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ON AN APPLICATION OF BINET'S SECOND FORMULA

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ABSTRACT. In this work we apply the second Binet formula for Euler's gamma function $\Gamma(x)$ and a Laplace transform formula to derive an infinite series expansion for the auxiliary function $f(x)$ in the computations of sine integral and cosine integral functions in terms of $\log\Gamma(x)$ and the Möbius function. Then we apply Möbius inversion to obtain a Kummer type series expansion for $\log\Gamma(x)$. Unlike the original Kummer formula, our formula is not a Fourier series anymore. By differentiating the series expansion for $f(x)$ we obtain an infinite series expansion for the auxiliary function $g(x)$ associated with sine integral and cosine integral functions as well.

1. Introduction

Let the $\Gamma(z)$ function be the Euler gamma function; for $z \neq 0, -1, -2, \dots$ it is given by [1, 3, 6]

$$\Gamma(z) = \sum_{n\geq 0}^{\infty} \frac{(-1)^n}{n!(z+n)} + \int_1^{\infty} e^{-t} t^{z-1} dt, \tag{1.1}$$

and the digamma function $\psi(z)$ is defined by $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$. Binet's second formula is given by

$$\log\Gamma(x) = x\log\frac{x}{e} + \frac{1}{2}\log\frac{2\pi}{x} + 2\int_0^{\infty} \frac{\arctan(t/x)dt}{e^{2\pi t}-1}, \quad |\arg x| < \frac{\pi}{2}. \tag{1.2}$$

Then for $\Re(x) > 0$, by the Stirling formula [1, 3], we have

$$\log\Gamma(x) - x\log\frac{x}{e} - \frac{1}{2}\log\frac{2\pi}{x} = \frac{1}{2x} + \mathcal{O}\left(\frac{1}{x^3}\right), \tag{1.3}$$

as $x \to \infty$.

Let $\mu(n)$ be the Möbius function, which is defined by $\mu(1) = 1$, $\mu(n) = 0$ if n has a repeated prime factor, and $\mu(n) = (-1)^k$ if n possesses k distinct prime factors. For $n \in \mathbb{N}$ it is well-known that

$$\sum_{m|n} \mu(m) = \begin{cases} 1 & n = 1, \\ 0 & n \neq 1. \end{cases} \tag{1.4}$$

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A consequence of the above identity is the well-known Lambert type formula [2,3,6]

$$\sum_{n=1}^{\infty} \mu(n) \frac{q^n}{1-q^n} = q, \quad |q| < 1. \tag{1.5}$$

The sine and cosine integrals are defined by [3,6]

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t} dt, \ \mathrm{si}(x) = -\int_x^\infty \frac{\sin(t)}{t} dt, \ \mathrm{Ci}(x) = -\int_x^\infty \frac{\cos t}{t} dt \tag{1.6}$$

respectively. Then we have

$$\mathrm{Si(x)} - \mathrm{si}(x) = \int_0^\infty \frac{\sin(t)}{t} dt = \frac{\pi}{2}$$

and

$$\mathrm{Ci}(x) = \gamma + \log x + \int_0^x \frac{\cos t - 1}{t} dt, \quad \Re(x) > 0.$$

Let [6]

$$f(x) = \int_0^\infty \frac{\sin(t)}{t+x} dt = \int_0^\infty \frac{e^{-xt} dt}{t^2+1} \tag{1.7}$$

and

$$g(x) = \int_0^\infty \frac{\cos(t)}{t+x} dt = \int_0^\infty \frac{t e^{-xt} dt}{t^2+1}. \tag{1.8}$$

Then,

$$\mathrm{si}(x) = -f(x)\cos(x) - g(x)\sin(x), \tag{1.9}$$

$$\mathrm{Ci}(x) = f(x)\sin(x) - g(x)\cos(x) \tag{1.10}$$

and

$$f(x) = \mathrm{Ci}(x)\sin(x) - \mathrm{si}(x)\cos(x), \tag{1.11}$$

$$g(x) = -\mathrm{Ci}(x)\cos(x) - \mathrm{si}(x)\sin(x). \tag{1.12}$$

Recall that for $0 < x < 1$ Kummer's Fourier series expansion for $\log\Gamma(x)$ is given by [1,3,6]

$$\log \frac{\Gamma(x)}{\sqrt{2\pi}} = -\frac{\log(2\sin \pi x)}{2} + (\gamma + \log 2\pi)\left(\frac{1}{2} - x\right) + \sum_{k=1}^{\infty} \frac{\log k}{\pi k} \sin(2k\pi x), \tag{1.13}$$

where $\gamma = -\Gamma'(0)$ is the Euler constant. In this work we are going to prove an infinite series expansion for the auxiliary function $f(x)$, then apply Möbius inversion to obtain an infinite series expansion for $\log\Gamma(x)$ which is reminiscent of Kummer's series expansion (1.13). Unlike (1.13), it is not a true Fourier series anymore. By differentiating the series expansion for $f(x)$ we obtain an infinite series expansion for the auxiliary function $g(x)$.

2. Main results

Theorem 1. *For $x>0$ we have*

$$f(2\pi x)=\pi\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\left(\log\Gamma(nx)-(nx)\log(nx)+nx-\frac{1}{2}\log\left(\frac{2\pi}{nx}\right)\right) \tag{2.1}$$

and

$$\log\Gamma(x)-x\log(x)+x-\frac{1}{2}\log\left(\frac{2\pi}{x}\right)=\sum_{n=1}^{\infty}\frac{f(2n\pi x)}{n\pi}. \tag{2.2}$$

In particular for any $m\in\mathbb{N}$ we have

$$\begin{aligned}&\frac{(-1)^{m-1}}{\pi}\mathrm{si}(m\pi)\\&=\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\left(\log\Gamma\left(\frac{nm}{2}\right)-\left(\frac{nm}{2}\right)\log\left(\frac{nm}{2}\right)+\frac{nm}{2}-\frac{1}{2}\log\left(\frac{4\pi}{nm}\right)\right)\end{aligned} \tag{2.3}$$

and

$$\log\Gamma\left(\frac{m}{2}\right)-\frac{m}{2}\log\frac{m}{2}+\frac{m}{2}-\frac{1}{2}\log\left(\frac{4\pi}{m}\right)=\sum_{n=1}^{\infty}(-1)^{mn-1}\frac{\mathrm{si}(mn\pi)}{n\pi}. \tag{2.4}$$

Corollary 2. *For $x>0$ we have*

$$g(2\pi x)=\frac{1}{2}\sum_{n=1}^{\infty}\mu(n)\left(\log(nx)-\psi(nx)-\frac{1}{2nx}\right) \tag{2.5}$$

and

$$\log(x)-\psi(x)-\frac{1}{2x}=2\sum_{n=1}^{\infty}g(2n\pi x). \tag{2.6}$$

In particular, for any $m\in\mathbb{N}$ we have

$$\mathrm{Ci}(m\pi)=\frac{(-1)^m}{2}\sum_{n=1}^{\infty}\mu(n)\left(\psi(mn/2)-\log(mn/2)+\frac{1}{mn}\right) \tag{2.7}$$

and

$$\psi(m/2)-\log(m/2)+\frac{1}{m}=2\sum_{n=1}^{\infty}(-1)^{mn}\,\mathrm{Ci}(mn\pi). \tag{2.8}$$

3. Proofs

We observe that the left hand side of (3.1) below is essentially a special value of a q-analogue for the digamma function $\psi(x)$. Here we present a short proof of (3.1) by applying the ubiquitous AGM inequality, even though a corollary of problem 32 in [7] would suffice, as has been kindly pointed out by a referee. The same referee also recommended reference [4]; a monotonicity argument similar to the one used there may be applied to prove a more general result than (3.1).

Lemma 3. *Let $0<q<1$. Then we have*

$$\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\le\frac{q^{1/2}\log\left(1-q^{1/2}\right)}{q-1}. \tag{3.1}$$

Proof. For $0 < q < 1$ and $n \in \mathbb{N}$, by the AGM inequality we have

$$\frac{1+q+\cdots+q^{n-1}}{n} \geq \sqrt[n]{1\cdot q\cdots q^{n-1}} = q^{(n-1)/2},$$

hence

$$1-q^n \geq (1-q)nq^{(n-1)/2}, \quad n\in\mathbb{N},\ 0<q<1.$$

Then,

$$\sum_{n=1}^{\infty}\frac{q^n}{1-q^n} \leq \frac{q^{1/2}}{1-q}\sum_{n=1}^{\infty}\frac{q^{n/2}}{n} = -\frac{q^{1/2}}{1-q}\log\left(1-q^{1/2}\right),$$

which gives (3.1). □

Lemma 4. *For all $x > 0$, let*

$$u(x) = \sum_{m=1}^{\infty}\mu(m)v(mx). \tag{3.2}$$

If there exist certain $\epsilon > 0$ such that

$$\sum_{k=1}^{\infty}k^{\epsilon}\left|v(kx)\right| < \infty, \tag{3.3}$$

then

$$v(x) = \sum_{m=1}^{\infty}u(mx). \tag{3.4}$$

Proof. For any $N \in \mathbb{N}$ we have

$$\sum_{n=1}^{N}u(nx)=\sum_{n=1}^{N}\sum_{m=1}^{\infty}\mu(m)v(mnx)=\sum_{k=1}^{\infty}\left(\sum_{n|k,1\leq n\leq N}\mu(n)\right)v(kx)=\sum_{k=1}^{\infty}a_k(N)v(kx),$$

where

$$a_k(N) = \sum_{n|k,1\leq n\leq N}\mu(n), \quad k\in\mathbb{N}.$$

Clearly, we have [2]

$$|a_k(N)| \leq \sum_{n|k,1\leq n\leq N}1 \leq \sum_{n|k}1 = d(k),$$

where $d(k)$ counts the number of divisors in $k \in \mathbb{N}$. Since it is known that for any $\epsilon > 0$ we have $d(k) = \mathcal{O}(k^{\epsilon})$ as $k \to \infty$, then by (3.3) we see that the series $\sum_{k=1}^{\infty}a_k(N)v(kx)$ converges absolutely, and it converges uniformly with respect to the parameter N. Then by (1.4) we have

$$\begin{aligned}\sum_{n=1}^{\infty}u(nx) &= \lim_{N\to\infty}\sum_{n=1}^{N}u(nx) = \lim_{N\to\infty}\sum_{k=1}^{\infty}a_k(N)v(kx)\\ &= \sum_{k=1}^{\infty}\left(\lim_{N\to\infty}a_k(N)\right)v(kx) = v(x).\end{aligned}$$

□

3.1. **Proof of Theorem 1.**

Proof. For each $n \in \mathbb{N}$ and $x > 0$, by (1.2) we have

$$\begin{aligned} \log\Gamma(nx) - (nx)\log(nx) + nx - \frac{1}{2}\log\left(\frac{2\pi}{nx}\right) &= \int_0^\infty \frac{2\arctan(\frac{t}{nx})}{e^{2\pi t}-1}\mathrm{d}t \\ &= 2n\int_0^\infty \frac{\arctan(\frac{t}{x})}{e^{2n\pi t}-1}\mathrm{d}t, \end{aligned}$$

i.e.

$$\frac{1}{2n}\left(\log\Gamma(nx) - (nx)\log(nx) + nx - \frac{1}{2}\log\left(\frac{2\pi}{nx}\right)\right) = \int_0^\infty \frac{\arctan(\frac{t}{x})}{e^{2n\pi t}-1}\mathrm{d}t. \tag{3.5}$$

Taking $q = e^{-2\pi t},\ t > 0$, in (1.5) we get

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{e^{2n\pi t}-1} = e^{-2\pi t}. \tag{3.6}$$

Applying (3.5), (3.6), the Laplace integral (formula (8.4) on page 82 of [5])

$$\int_0^\infty e^{-st}\arctan(t)dt = \frac{\mathrm{Ci}(s)\sin(s) - \mathrm{si}(x)\cos(s)}{s} \tag{3.7}$$

and (1.11) we get

$$\begin{aligned} &\sum_{n=1}^{\infty}\frac{\mu(n)}{2n}\left(\log\Gamma(nx) - (nx)\log(nx) + nx - \frac{1}{2}\log\left(\frac{2\pi}{nx}\right)\right) \\ &= \sum_{n=1}^{\infty}\mu(n)\int_0^\infty \frac{\arctan(\frac{t}{x})}{e^{2n\pi t}-1}\mathrm{d}t = \int_0^\infty \sum_{n=1}^{\infty}\frac{\mu(n)}{e^{2n\pi t}-1}\arctan(\frac{t}{x})dt \\ &= \int_0^\infty e^{-2\pi t}\arctan(\frac{t}{x})dt = x\int_0^\infty e^{-2\pi x t}\arctan(t)dt \\ &= \frac{\mathrm{Ci}(2\pi x)\sin(2\pi x) - \mathrm{si}(2\pi x)\cos(2\pi x)}{2\pi} = \frac{f(2\pi x)}{2\pi}, \end{aligned} \tag{3.8}$$

which proves (2.1). The exchange order of summation and integration in the above proof can be justified by applying Fubini's theorem since for each fixed $x > 0$ we have

$$\sum_{n=1}^{\infty}\left|\frac{\mu(n)}{e^{2n\pi t}-1}\arctan(\frac{t}{x})\right| \le \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{e^{2n\pi t}-1} = \mathcal{O}\left(e^{-2\pi t}\right), \quad t \uparrow +\infty$$

and

$$\sum_{n=1}^{\infty}\left|\frac{\mu(n)}{e^{2n\pi t}-1}\arctan(\frac{t}{x})\right| \le \mathcal{O}\left(\frac{t}{x}\sum_{n=1}^{\infty}\frac{1}{e^{2n\pi t}-1}\right) = \mathcal{O}\left(\log t\right), \quad t \downarrow 0,$$

which are the consequences of Lemma 3 with $q = e^{-2\pi t},\ t > 0$, and

$$\arctan(z) = \mathcal{O}\left(z\right), \quad z \downarrow 0$$

and

$$\arctan(z) = \mathcal{O}\left(1\right), \quad z \uparrow \infty.$$

For $x > 0$, from

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{2nx} \left(\log \Gamma(nx) - (nx)\log(nx) + nx - \frac{1}{2}\log\left(\frac{2\pi}{nx}\right) \right) = \frac{f(2\pi x)}{2\pi x}$$

and (1.3), we get (2.2) by applying Lemma 4.

Formulas (2.3) and (2.4) are obtained by applying

$$\cos k\pi = (-1)^k, \; \sin k\pi = 0, \quad k \in \mathbb{N}.$$

□

3.2. **Proof of Corollary 2.** Since [1, 3, 6]

$$\psi(x) - \log(x) + \frac{1}{2x} = -\frac{1}{12x^2} + \mathcal{O}\left(\frac{1}{x^4}\right)$$

as $x \to \infty$, the series in (2.5) converges absolutely and uniformly in x for x in any compact subset of $(0, \infty)$. Then (2.5) is obtained by differentiating formulas (2.1) with respect to x. Formula (2.6) follows from (2.5) by applying Lemma 4, and the formulas (2.7), (2.8) are obtained by letting $x = m/2$, $m \in \mathbb{N}$, in formulas (2.5) and (2.6) respectively.

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