{k}[[x]][Z]$ . Let  $\omega \in \mathbb{R}_{>0}^n$  and let  $\omega'$  be defined as in Definition 2.1. If  $\text{In}_{\omega'}(R_1), \text{In}_{\omega'}(R_2) \in \mathbb{k}[x^q, Z]$  and if they are coprime, then  $R_1, R_2 \in \mathbb{k}[[x^q]][Z]$ .*

*Proof.* If  $\text{char}(\mathbb{k}) = p > 0$  let us write  $q = p^e m$  with  $m \wedge p = 1$ . If  $\text{char}(\mathbb{k}) = 0$ , we set  $m := q$  and  $p := 1$ . Then we define

$$Q := \prod R_1(\xi_1 x_1, \dots, \xi_n x_n, Z)^{p^e},$$

where  $(\xi_1, \dots, \xi_n)$  runs over the  $n$ -uples of  $m$ -th roots of unity in an algebraic closure of  $\mathbb{k}$ . Then  $Q \in \mathbb{k}[[x^q]][Z]$  and  $\text{In}_{\omega'}(Q) = \text{In}_{\omega'}(R_1)^{m^n p^e}$ . Thus,  $\text{In}_{\omega'}(R_2)$  and  $\text{In}_{\omega'}(Q)$  are coprime. Therefore,  $R_2$  and  $Q$  are coprime. Since  $R_1$  divides  $Q$  and  $P_0 = R_1 R_2$ , then the greatest common divisor of  $P_0$  and  $Q$  in  $\mathbb{k}((x))[Z]$  is  $R_1$ . But the greatest common divisor does not depend on the base field, so  $R_1$  is also the greatest common divisor of  $P_0$  and  $Q$  in  $\mathbb{k}((x^q))[Z]$ . Hence there is an element  $F \in \mathbb{k}((x))^*$  such that  $FR_1 \in \mathbb{k}((x^q))[Z]$ . Since  $R_1(Z)$  is monic, then  $F \in \mathbb{k}((x^q))$ , thus  $R_1 \in \mathbb{k}[[x^q]][Z]$ . By symmetry we also get  $R_2 \in \mathbb{k}[[x^q]][Z]$ .  $\square$

**Corollary 2.7.** *Let us assume that  $P(Z) = Z^d + a_1 Z^{d-1} + \dots + a_d \in \mathbb{k}[[x]][Z]$  is an irreducible Weierstraß polynomial,  $P(Z) \neq Z^d$ . Then the following properties hold:*

- i) If  $P(Z)$  has an orthant associated polyhedron, then  $P_\Gamma$  is not the product of two coprime polynomials.*
- ii) If  $P(Z)$  has an orthant associated polyhedron, the convex hull of  $\text{Supp}(\text{In}_{\omega'}(P))$  is a segment joining  $(0, d)$  to  $(d\gamma, 0)$ , and  $d\gamma$  is the initial exponent of  $a_d$  for the valuation  $\nu_\omega$ , for every  $\omega' \in \mathbb{R}_{>0}^{n+1}$  given as in Definition 2.1 for the polynomial  $P(Z)$ .*
- iii) If  $P(Z)$  has an orthant associated polyhedron, let  $u \in \mathbb{Z}^{n+1}$  be the primitive vector such that  $mu = (-d\gamma, d)$  for some  $m \in \mathbb{N}$  and set  $y := (x, Z)$ . Then we can write*

$$P_\Gamma(x, Z) = x^{d\gamma} Q(y^u),$$

where  $Q(T) \in \mathbb{k}[T]$  is not the product of two coprime polynomials. In particular, if  $\mathbb{k}$  is algebraically closed, then  $Q(T)$  has only one root in  $\mathbb{k}$ .

- iv) If the Newton polyhedron of  $P(Z)$  has no compact face of dimension  $> 1$ , then  $P(Z)$  has an orthant associated polyhedron and its Newton polyhedron has only one compact face of dimension one which is the segment of *i*).*

*Proof.* Suppose  $P_\Gamma$  is the product of two coprime factors. Then Theorem 2.4 implies that we could factor  $P(Z)$  as the product of two monic polynomials (since  $P_\Gamma$  is monic). This contradicts the irreducibility of  $P(Z)$ . So *i*) is proven.

If  $P_\Gamma(x, 0) = 0$ , then  $Z$  divides  $P_\Gamma(x, Z)$ . But by *i*)  $P_\Gamma(x, Z)$  is not the product of two coprime polynomials, thus  $P_\Gamma(x, Z) = Z^d$ . This contradicts the fact that  $P_\Gamma(x, Z)$  has a nonzero monomial of the form  $x^\alpha Z^j$  for  $j < d$ . Hence,  $P_\Gamma(x, 0) \neq 0$  and *ii*) is proven.

We can write

$$P_\Gamma(x, Z) = Z^d + \sum_{j=0}^{d-1} c_{(d-j)\gamma, j} x^{(d-j)\gamma} Z^j.$$

So we have that

$$P_\Gamma(x, Z) = x^{d\gamma} \left( \frac{Z^d}{x^{d\gamma}} + \sum_{j=0}^{d-1} c_{(d-j)\gamma, j} \frac{Z^j}{x^{j\gamma}} \right).$$

By *ii*) we have that  $d\gamma \in \mathbb{Z}_{\geq 0}^n$ . This implies that  $j\gamma \in \mathbb{Z}_{\geq 0}^n$  as soon as  $c_{(d-j)\gamma, j} \neq 0$ . For any such  $j$ , let  $i \geq 0$  be such that

$$(2) \quad iu = (-j\gamma, j).$$

Then  $i \in \mathbb{Z}_{\geq 0}$  since  $u$  is primitive.

Thus,  $P_\Gamma(x, Z) = x^{d\gamma}(y^{mu} + \sum_{i < m} c_i y^{iu})$ , where

$$c_i := c_{iu+(d\gamma, 0)} \quad \forall i.$$

We set  $Q(T) := T^m + \sum_{i < m} c_i T^i \in \mathbb{k}[T]$ . If  $Q(T)$  factors as the product of two coprime monic polynomials, let us say  $Q(T) = Q_1(T) \cdot Q_2(T)$ , where  $Q_1(T)$  and  $Q_2(T) \in \mathbb{k}[T]$  are coprime and monic. Let  $m_1$  and  $m_2$  be the respective degrees of  $Q_1$  and  $Q_2$  and define  $d_i \in \mathbb{Z}_{\geq 0}$  by

$$(-d_i\gamma, d_i) = m_i u \text{ for } i = 1, 2.$$

Then we have

$$P_\Gamma(x, Z) = x^{d\gamma}Q(y^u) = (x^{d_1\gamma}Q_1(y^u)) \cdot (x^{d_2\gamma}Q_2(y^u)).$$

Moreover, by (2), a monomial of  $x^{d_1\gamma}Q_1(y^u)$  has the form

$$c x^{d_1\gamma} y^{iu} = c x^{d_1\gamma} \left( \frac{Z^j}{x^{j\gamma}} \right) = c x^{(d_1-j)\gamma} Z^j,$$

for  $0 \leq i \leq m_1$ , i.e., for  $0 \leq j \leq d_1$ . Hence,  $x^{d_1\gamma}Q_1(y^u) \in \mathbb{k}[x, Z]$ . By symmetry we also have that  $x^{d_2\gamma}Q_2(y^u) \in \mathbb{k}[x, Z]$ .

Then the polynomials  $P_1(x, Z) := x^{d_1\gamma}Q_1(y^u)$  and  $P_2(x, Z) := x^{d_2\gamma}Q_2(y^u)$  are coprime, which contradicts Theorem 2.4. Thus, *iii*) is proven.

Let us assume that the Newton polyhedron of  $P(Z)$  does not have an orthant associated polyhedron. This means that  $\Delta_P$  has at least two distinct vertices denoted by  $\gamma_1$  and  $\gamma_2$  such that the segment  $[\gamma_1, \gamma_2]$  is included in the boundary of  $\Delta_P$ . Thus the Newton polyhedron of  $P$  has at least three different vertices  $a := (0, d)$ ,  $b := (\frac{d-j}{d}\gamma_1, j)$  and  $c := (\frac{d-k}{k}\gamma_2, k)$ . Since  $a, b, c$  are vertices of  $\text{NP}(P)$  the triangle delimited by these three points is a face of  $\text{NP}(P)$ , so the Newton polyhedron of  $P$  has at least one face of dimension two. □

Theorem 2.4 also provides an elementary proof for the following generalization of [ACLM1, Theorem 1.5].

**Corollary 2.8.** *Let  $P(Z) \in \mathbb{k}[[x]][Z]$  be a Weierstraß polynomial not equal to a power of  $Z$  which has an orthant polyhedron and which is not necessarily irreducible. Then there exists at least one monic polynomial  $Q \in \mathbb{k}[[x]][Z]$  dividing  $P$  such that:*

- i) The polynomial  $Q$  has an orthant polyhedron.*
- ii) If  $\Gamma'$  denotes the compact edge of  $\text{NP}(Q)$  projecting to the unique vertex of  $\Delta_Q$ , then  $Q_{\Gamma'}$  is not the product of two coprime polynomials.*
- iii) The segment  $\Gamma'$  is parallel to  $\Gamma$ .*

*Proof.* If  $P_\Gamma(x, Z)$  is the power of some irreducible polynomial in  $\mathbb{k}[x, Z]$ , then we choose  $Q = P$ . Clearly, all required properties hold. (Note: even in this case  $P(Z)$  is not necessarily irreducible nor a power of an irreducible polynomial, e.g.,  $P(Z) = (Z^2 - x^3)^2 - x^8$ ).

Suppose  $P_\Gamma(x, Z)$  is the product of two coprime polynomials in  $\mathbb{k}[x, Z]$ . Since  $P$  is monic in  $Z$  we may assume that these two polynomials are monic. Denote by  $d$  the degree of  $P(Z)$ . By Theorem 2.4,  $P(Z) = S_1(Z) \cdot S_2(Z)$ , where  $S_i \in \mathbb{k}[[x]][Z]$

are polynomials of degree  $1 \leq d_i < d, i \in \{1, 2\}$ , and  $d_1 + d_2 = d$ . Moreover, we may assume that  $S_1$  has an orthant associated polyhedron such that its only compact face containing  $(0, d_1)$  is parallel to  $\Gamma$ . So the existence of  $Q$  is given by induction on  $d$  by replacing  $P$  by  $S_1$ .  $\square$

*Remark 2.9.* If we continue applying the theorem for those  $S_i$  in the proof having property  $i$ ), we obtain a factorization  $P = S_+ Q_1 \cdots Q_k$ , where each  $Q_i$  fulfills the properties  $i$ ) –  $iii$ ) whereas  $S_+$  does not. Moreover, using the obvious notation, the polynomials  $(Q_i)_{\Gamma_i}$  are pairwise coprime.

Let us mention that, using the notation of [ACLM1, Theorem 1.5], is valid only if all  $\mu_i$  are different from zero (which is implicitly assumed in the proof). But it is not true as stated in [ACLM1]. A correct version is given in [ACLM2]; see factorization Theorem 1.13 and, in particular, formula (1.3). Using our notation, this means that we cannot assume that  $S_+ = 1$ . For instance, set

$$P(Z) = (Z^2 + x)(Z^2 + x^3) = Z^4 + x(1 + x^2)Z^2 + x^4.$$

Then  $P(Z)$ ,  $Z^2 + x$  and  $Z^2 + x^3$  have an orthant associated polyhedron (they are even  $\nu$ -quasi-ordinary and the last two polynomials are irreducible) and

$$P_\Gamma = Z^4 + xZ^2 = Z^2(Z^2 + x).$$

But  $P$  does not factor as the product of two polynomials having an orthant associated polyhedron and whose compact faces are parallel to  $\Gamma$ . Indeed, the polynomial  $Z^2 + x^3$  has an orthant associated polyhedron but its compact face is not parallel to  $\Gamma$ .

Moreover, being Weierstraß prepared is a crucial assumption which is not stated in [ACLM1] (but appears in [ACLM2], see Notation 1.4). For example,  $P(Z) = Z^3 + Z^2 - x^4 \in \mathbb{k}[[x]][Z]$  is irreducible, but  $P_\Gamma = (Z - x^2)(Z + x^2)$ .

*Remark 2.10.* As we pointed out at the beginning, Hensel’s Lemma is the crucial ingredient for the proof of our result. In fact, we can replace  $\mathbb{k}[[x]]$  by any regular Henselian local ring  $(R, \mathfrak{m})$  with residue field  $\mathbb{k} = R/\mathfrak{m}$  and regular system of parameters  $(x) = (x_1, \dots, x_n)$ . (Note that  $R$  is not necessarily equi-characteristic).

For instance, using that  $R$  is Noetherian, one can see that any element  $f \in R$  can be written as a finite sum  $f = \sum_\alpha c_\alpha x^\alpha$ , with units  $c_\alpha \in R^*$ . Thus,  $\Delta_P$  is defined for a polynomial  $P = P(Z) \in R[Z]$ . Furthermore,  $\nu_\omega, \omega \in \mathbb{R}_{>0}^n$ , defines a monomial valuation on  $R$  whose associated graded ring is isomorphic to a polynomial ring  $\mathbb{k}[X_1, \dots, X_n]$ , where  $X_j := \text{In}_\omega(x_j), 1 \leq j \leq n$ . Of course, one needs to be more careful in the proofs, e.g.,  $\mathbb{k}[[x^q]]$  needs to be replaced by the image of  $R$  under the map that sends each  $x_i$  to  $x_i^q$ .

### 3. AN EXAMPLE CONCERNING COMPACT FACES OF DIMENSION $> 1$

Let  $n = 2$  and let us replace the variables  $(x_1, x_2)$  by  $(x, y)$  for simplicity. We set

$$P(Z) := Z^2 - (x^3 - y^5)^2 + y^{11} = (Z - x^3 + y^5)(Z + x^3 - y^5) + y^{11},$$

seen as a polynomial of  $\mathbb{k}[[x, y]][Z]$ , where  $\mathbb{k}$  is an algebraically closed field of characteristic different from 2.

We will show that  $P$  does not have an orthant associated polyhedron, since  $\Delta_P$  has two different vertices. On the other hand, we will prove that  $P(Z)$  is irreducible while for every  $\omega \in \mathbb{R}_{>0}^2$  the polynomial  $\text{In}_\omega(P)$  is always the product of two

coprime monic polynomials. This shows that Theorem 2.4 cannot be extended to polynomials without an orthant associated polyhedron.

The Newton polyhedron of  $P(Z)$  is the convex hull of

$$\{(6, 0, 0), (0, 10, 0), (0, 0, 2)\} + \mathbb{R}_{\geq 0}^3.$$

The associated polyhedron  $\Delta_P$  of  $P(Z)$  is the convex hull of

$$\{(6, 0), (0, 10)\} + \mathbb{R}_{\geq 0}^2$$

and has two vertices  $v = (6, 0)$  and  $u = (0, 10)$ . For  $\omega \in \mathbb{R}_{> 0}^2$ , if  $6\omega_1 < 10\omega_2$ , then

$$\text{In}_{\omega'}(P) = Z^2 - x^6 = (Z - x^3)(Z + x^3).$$

If  $6\omega_1 > 10\omega_2$ , then we have that

$$\text{In}_{\omega'}(P) = Z^2 - y^{10} = (Z - y^5)(Z + y^5).$$

If  $6\omega_1 = 10\omega_2$ , we have that

$$\text{In}_{\omega'}(P) = Z^2 - (x^3 - y^5)^2 = (Z - x^3 + y^5)(Z + x^3 - y^5).$$

Thus, in all cases,  $\text{In}_{\omega'}(P)$  is the product of two coprime polynomials (since  $\text{char}(\mathbb{k}) \neq 2$ ).

On the other hand,  $P(Z)$  is irreducible since  $(x^3 - y^5)^2 - y^{11}$  is not a square in  $\mathbb{k}[[x, y]]$ .

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AIX-MARSEILLE UNIVERSITÉ, CNRS, CENTRALE MARSEILLE, I2M, UMR 7373, 13453 MARSEILLE, FRANCE

*E-mail address:* `guillaume.rond@univ-amu.fr`

INSTITUT FÜR ALGEBRAISCHE GEOMETRIE, LEIBNIZ UNIVERSITÄT HANNOVER, WELFENGARTEN 1, 30167 HANNOVER, GERMANY

*Current address:* The Fields Institute, 222 College Street, Toronto, Ontario, M5T 3J1, Canada  
— and — Department of Mathematics, University of Toronto, 40 St. George Street, Toronto, Ontario, M5S 2E4, Canada

*E-mail address:* `schober@math.toronto.edu`