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ABOUT THE COHOMOLOGICAL DIMENSION OF CERTAIN STRATIFIED VARIETIES

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ABSTRACT. We determine an upper bound for the cohomological dimension of the complement of a closed subset in a projective variety which possesses an appropriate stratification. We apply the result to several particular cases, including the Bialynicki-Birula stratification; in this latter case, the bound is sharp.

INTRODUCTION

The cohomological dimension of a quasi-projective variety X is defined as

$$\operatorname{cd}(X) := \max\{t \geqslant 0 \mid H^t(X, \mathscr{F}) \neq 0 \text{ for some quasi-coherent sheaf } \mathscr{F} \text{ on } X\}.$$

Intuitively, it measures at which extent X is affine or complete: on one hand, affine varieties are characterized by the fact that their cohomological dimension vanishes (Serre's criterion); on the other hand, $\operatorname{cd}(X) \leqslant \dim(X)$ (cf. [8]), and the inequality is strict unless X is proper (Lichtenbaum's criterion of properness [3, 11, 14]).

There is no general formula for the cohomological dimension of non-complete varieties, although it can be computed algorithmically (cf. [21]). However, there are a number of articles in the mathematical literature which determine upper bounds (sharp; cf. [13]) for the cohomological dimension [6, 7, 11, 12, 15, 17, 19]. An exact result is obtained in [18], where it is proved that for an ample subvariety Z of a projective variety X defined over an algebraically closed field of characteristic zero, $\operatorname{cd}(X \setminus Z) = \operatorname{codim}_X(Z) - 1$.

The goal of this article is to compute the cohomological dimension of non-complete varieties which admit appropriate stratifications, which we call *affine bundle stratifications*; the definition is inspired from [5]. The main result is Theorem 2.3, which is an upper bound for the cohomological dimension of the varieties admitting lci affine bundle stratifications (see Definition 1.3 for the definition). The proof involves standard results about local cohomology groups. However, to our knowledge, the applications below have not been studied before in this manner.

The most common situation which gives rise to such stratifications is when the multiplicative group acts on a projective variety; the stratification we are referring to is known as the Bialynicki-Birula decomposition [1]. In Section 3.1 we do *exactly* determine the cohomological dimension of the complement of the sink of the action; in particular, the upper bound obtained in the theorem is sharp. Another class of

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examples which is not related to group actions and leads to affine bundle stratifications arises when one considers complements of zero loci of sections in globally generated vector bundles; this is discussed in section 3.2.

Throughout this note, the varieties are defined over an algebraically closed field $\Bbbk$.

1. Background material

Definition 1.1. The *cohomological dimension* of a quasi-projective scheme X, denoted by $\mathrm{cd}(X)$, is the least integer n such that, for any quasi-coherent sheaf $\mathscr{F}$ on X,

$$H^t(X, \mathscr{F}) = 0,\ \forall\, t > n,$$

holds.

Although we are ultimately interested in the cohomology of *coherent* sheaves, the necessity to apply Leray-type spectral sequences for affine morphisms (see below) forces us to work in the quasi-coherent setting.

Lemma 1.2. *If $f : X \to B$ is an affine morphism, then* $\mathrm{cd}(X) \leqslant \mathrm{cd}(B)$.

Example 3.12 shows that the inequality is strict in general.

Proof. This is a direct consequence of [9, Ch. III, Corollaire 1.3.3]; since f is affine, for any quasi-coherent sheaf $\mathscr{F}$ on X, $H^t(X, \mathscr{F}) \cong H^t(B, f_*\mathscr{F})$ holds, for all $t \geqslant 0$. □

Definition 1.3. Let X be a quasi-projective variety. An *affine bundle stratification* of X is a triple $(Z_\bullet, Y_\bullet, f_\bullet)$ as follows:

(i) a filtration by *closed* subsets $\emptyset = Z_0 \subset Z_1 \subset \ldots \subset Z_r = X$;
(ii) a collection $\{Y_1, \ldots, Y_r\}$ of quasi-projective varieties;
(iii) affine morphisms $f_j : Z_j \setminus Z_{j-1} \to Y_j$, for $j = 1, \ldots, r$.

If, moreover,

(iv) $Z_j \setminus Z_{j-1} \subset X$ is a local complete intersection (lci for short), for $j \geqslant 1$,

then we have an *lci affine bundle stratification.*

The role of the varieties $Y_\bullet$ may not be clear at this point. *A priori*, one could take $Y_j = Z_j \setminus Z_{j-1}$ and f_j the identity; however, it is desirable to have Y_j as low dimensional as possible. The definition is inspired from [5], but it is not the same.

Example 1.4. The two notions introduced above are distinct. Indeed,

$$\begin{aligned} &\emptyset \subset \operatorname{Spec} \tfrac{\Bbbk[x,y,z]}{\langle xy, yz, zx\rangle} \subset \mathbb{A}^3_{\Bbbk} = \operatorname{Spec} \Bbbk[x,y,z], \\ \text{and} \quad &\emptyset \subset \{0\} \subset \operatorname{Spec} \tfrac{\Bbbk[x,y,z]}{\langle xy, yz, zx\rangle} \subset \mathbb{A}^3_{\Bbbk} = \operatorname{Spec} \Bbbk[x,y,z] \end{aligned}$$

are examples of non-lci and lci stratifications, respectively. In both cases, the varieties $Y_\bullet$ are the consecutive differences of the filtrations and $f_\bullet$ the identity morphisms.

2. The main result

Throughout this section we assume that $(Z_\bullet, Y_\bullet)$ is an lci affine bundle stratification of a quasi-projective variety X. The local cohomology groups and sheaves with locally closed supports are defined in [10, Exposé I]; for an algebraic introduction, the reader may consult [4]. The letters 'H' and '$\mathscr{H}$' will stand for the local cohomology groups and sheaves respectively. The following result is well known; we recall it for completeness.

Lemma 2.1. *Let A be an affine scheme and $S \subset A$ be a δ-codimensional, complete intersection subscheme. Then, for any quasi-coherent sheaf $\mathscr{F}$ on A, $H^t_S(A, \mathscr{F}) = 0$ holds for all $t > \delta$. Consequently, if S is a locally closed lci in a quasi-projective scheme X, then $\mathscr{H}^t_S(X, \mathscr{F}) = 0$, for all quasi-coherent sheaves $\mathscr{F}$ on X and $t >$* $\operatorname{codim}_X(S)$.

Proof. It is done by induction on δ; for $\delta = 0$, the statement is Serre's criterion. For the inductive step, write $S = S_1 \cap \ldots \cap S_\delta$, where all $S_j \subset A$ are hypersurfaces. Now consider the exact sequence

$$\ldots \to H^{t-1}_{S_1\cap\ldots\cap S_{\delta-1}\cap(A\setminus S_\delta)}(A \setminus S_\delta, \mathscr{F}) \to H^t_{S_1\cap\ldots\cap S_\delta}(A, \mathscr{F}) \to H^t_{S_1\cap\ldots\cap S_{\delta-1}}(A, \mathscr{F}) \to \ldots$$

and use that both A and $A \setminus S_\delta$ are affine. □

Lemma 2.2. *Let the situation be as in Definition* 1.3(i)-(iv). *Then for any quasi-coherent sheaf $\mathscr{F}$ on X,*

$$H^t_{Z_j\setminus Z_{j-1}}(X, \mathscr{F}) = 0,\ \forall\, t > \operatorname{cd}(Y_j) + \operatorname{codim}_X(Z_j \setminus Z_{j-1}),\ \forall\, j = 2, \ldots, r,$$

holds.

Proof. To simplify the notation, we denote $S := Z_j \setminus Z_{j-1}$, $U := X \setminus Z_{j-1}$, $B := Y_j$. Since $S \subset U$ is closed, we deduce that $H^t_S(X, \mathscr{F}) = H^t_S(U, \mathscr{F})$. The local cohomology groups can be computed by means of a spectral sequence (cf. [10, Exposé I, Théorème 2.6]):

$$H^b(U, \mathscr{H}^a_S(\mathscr{F})) \;\Rightarrow\; H^{b+a}_S(U, \mathscr{F}).$$

First, since $S \subset U$ is lci, Lemma 2.1 implies that $\mathscr{H}^a_S(\mathscr{F}) = 0$, for all $a > \operatorname{codim}(S)$. Second, $\mathscr{H}^a_S(\mathscr{F}) = \varinjlim_m \mathscr{E}xt^a(\mathcal{O}_U/\mathcal{I}^m_S, \mathscr{F})$ (cf. [8, Exposé II, Théorème 2]), which implies that

$$H^b(U, \mathscr{H}^a_S(\mathscr{F})) = \varinjlim_m H^b(U, \mathscr{E}xt^a(\mathcal{O}_U/\mathcal{I}^m_S, \mathscr{F})).$$

The $\mathscr{E}xt$ groups are supported on (thickenings of) S and $\operatorname{cd}(S) \leqslant \operatorname{cd}(Y)$, so the expression above vanishes for $b > \operatorname{cd}(Y)$. The spectral sequence yields $H^t_S(U, \mathscr{F}) = 0$ for t in the indicated range. □

Theorem 2.3. *Let $(Z_\bullet, Y_\bullet)$ be an lci affine bundle stratification of a quasi-projective variety X. Then*

$$\operatorname{cd}(X \setminus Z_1) \leqslant \max_{j=2,\ldots,r} \{\operatorname{cd}(Y_j) + \operatorname{codim}_X(Z_j \setminus Z_{j-1})\} \tag{2.1}$$

holds.

As we will see, in many geometric situations, the inequality above is actually an equality. However, the inequality is strict in general (cf. Example 3.12).

Proof. Let $\mathscr{F}$ be an arbitrary quasi-coherent sheaf on $X\setminus Z_1$; denote by t_0 the right-hand side of (2.1). We show by decreasing recurrence that $H^t(X\setminus Z_j,\mathscr{F})=0$, for $j=r-1,\dots,1$ and $t>t_0$. For $j=r-1$, we have $H^t(X\setminus Z_{r-1},\mathscr{F})\overset{1.2}{=}H^t(Y_r,f_{r*}\mathscr{F})=0$, for $t>\mathrm{cd}(Y_r)$. Now we prove the recursive step. The exact sequence in local cohomology (cf. [10, Exposé I, Théorème 2.8]) yields

$$\ldots\to H^t_{Z_j\setminus Z_{j-1}}(X\setminus Z_{j-1},\mathscr{F})\to H^t(X\setminus Z_{j-1},\mathscr{F})\to H^t(X\setminus Z_j,\mathscr{F})\to\ldots\ .$$

By Lemma 2.2, the left-hand side vanishes for $t>t_0$, while the right-hand side vanishes by assumption; thus the cohomology vanishing holds for $j-1$ too. $\square$

Remark 2.4. In Definition 1.3, the condition that $Z_j\setminus Z_{j-1}$ is an lci in X is imposed to avoid technical estimates. The theorem above holds in general if one replaces $\mathrm{codim}_X(Z_j\setminus Z_{j-1})$ by the local cohomological dimension of $Z_j\setminus Z_{j-1}$ in X (see [17]).

Henceforth we assume that $X,Y_\bullet$ are projective.

Is (2.1) an equality?

Under which further assumptions is the equality reached?

Assume that the right-hand side of (2.1) attains its maximum at $j_0\in\{2,\dots,n\}$. Clearly, the answer to the question is affirmative if there is a

$$\big(\dim Y_{j_0}+\mathrm{codim}(Z_{j_0}\setminus Z_{j_0-1})\big)\text{-dimensional,}$$

irreducible *projective* variety $Z'\subset X\setminus Z_1$. Indeed, the inequality '$\geqslant$' holds too, by the Hartshorne-Lichtenbaum criterion [11, Corollary 3.2]: $\mathrm{cd}(X\setminus Z_1)\geqslant\mathrm{cd}(Z')=\dim Z'$.

To understand how realistic this naive answer is, we analyze the right-hand side of (2.1). There is an open affine subset $A\subset X$ and an irreducible projective variety $\tilde{Y}_j\subset Z_j$ (a multi-section of f_j) with the following properties:

– $\tilde{Y}_j\cap(Z_j\setminus Z_{j-1})\neq\emptyset$ is lci in $Z_j\setminus Z_{j-1}$;

– $f_j:\tilde{Y}_j\cap(Z_j\setminus Z_{j-1})\to Y_j$ is generically finite, in particular $\dim\tilde{Y}_j=\dim Y_j$, for all j.

Remark 2.5. Suppose that X is smooth. Then the following hold:

- One can choose A such that both $(Z_j\setminus Z_j-1)\cap A$ and $\tilde{Y}_j\cap A$ are smooth.
- The right-hand side of (2.1) equals $d_j:=\dim\big(\underline{\mathsf{N}}_{(Z_j\setminus Z_{j-1})\cap A/A}\big|_{\tilde{Y}_j\cap A}\big)$, which is the restriction to $\tilde{Y}_j\cap A$ of the *total space* of the normal sheaf of $Z_j\setminus Z_{j-1}$ in X.
- There is a closed subvariety $A_j\subset A$ which 'extends' $\tilde{Y}_j\cap A$ in the normal direction to $(Z_j\setminus Z_{j-1})\cap A$, that is,

$$\dim A_j=d_j,\ A_j\cap(Z_j\setminus Z_{j-1})\cap A=\tilde{Y}_j\cap A.$$

Proof. The first claim holds for A sufficiently small. Then use that the normal sheaf of $(Z_j\setminus Z_{j-1})\cap A$ in A is locally free, of rank $\mathrm{codim}_X(Z_j\setminus Z_{j-1})$.

Finally, let $A=\mathrm{Spec}(R)$ and $(Z_j\setminus Z_{j-1})\cap A=\mathrm{Spec}(A/I)$, where R is a regular local ring and $I\subset R$ is a complete intersection ideal. Then I^n/I^{n+1} are free R/I-modules, for $n\geqslant 1$, and one can successively lift (not canonically) the identity $R/I\to R/I$ to a ring homomorphism $R/I\to\hat{R}_I$. This latter induces an isomorphism $\psi:\frac{R}{I}\big[\big[\frac{I}{I^2}\big]\big]\overset{\cong}{\to}\widehat{R}_I$ between the formal completions.

Let $\frac{J}{I} \subset \frac{R}{I}$ be the ideal defining $\tilde{Y}_j \cap A$. Then $\psi\big(\frac{J}{I}[[\frac{I}{I^2}]]\big) \subset \hat{A}_I$ is a closed ideal (for the I-adic topology), so it lifts to an ideal $J' \subset A$ with the desired properties. ☐

Consequently, if one can ensure that $Z'_{n-j+1} := \overline{A_j}^X$ is contained in $X \setminus Z_1$ (intuitively, if A_j does not 'bend backwards'), then Z'_j satisfies the conditions in the naive answer.

Corollary 2.6. *Let $(Z_\bullet, Y_\bullet)$ be an lci affine bundle stratification of X, with $X, Y_\bullet$ projective. Assume that for $j = 2, \ldots, n$, there is $Z'_{n+1-j} \subset X \setminus Z_{j-1}$ irreducible, closed, such that the following hold:*

$$Z'_{n+1-j} \cap Z_j = Y_j, \quad \dim Z'_{n+1-j} + \dim Z_j = \dim Y_j + \dim X.$$

Then we have $\operatorname{cd}(X \setminus Z_1) = \max\limits_{j=2,\ldots,r} \{\operatorname{cd}(Y_j) + \operatorname{codim}_X(Z_j \setminus Z_{j-1})\}$.

As we will see, the BB-stratification of a projective variety satisfies this condition.

3. Applications

3.1. **The complement of the sink of a G_m-action.** The Bialynicki-Birula (BB for short) decomposition arises in the context of the actions of the multiplicative group G_m. Let X be a smooth projective G_m-variety; we denote the action by $\lambda : G_m \times X \to X$ and assume that it is effective. In this situation X admits two (the plus and minus) decompositions into smooth, locally closed subsets (cf. [1]):

- For any $x \in X$, the specializations at $\{0, \infty\} = \mathbb{P}^1 \setminus G_m$ are denoted $\lim\limits_{t\to 0} \lambda(t) \times x$ and $\lim\limits_{t\to\infty} \lambda(t) \times x$; they are both fixed by λ.
- The fixed locus X^λ of the action is a disjoint union $\coprod\limits_{s\in F_{BB}} Y_s$ of smooth subvarieties. For $s \in F_{BB}$, $Y_s^\pm := \{x \in X \mid \lim\limits_{t\to 0 \text{ resp. } \infty} \lambda(t) \times x \in Y_s\}$ is locally closed in X (a BB-cell).
- $X = \coprod\limits_{s\in F_{BB}} Y_s^+ = \coprod\limits_{s\in F_{BB}} Y_s^-$, and the morphisms
$$Y_s^\pm \to Y_s,\ x \mapsto \lim_{t\to 0 \text{ resp. } \infty} \lambda(t) \times x$$
are locally trivial, affine space fibrations. They are not necessarily vector bundles; that is, the transition functions may be non-linear.
- The *source* Y_{source} and the *sink* Y_{sink} of the action are characterized by the facts that $Y^+_{\text{source}} \subset X$ is open, $Y^-_{\text{source}} = Y_{\text{source}}$, and $Y^+_{\text{sink}} = Y_{\text{sink}}$, $Y^-_{\text{sink}} \subset X$ is open.
- There is a (not unique) partial order '$\prec$' on F_{BB} such that
$$\overline{Y_s^+} \subset \bigcup_{t\preceq s} Y_t^+ =: Z_s \quad \text{(cf. [2])};$$
the difference of two consecutive terms of the filtration $Z_\bullet$ is of the form Y_s^+. The minimal element of this (plus) filtration (the Z_1 in Definition 1.3) is Y_{sink}. A similar statement holds for the minus-decomposition.

In this case, Theorem 2.3 yields the following.

Theorem 3.1. *Let the situation be as above. Then*

$$\operatorname{cd}(X \setminus Y_{\text{sink}}) = \max\{\dim Y_s^- \mid s \neq \text{sink}\} = \dim(X \setminus Y_{\text{sink}}^-)$$

holds, and similarly $\operatorname{cd}(X \setminus Y_{\text{source}}) = \dim(X \setminus Y_{\text{source}}^+)$.

Proof. Indeed, for all $s \in F_{BB}$, we have $\dim Y_s^+ + \dim Y_s^- = \dim Y_s + \dim X$, and we apply Corollary 2.6. □

In the paragraph 3.1.2 below we will need a few more details about the BB-decomposition. Let us define:

(3.1)
$$\begin{aligned}
G(\lambda) &:= \{g \in G \mid g^{-1}\lambda(t)g = \lambda(t),\ \forall\, t \in G_m\}, \text{ the centralizer of } \lambda \text{ in } G,\\
P(\pm\lambda) &:= \{g \in G \mid \lim_{t\to 0}\big(\lambda(t)^{\pm} g\, \lambda(t)^{\mp}\big) \text{ exists in } G \ \big(\text{and belongs to } G(\lambda)\big)\},\\
U(\pm\lambda) &:= \{g \in G \mid \lim_{t\to 0}\big(\lambda(t)^{\pm} g\, \lambda(t)^{\mp}\big) = e \in G\}.
\end{aligned}$$

Then $G(\lambda)$ is a connected, reductive subgroup of G, $P(\pm\lambda) \subset G$ are parabolic subgroups, $G(\lambda)$ is their Levi-component, and $U(\pm\lambda)$ is the unipotent radical (cf. [20, §13.4]).

Lemma 3.2.

(i) $Y_{\text{source}}, Y_{\text{sink}}$ *are invariant under* $P(-\lambda)$ *and* $P(\lambda)$ *respectively.*

(ii) *The BB-cell* Y_s^+ *is* $P(\lambda)$*-invariant, and* $U(\lambda)$ *preserves the fibration* $Y_s^+ \to Y_s$, *for all* $s \in F_{BB}$.

Proof. (i) We prove the statement for Y_{source}; the case Y_{sink} is analogous. We claim that $G(\lambda)$ leaves Y_{source} invariant; for $y \in Y_{\text{source}}$ and $c \in G(\lambda)$,

$$cy = \lambda(t)\cdot(cy)\ \forall\, t \in G_m \ \Rightarrow\ cy \in X^{\lambda}; \quad \text{thus } G(\lambda)y \subset X^{\lambda}$$

holds.

But X^{λ} is the disjoint union of its components, and $G(\lambda)y$ is connected and contains $y \in Y_{\text{source}}$, so $G(\lambda)Y_{\text{source}} = Y_{\text{source}}$. The same argument shows that $G(\lambda)Y_s = Y_s$, for any $s \in F_{BB}$. For $g \in P(-\lambda)$, $c := \lim_{t\to 0}\lambda(t)^{-1}g\lambda(t) \in G(\lambda)$ holds, so

$$cy = \lim_{t\to 0}\lambda(t)^{-1}g\lambda(t)y \in Y_{\text{source}} \ \Rightarrow\ \lim_{t\to 0}\lambda(t)^{-1}(gy) \in Y_{\text{source}}.$$

We claim that $gy \in Y_{\text{source}}$; otherwise $gy \in X \setminus Y_{\text{source}}$ is 'repelled' from Y_{source}, and the limit belongs to another component of X^{λ}.

(ii) Take $x \in X$ with $\lim_{t\to 0}\lambda(t)x = y \in Y_s \subset X^{\lambda}$, $g \in P(\lambda)$. Then

$$\begin{aligned}
&c := \lim_{t\to 0}\lambda(t)g\lambda(t)^{-1} \in G(\lambda),\\
&\lim_{t\to 0}\lambda(t)gx = \lim_{t\to 0}\big(\lambda(t)g\lambda(t)^{-1}\cdot\lambda(t)x\big) = cy \ \Rightarrow\ gx \in Y_s^+
\end{aligned}$$

hold.

Similarly, for $g \in U(\lambda)$, one finds that $\lim_{t\to 0}\lambda(t)x = y$ implies $\lim_{t\to 0}\lambda(t)(gx) = y$. □

3.1.1. *The case of toric varieties.* Let N be a lattice of rank d, $\Sigma \subset N_{\mathbb{R}}$ be a projective simplicial fan, and X_Σ be the corresponding toric variety. Denote by $\{D_\rho\}_{\rho\in\Sigma(1)}$ the invariant divisors. We denote by $T \subset X_\Sigma$ the big torus and consider a 1-PS λ of T. Note that λ is determined by an element of N, which will be denoted the same (so $\lambda \in N$). Further details about toric varieties can be found in [16].

The fixed components of the G_m-action on X determined by λ is a disjoint union of toric subvarieties of X_Σ (they are all T-invariant), which are intersections of T-invariant divisors. Hence they are of the form

$$Y_\sigma := \bigcap_{\rho\in\sigma(1)} D_\rho, \quad \sigma \in \Sigma.$$

Since Σ is simplicial, Y_σ is a complete intersection.

An open affine, T-invariant neighbourhood of the generic point of Y_σ in X_Σ is $\operatorname{Spec}(\Bbbk[\sigma^\vee])$, where $\sigma^\vee := \{m \in N^\vee \mid m(\xi) \geqslant 0,\ \forall\, \xi \in \sigma(1)\}$. Consider the cones:

$$\sigma^\perp := \{m \in \sigma^\vee \mid m(\xi) = 0,\ \forall\, \xi \in \sigma\}, \quad \sigma^{>0} := \{m \in \sigma^\vee \mid \exists\, \xi \in \sigma,\ m(\xi) > 0\}.$$

Then $\sigma^{>0}$ determines the ideal $\mathcal{I} \subset \Bbbk[\sigma^\vee]$ with quotient ring $\Bbbk[\sigma^\perp]$, which induces the inclusion $\operatorname{Spec}(\Bbbk[\sigma^\perp]) \subset \operatorname{Spec}(\Bbbk[\sigma^\vee])$. The left-hand side is an affine chart for Y_σ (more precisely, its big torus).

Lemma 3.3. *The fixed components X_Σ are $Y_\sigma = \bigcap_{\rho\in\sigma(1)} D_\rho$, where $\sigma \in \Sigma$ are* the minimal cones *such that $\lambda \in \langle\sigma(1)\rangle$ (the vector space generated by $\sigma(1)$).*

Let σ_{source} (respectively σ_{sink}) be the minimal cones such that $\lambda \in \operatorname{interior}(\sigma_{\text{source}})$ (respectively $-\lambda \in \operatorname{interior}(\sigma_{\text{sink}})$). Then we have

$$Y_{\text{source}} = Y_{\sigma_{\text{source}}} \quad \text{and} \quad Y_{\text{sink}} = Y_{\sigma_{\text{sink}}}.$$

Proof. Let $\sigma \in \Sigma$ be such that Y_σ is fixed by λ. The closed points of $\operatorname{Spec}(\Bbbk[\sigma^\vee])$ correspond to semi-group homomorphisms $x : (\sigma^\vee, +) \to (\Bbbk, \cdot)$. The distinguished element

$$x_\sigma : \sigma^\vee \to \Bbbk, \quad x_\sigma(m) := \begin{cases} 1, & \text{if } m \in \sigma^\perp; \\ 0, & \text{if } m \in \sigma^{>0} \end{cases}$$

corresponds to the generic point of $\operatorname{Spec}(\Bbbk[\sigma^\perp])$. The action of λ on x_σ is the following:

$$\big(\lambda(t)\cdot x_\sigma\big)(m) = t^{m(\lambda)} \cdot x_\sigma(m),\ \forall\, m \in \sigma^\vee.$$

By assumption, it holds that $\lambda(t)\cdot x_\sigma = x_\sigma$, $\forall\, t \in G_m$. For $m \in \sigma^{>0}$, both sides vanish. For $m \in \sigma^\perp$, we deduce that $m(\lambda) = 0$; since m is arbitrary, it follows that $\lambda \in \langle\sigma(1)\rangle$. $\square$

Let $\Sigma_{BB} \subset \Sigma$ be the subset consisting of the cones σ as above; for $\rho \in \Sigma(1)$, denote by ξ_ρ the generator of $\rho \cap N$. Then, for any $\sigma \in \Sigma_{BB}$, we can (uniquely) write

$$\lambda = \sum_{\rho\in\sigma(1)^-} \underbrace{c_\rho \cdot}_{<0} \xi_\rho + \sum_{\rho\in\sigma(1)^+} \underbrace{c_\rho \cdot}_{>0} \xi_\rho.$$

Corollary 3.4. *Let the situation be as above. Then*

$$\operatorname{cd}(X \setminus Y_{\text{sink}}) = d - \min\{\, \#\,\sigma(1)^+ \mid \sigma \in \Sigma_{BB}\}$$

holds.

Proof. Note that $\operatorname{codim} Y_\sigma^- = \#\sigma(1)^+$, for all $\sigma \in \Sigma_{BB}$, and apply Theorem 3.1. □

3.1.2. *The case of homogeneous varieties.* Consider the homogeneous variety $X = G/P$, where G is connected, reductive, and P is a parabolic subgroup, and consider a subgroup $\lambda : G_m \to G$; it induces a G_m-action on X. The adjoint action of λ on $\mathrm{Lie}(G)$ decomposes it into the positive/negative weight spaces

$$\mathrm{Lie}(G) = \mathrm{Lie}(G)^-_\lambda \oplus \mathrm{Lie}(G)^0 \oplus \mathrm{Lie}(G)^+_\lambda.$$

Lemma 3.5. *The following statements hold:*

(i) *The connected components of the fixed locus X^λ are homogeneous for the action of $G(\lambda)$.*

(ii) *The sink Y_{sink} contains $\hat{e} \in G/P$ if and only if $\lambda \subset P$ and $\mathrm{Lie}(G)^+_\lambda \subset \mathrm{Lie}(P)$.*

Proof. (i) The differential of the multiplication $\mathrm{Lie}(G) \to T_yX$ is surjective at any $y \in X^\lambda$, and it is λ-equivariant for the adjoint action on $\mathrm{Lie}(G)$. Both sides decompose into direct sums of weight spaces; in particular,

$$\mathrm{Lie}(G(\lambda)) = \mathrm{Lie}(G)^0_\lambda \to (T_yX)^0$$

is surjective too. Therefore all the $G(\lambda)$-orbits are open in X^λ; hence the components of X^λ are homogeneous under $G(\lambda)$.

(ii) The point $\hat{e}$ belongs to Y_{sink} if and only if:

– $\hat{e}$ is fixed by λ, that is, $\mathrm{Im}(\lambda) \subset P$;

– the weights of λ on $T_{\hat{e}}X \cong \mathrm{Lie}(G)/\mathrm{Lie}(P)$ are negative. □

If $Q, P \subset G$ are two parabolic subgroups, then G/P decomposes into the following finite disjoint union (the Bruhat decomposition; cf. [20, §8]) of locally closed orbits under the action of Q:

$$G/P = \coprod_{w \in F_{\mathrm{Bruhat}}} QwP, \text{ with } F_{\mathrm{Bruhat}} = \mathrm{Weyl}(Q)\big\backslash \mathrm{Weyl}(G)\big/\mathrm{Weyl}(P).$$

Actually, F_{Bruhat} parametrizes the $\mathrm{Weyl}(Q)$-orbits in $(G/P)^T$. Each double coset in F_{Bruhat} contains a unique representative of minimal length; for each $w \in F_{\mathrm{Bruhat}}$ of minimal length,

$$\dim(QwP) = \mathrm{length}(w) + \dim\big(\mathrm{Levi}(Q)/\mathrm{Levi}(Q) \cap wPw^{-1}\big).$$

Proposition 3.6. *The BB-decomposition of G/P for the action of λ coincides with the Bruhat decomposition for the action of $P(\lambda)$. If $\mathrm{Lie}(G)^+_\lambda \subset P$, then the sink of the action is $P(\lambda)P \cong \frac{G(\lambda)}{G(\lambda)\cap P}$.*

We remark that any standard parabolic $Q \subset G$ is of the form $P(\lambda)$, for some λ, such that $\mathrm{Lie}(G)^+_\lambda$ is contained in the Borel subgroup of $\mathrm{Lie}(G)$.

Proof. Each Bruhat cell is the $P(\lambda)$-orbit of some $x \in (G/P)^T$; any such x belongs to a component $Y_s \subset (G/P)^\lambda$. Finally, the BB-cell Y_s^+ is $P(\lambda)$-invariant (cf. Lemma 3.2). Hence each Bruhat cell is contained in a unique BB-cell. But the union of the former is G/B, and the latter cells are pairwise disjoint. It follows that each Bruhat cell equals some BB-cell. □

Theorem 3.7. *Let the situation be as above. Then*
$$\operatorname{cd}\left(\frac{G}{P}\setminus\frac{G(\lambda)}{G(\lambda)\cap P}\right)=\max_{w\in F_{\mathrm{Bruhat}}\setminus\{e\}}\dim\big(P(-\lambda)wP\big)$$
holds.

Proof. It follows directly from Theorem 3.1. The minus BB-decomposition corresponds to the action of $-\lambda$, so we must replace $P(\lambda)\rightsquigarrow P(-\lambda)$. $\square$

Example 3.8. Let $W:=\Bbbk\oplus W'$ with $W'=\Bbbk^w$, $0\leqslant u<w$, and consider
$$\lambda: G_m\to \mathrm{GL}(W),\quad \lambda(t):=\operatorname{diag}(t^{-1},1,\dots,1).$$
It induces an action on the Grassmannian $X:=\mathrm{Gr}(u+1;W)$ whose fixed point set has two components:
$$Y_{\mathrm{source}}=\{U\subset W\mid \langle 1,\underline{0}\rangle\in U\}\cong \mathrm{Gr}(u;W'),\quad Y_{\mathrm{sink}}=\mathrm{Gr}(u+1;W').$$
Theorem 3.7 implies that
$$\begin{aligned}\operatorname{cd}(X\setminus Y_{\mathrm{sink}}) &=\operatorname{cd}\big(\mathrm{Gr}(u+1;w+1)\setminus \mathrm{Gr}(u+1;w)\big)=\dim \mathrm{Gr}(u;W')\\ &=u(w-u)=\dim X-(w-u).\end{aligned}$$
Note that, except for $u=0$ and $u=w-1$, this is much larger than the 'universal' lower bound $\operatorname{codim}_X(Y_{\mathrm{sink}})-1=u$.

3.2. **Complements of zero loci in globally generated vector bundles.** The previous examples might give the impression that lci affine bundle stratifications belong to the realm of group actions. Here we show that this is not the case.

Proposition 3.9. *Let Z be a closed subscheme of a projective variety X. We assume that there is a modification $\tilde{X}$ of X along Z such that the following holds: there are a projective variety X' and a morphism $f:\tilde{X}\to X'$ such that the exceptional divisor E_Z is f-relatively ample. Then* $\operatorname{cd}(X\setminus Z)\leqslant \dim X'$.

Proof. The assumption that E_Z is f-relatively ample implies that $f:\tilde{X}\setminus E_Z\to X'$ is an affine morphism. Since $\tilde{X}\setminus E_Z\cong X\setminus Z$, in this case we obtain a filtration with two strata: $Z_1:=E_Z$ and $Z_2:=\tilde{X}$. $\square$

The previous situation arises as follows. Consider a globally generated vector bundle $\mathcal{N}$ on a projective variety X of rank $\nu\leqslant\dim X$, such that $\det(\mathcal{N})$ is ample. Let Z be the zero locus of an arbitrary (non-zero) section in $\mathcal{N}$. The blow-up $\tilde{X}:=\mathrm{Bl}_Z(X)$ of X along Z fits in the diagram

$$\begin{tikzcd}
\tilde{X} \arrow[r, hook] \arrow[d, "\pi"'] \arrow[rrd, "f"] & \mathbb{P}(\mathcal{N})=\mathbb{P}\Big(\bigwedge^{\nu-1}\mathcal{N}^\vee\otimes\det(\mathcal{N})\Big) \arrow[r, hook] & X\times\mathbb{P}\Big(\bigwedge^{\nu-1}\Gamma(\mathcal{N})^\vee\Big) \arrow[d] \\
X & & \mathbb{P}:=\mathbb{P}\Big(\bigwedge^{\nu-1}\Gamma(\mathcal{N})^\vee\Big),
\end{tikzcd} \tag{3.2}$$

and

$$\mathcal{O}_{\tilde{X}}(E_Z)=\mathcal{O}_{\mathbb{P}(\mathcal{N})}(-1)\big|_{\tilde{X}}=\big(\det(\mathcal{N})\boxtimes\mathcal{O}_{\mathbb{P}}(-1)\big)\big|_{\tilde{X}} \tag{3.3}$$

holds, so E_Z is f-relatively ample. (In particular, note that $Z\neq\emptyset$.)

Corollary 3.10. *Let the situation be as above. Then* $\operatorname{cd}(X \setminus Z) \leqslant \dim f(\tilde{X})$ *holds.*

Example 3.11. Let (W, ω) be a $(w+1)$-dimensional vector space endowed with a non-degenerate, skew-symmetric bilinear form. We consider the Grassmannian of ω-isotropic subspaces:

$$X := \text{sp-Gr}(u+1; W) = \{U \in \text{Gr}(u+1; W) \mid \omega|_U = 0\}.$$

It is a homogeneous variety under the action of the group $\mathrm{Sp}(W) \subset \mathrm{GL}(W)$ which preserves the symplectic form ω. For $s \in W \setminus \{0\}$, let $s^\perp := \{t \in W \mid \omega(s,t) = 0\}$, and consider

$$Z := \text{sp-Gr}(u+1; s^\perp) = \{U \in \text{sp-Gr}(u+1; W) \mid U \subset s^\perp\}.$$

It is the zero locus of a section in the dual of the tautological (locally free) sheaf on X. Then the previous corollary implies that

$$\operatorname{cd}(X \setminus Z) \leqslant \dim \text{sp-Gr}(u; s^\perp) = \frac{w(2w-3u+1)}{2} = \dim X - (w - 2u).$$

The estimate cannot be obtained by using G_m-actions, as in subsection 3.1.2, because Z is not a homogeneous variety (cf. Lemma 3.5(i)).

Example 3.12. In general the previous inequality is strict. Let $Z \subset \mathbb{P}^3$ be the zero locus of a general section in $\mathcal{N} := \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2)$: it is the intersection of a plane $\{s_1 = 0\}$ with a quadric $\{s_2 = 0\}$. On one hand, $Z \subset \mathbb{P}^3$ is an ample, 2-codimensional subvariety, so $\operatorname{cd}(\mathbb{P}^3 \setminus Z) = 1$ (cf. [18, Theorem 7.1, 5.4]).

On the other hand, the morphism f in (3.2) is the following:

$$\begin{aligned} &\mathbb{P}^3 \dashrightarrow \mathbb{P}\big(\mathcal{O}(1)_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2)\big) \cong \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}\big) \to \mathbb{P}^4, \\ &\underline{x} \mapsto [x_0 s_1(x) : x_1 s_1(x) : x_2 s(x) : x_3 s_1(x) : s_2(x)]. \end{aligned}$$

We claim that the image of this morphism is 2-dimensional. Indeed, consider

$$\underline{y} = [y_0 : \ldots : y_4] = f(\underline{x}) \in \operatorname{Im}(f), \quad y_0 = 1.$$

One computes

$$x_0 = 1, x_1 = y_1, x_2 = y_2, x_3 = y_3, s_1(1, y_1, y_2, y_3) = 1, s_2(1, y_1, y_2, y_3) = y_4.$$

Thus $\underline{y}$ satisfies two independent equations in $\mathbb{P}^4$.

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