DISTORTION OF EMBEDDINGS OF BINARY TREES INTO DIAMOND GRAPHS

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Abstract. Diamond graphs and binary trees are important examples in the theory of metric embeddings and also in the theory of metric characterizations of Banach spaces. Some results for these families of graphs are parallel to each other; for example superreflexivity of Banach spaces can be characterized both in terms of binary trees (Bourgain, 1986) and diamond graphs (Johnson-Schechtman, 2009). In this connection, it is natural to ask whether one of these families admits uniformly bilipschitz embeddings into the other. This question was answered in the negative by Ostrovskii (2014), who left it open to determine the order of growth of the distortions. The main purpose of this paper is to get a sharp up-to-a-logarithmic-factor estimate for the distortions of embeddings of binary trees into diamond graphs and, more generally, into diamond graphs of any finite branching \( k \geq 2 \). Estimates for distortions of embeddings of diamonds into infinitely branching diamonds are also obtained.

1. Introduction

Binary trees and diamond graphs play an important role in the theory of metric embeddings and metric characterizations of properties of Banach spaces; see \cite{bourgain, johnson_schechtman, ostrovskii, ostrovskii2014, ostrovskii2016} and also presentations in the books \cite{odet, schm}. Some results for these families of graphs are parallel to each other; for example superreflexivity of Banach spaces can be characterized both in terms of binary trees (Bourgain \cite{bourgain}) and diamond graphs (Johnson-Schechtman \cite{johnson_schechtman}). In this connection, it is natural to ask whether these families of graphs admit bilipschitz embeddings with uniformly bounded distortions one into another. In one direction the answer is clear: The fact that diamond graphs do not admit uniformly bilipschitz embeddings into binary trees follows immediately from the combination of the result of Rabinovich and Raz \cite[Corollary 5.3]{rabinovich} stating that the distortion of any embedding of an \( n \)-cycle into any tree is \( \geq \frac{n}{3} - 1 \) and the observation that large diamond graphs contain large cycles isometrically. As for the opposite direction, it was proved in \cite{ostrovskii2014} that binary trees do no admit uniformly bilipschitz embeddings into diamond graphs. The goal of this paper is to get a sharp-up-to-a-logarithmic-factor estimate for the distortions of embeddings of binary trees into diamond graphs and, more generally, into diamond graphs of any finite branching \( k \geq 2 \). In addition, estimates for distortions of embeddings of diamonds into infinitely branching diamonds are obtained.
2. Definitions and the main result

To begin with, let us present the necessary definitions.

**Definition 2.1.** A binary tree of depth $n$, denoted $T_n$, is a finite graph in which each vertex is represented by a finite (possibly empty) sequence of 0’s and 1’s, of length at most $n$. Two vertices in $T_n$ are adjacent if the sequence representing one of them is obtained from the sequence representing the other by adding one term on the right. (For example, vertices corresponding to $(1, 1, 0)$ and $(1, 1, 0, 1)$ are adjacent.) Vertices which correspond to sequences of length $k$ are called vertices of $k$-th generation. The vertex corresponding to the empty sequence is called a root. If a sequence $\tau$ is an initial segment of the sequence $\sigma$ we say that $\sigma$ is a descendant of $\tau$ and that $\tau$ is an ancestor of $\sigma$. See Figure 1 for a sketch of $T_3$.

**Definition 2.2.** Diamond graphs $\{D_n\}_{n=0}^\infty$ are defined inductively as follows: The diamond graph of level 0 is denoted by $D_0$. It has two vertices joined by an edge. The diamond graph $D_n$ is obtained from $D_{n-1}$ as follows. Given an edge $uv \in E(D_{n-1})$, it is replaced by a quadrilateral $u,a,v,b$ with edges $ua$, $av$, $vb$, $bu$. See Figure 2 for a sketch of $D_2$.

All graphs considered in this paper are endowed with the shortest path distance: the distance between any two vertices is the number of edges in a shortest path between them.

**Definition 2.3.** Let $M$ be a finite metric space and $\{R_n\}_{n=1}^\infty$ be a sequence of finite metric spaces with increasing cardinalities. The distortion $c_R(M)$ of embeddings of $M$ into $\{R_n\}_{n=1}^\infty$ is defined as the infimum of $C \geq 1$ for which there is $n \in \mathbb{N}$, a map $f : M \to R_n$, and a number $r = r(f) > 0$ — called the scaling factor — satisfying the condition:

$$\forall u, v \in M \quad rd_M(u, v) \leq d_{R_n}(f(u), f(v)) \leq rCd_M(u, v).$$

Therefore, $c_D(T_n)$ is the infimum of distortions of embeddings of the binary tree $T_n$ into diamond graphs. Our main result is expressed by the following assertion:
Figure 2. Diamond $D_2$ in which generations of vertices are shown.

**Theorem 2.4.** There exists a constant $c > 0$ such that

$$c \frac{n}{\log_2 n} \leq c_D(T_n) \leq 2n$$

for all $n \geq 2$.

In recent years [1,9,14,15] we see an increasing interest in diamonds of high branching (see Definition 2.5). In view of this, we prove versions of Theorem 2.4 for such graphs (Theorems 2.6 and 2.7).

**Definition 2.5.** Fix $k \in \mathbb{N} \cup \{\infty\}$, $k \geq 2$. Let $D_{0,k}$ be a graph consisting of two vertices joined by one edge. The graph $D_{n+1,k}$ is obtained from $D_{n,k}$ if we replace each edge $uv$ in $D_{n,k}$ by a set of $k$ paths of length 2 joining $u$ and $v$. We call the graphs $D_{n,k}$ diamonds of branching $k$ if $k$ is finite and diamonds of infinite branching if $k = \infty$.

Call one of the vertices of $D_{0,k}$ the top and the other the bottom. Define the top and the bottom of $D_{n,k}$ as vertices which evolved from the top and the bottom of $D_{0,k}$, respectively. A subdiamond of $D_{n,k}$ is a subgraph which evolved from an edge of some $D_{m,k}$ for $0 \leq m \leq n$. The top and bottom of a subdiamond of $D_{n,k}$ are defined as the vertices of the subdiamond which are the closest to the top and bottom of $D_{n,k}$, respectively.

It can be noticed that $D_n = D_{n,2}$. Let $c_{(D,k)}(M)$ denote the distortion of embeddings of a finite metric space $M$ into $\{D_{n,k}\}$, as in Definition 2.3. The next generalization of Theorem 2.4 holds.
Theorem 2.6. If $k$ is finite, then there exists $c(k) > 0$ such that
\[ c(k) \frac{n}{\log_2 n} \leq c_{(D,k)}(T_n) \leq 2n \]
for all $n \geq 2$.

For infinitely branching diamonds, the following weaker version of Theorem 2.4 is valid:

Theorem 2.7. There exists constant $c(\infty) > 0$ such that
\[ c(\infty) \sqrt{n} \leq c_{(D,\infty)}(T_n) \leq 2n. \]

We refer to [2] for graph-theoretical terminology and to [12] for terminology of the theory of metric embeddings.

3. Estimates from above

Since $D_n$ is isometric to a subset of $D_{n,k}$ whenever $k \geq 2$, it suffices to prove the estimate from above for the binary diamonds $\{D_n\}$.

Proof of $c_D(T_n) \leq 2n$. Observe that the diamond $D_k$ contains isometrically the tree which is customarily denoted $K_{1,2^k}$. This tree has $2^k + 1$ vertices, and one of the vertices is incident to the remaining $2^k$ vertices. In fact, one can easily establish by induction that the bottom of the diamond $D_k$ has degree $2^k$, and the bottom together with all of its neighbors forms the desired tree.

Choose $k$ in such a way that $2^k + 1 \geq 2^{n+1} - 1$, where $2^{n+1} - 1$ is the number of vertices in $T_n$.

Now, map the root of $T_n$ to the bottom of $D_k$ and all of the other vertices of $T_n$ to distinct vertices adjacent to the bottom. Denote the obtained map by $F_n$ and the vertex set of $T_n$ by $V(T_n)$. We claim that the following inequalities are true:

\[ \forall u, v \in V(T_n) \quad \frac{1}{n} d_{T_n}(u, v) \leq d_{D_k}(F_n(u), F_n(v)) \leq 2d_{T_n}(u, v), \]

yielding $c_D(T_n) \leq 2n$.

Indeed, the right-hand side inequality follows from the fact that any distance between two distinct vertices in $K_{1,2^k}$ does not exceed 2.

To justify the left-hand side inequality consider the two cases:

1. One of the vertices, say $u$, is the root of $T_n$. Then $d_{D_k}(F_n(u), F_n(v)) = 1$ and $d_{T_n}(u, v) \leq n$. The left-hand side inequality in this case follows.

2. Neither $u$ nor $v$ is the root of $T_n$. Then $d_{D_k}(F_n(u), F_n(v)) = 2$ and $d_{T_n}(u, v) \leq 2n$. The left-hand side inequality follows in this case, too. □

4. Estimates from below

4.1. Diamonds of finite branching. Observe that Theorem 2.4 is a special case of Theorem 2.6. For this reason, only the lower estimate of Theorem 2.6 has to be proved.

Proof of $c_{(D,k)}(T_n) \geq c(k) \frac{n}{\log_2 n}$. Fix an integer $k$ ($2 \leq k < \infty$) for the whole proof and omit from most of our notation dependence on $k$, as it is clear that almost all of the introduced objects depend on $k$. If $\alpha_n = 3c_{(D,k)}(T_n)$, then there exists a
Lemma 4.2. The cardinality of a \( 2^{p(n)} \)-separated set — i.e., a set satisfying \( d(u,v) \geq 2^{p(n)} \) for any \( u \neq v \) — in a subdiamond of \( D_{m,k} \) of diameter \( 2^d \) does not exceed \( k \cdot (2k)^{d-p(n)} \) if \( q \geq p(n) \).
Proof. It is easy to see that each subdiamond of $D_{m,k}$ of diameter $2^p(n)$ contains at most $k$ vertices out of each $2^p(n)$-separated set. The number of subdiamonds of diameter $2^p(n)$ in a diamond of diameter $2^q$ is equal to the number of edges in the diamond of diameter $2^q-p(n)$. This number of edges is $(2k)^{q-p(n)}$, because in each step of the construction of diamonds the diameter doubles and the number of edges is multiplied by $(2k)$.

This contradicts (5) because, on one hand, the vertex $τ_i$ has more than $2^r$ descendants in the next $r$ generations, and the images of these descendants, by the bilipschitz condition (3), should form a $2^p(n)$-separated set. On the other hand, Lemma 4.2 implies that a $2^p(n)$-separated set in a union of two diamonds of diameters $2^{d-1}$ does not exceed $2 \cdot k \cdot (2k)^{d-1-p(n)} = (2k)^{d-p(n)}$.

Since $α_n = 3c_{(D,k)}(T_n)$, to prove the existence of a constant $c(k) > 0$ such that $c_{(D,k)}(T_n) \geq c(k) \log_2 n$, it suffices to show that the existence of the subsequence of values of $n$ for which $α_n = o\left(\frac{n}{\log_2 n}\right)$ leads to a contradiction. This will be done by demonstrating that the existence of such a subsequence implies the existence of $n$, $r$ and $d$ satisfying $1 \leq r < n$ and (1)-(6). Let us rewrite inequalities (1)-(6) as

\begin{align}
&2^{d-p(n)} > 2α_n(r+1), \\
&2^{(d-p(n))\log_2 (2k)} < 2^r, \\
&2^{d-p(n)} < n - r.
\end{align}

Set $r = r(n) = [\log_2 (2k) \cdot \log_2 n]$. Since $k$ is fixed, for sufficiently large $n$, one has $n - r > 2$. Define $d = d(n) \in \mathbb{N}$ to be the largest integer for which (9) holds. It has to be pointed out that with this choice of $d$, the inequality $2^{d-p(n)} > \frac{n}{4}$ holds when $n$ is sufficiently large. Since for our choice of $r$ we have $2α_n(r(n)+1) = o(n)$ for the corresponding subsequence of values of $n$, it is clear that, for sufficiently large $n$ in the subsequence, the condition (7) is also satisfied. It remains to observe that, with the described choice of $r$, the inequality (8) follows from

\begin{align}
(2^{d-p(n)})^{\log_2 (2k)} < n^{\log_2 (2k)}.
\end{align}

Since by virtue of (9), $2^{d-p(n)} < n$, the last inequality is obvious.

4.2. Diamonds of infinite branching. In this case, the methods based on the upper bounds for cardinalities of $2^p(n)$-separated sets in subdiamonds are not applicable since the cardinalities are infinite. Consequently, the method of [13], which gives weaker estimates but works in the case of infinite branching, will be employed.

Proof of Theorem 2.7 Since the upper estimate has already been established in section 5, to complete the proof it has to be shown that $c_{(D,∞)}(T_n) \geq c(∞) \sqrt{n}$ for some constant $c(∞) > 0$.

If $α_n = 3c_{(D,∞)}(T_n)$, then there exists a map $F_n$ of $V(T_n)$ into $V(D_{m(n),∞})$ for some $m(n) \in \mathbb{N}$ satisfying (11) with $C = α_n$ and the scaling factor being an integer power of 2, that is,

\begin{align}
∀u, v \in V(T_n) \quad 2^{p(n)}d_{T_n}(u, v) \leq d_{D_{m(n),∞}}(F_n(u), F_n(v)) \leq α_n 2^{p(n)}d_{T_n}(u, v)
\end{align}

for some $p(n) \in \mathbb{Z}$. With the help of the same argument as in Theorem 2.6 it may be assumed that $p(n) \geq 0$. 

Since \( \{ c_{(D, \infty)}(T_n) \}_{n=1}^\infty \) as well as \( \{ \sqrt{n} \}_{n=1}^\infty \) are sequences of positive numbers, it suffices to prove that the inequality of the form \( c_{(D, \infty)}(T_n) \geq c\sqrt{n} \) holds for some \( c > 0 \) and sufficiently large \( n \).

Assume that \( n > 9 \) and denote by \( d = d(n) \) the largest integer satisfying
\[
2^d < 2^{p(n)} \cdot \left\lceil \frac{n}{3} \right\rceil.
\]

Consider any vertex \( \tau_{\left\lfloor \frac{n}{3} \right\rfloor} \) of generation \( \left\lfloor \frac{n}{3} \right\rfloor \) in \( T_n \). Let \( \tau_0, \ldots, \tau_{\left\lfloor \frac{n}{3} \right\rfloor} \) be a path joining the root \( \tau_0 \) and \( \tau_{\left\lfloor \frac{n}{3} \right\rfloor} \) in \( T_n \). Inequality (10) implies that
\[
d_{D_{m(n), \infty}}(F_n(\tau_i), F_n(\tau_{i+1})) \leq \alpha_n 2^{p(n)}
\]
and
\[
d_{D_{m(n), \infty}}(F_n(\tau_0), F_n(\tau_{\left\lfloor \frac{n}{3} \right\rfloor})) \geq 2^{p(n)} \left\lceil \frac{n}{3} \right\rceil.
\]
By combining these inequalities with condition (11) and Observation 4.1(2), we conclude that there exists \( i \in \{0, \ldots, \left\lfloor \frac{n}{3} \right\rfloor \} \) such that
\[
d_{D_{m(n), \infty}}(F_n(\tau_i), v) \leq \alpha_n 2^{p(n)}
\]
for some \( v \) of generation \( d \) in \( D_{m(n), \infty} \).

Inequality (10) together with (12) implies that descendants of \( \tau_i \) of generation \( n \) in \( T_n \) will be mapped onto vertices whose distances from \( v \) are at least \( (n-i)2^{p(n)} - \alpha_n 2^{p(n)} \). One has
\[
(n-i)2^{p(n)} - \alpha_n 2^{p(n)} \geq \left( \frac{2}{3} n - \alpha_n \right) 2^{p(n)}.
\]
If \( \alpha_n \geq \frac{n}{3} \), the conclusion of the theorem holds with \( c(\infty) = \frac{1}{5} \). Therefore, assume that \( \alpha_n \leq \frac{n}{3} \). In this case, the right-hand side of (13) is not less than
\[
\frac{n}{3} 2^{p(n)} \geq 2^d > 2^{d-1}.
\]

Thence, on each path joining \( \tau_i \) with one of its descendants of generation \( n \) (in \( T_n \)) there is a vertex which is mapped by \( F_n \) outside the union of two subdiamonds of height \( 2^{d-1} \) with the common vertex at \( v \).

Let \( x_1 \) and \( x_2 \) be the different from \( v \) tops/bottoms of the subdiamonds mentioned in the previous paragraph. The statement about the paths mentioned in the previous paragraph implies that on each path joining \( \tau_i \) with one of its descendants (in \( T_n \)) of generation \( n \) there is a vertex, the \( F_n \)-image of which is at distance at most \( \alpha_n 2^{p(n)} \) from either \( x_1 \) or \( x_2 \). In fact, it is clear that this condition holds for the first vertex on the path whose \( F_n \)-image is outside the union of the subdiamonds.

Now, let us fix such a path and estimate from below the generation \( r \) of the first vertex on this path, whose \( F_n \)-image is outside the union of the subdiamonds. It can be seen by using inequality (10) that the earliest generation \( r \) for which it is possible for such image to be outside the union of the subdiamonds has to satisfy
\[
(r-i)2^{p(n)}\alpha_n + 2^{p(n)}\alpha_n > 2^{d-1},
\]
whence
\[
(r-i) \geq \frac{2^{d-1}}{2^{p(n)}\alpha_n} - 1.
\]

The choice of \( d \) (see the line preceding (11)) implies that
\[
2^{d+1} \geq 2^{p(n)} \cdot \left\lceil \frac{n}{3} \right\rceil,
\]
and hence
\[(r - i) \geq \frac{1}{4\alpha_n} \left\lfloor \frac{n}{3} \right\rfloor - 1.\]

Next, consider four different descending paths in $T_n$ starting at different descendants of $\tau_i$ of generation $(i + 2)$. Along each of these paths we pick the first vertex whose $F_n$-image is outside the union of the two subdiamonds. Let $v_1, v_2, v_3,$ and $v_4$ be the picked vertices and suppose that these vertices belong to generations $r_1, r_2, r_3,$ and $r_4$, respectively, in $T_n$.

First, assume that $r_j > i + 2$ for $j = 1, 2, 3, 4$, while the case where $r_j = i + 2$ for some $j$ will be considered at the very end of the proof. By the argument above, each $r_j$ satisfies
\[(r_j - i) \geq \frac{1}{4\alpha_n} \left\lfloor \frac{n}{3} \right\rfloor - 1.\]

Therefore the pairwise distances between vertices $v_1, v_2, v_3,$ and $v_4$ are at least:
\[r_{j_1} + r_{j_2} - 2i - 2 \geq \frac{1}{2\alpha_n} \left\lfloor \frac{n}{3} \right\rfloor - 4, \quad j_1, j_2 \in \{1, 2, 3, 4\}, \quad j_1 \neq j_2.\]

The argument above implies that under the assumption $r_j > i + 2$ the image $F_n(v_j)$ is at distance at most $2^{p(n)}\alpha_n$ to either $x_1$ or $x_2$. As a result, at least two of these images are at distance at most $2 \cdot 2^{p(n)}\alpha_n$ from each other. Using (10) one concludes that
\[2^{p(n)} \left( \frac{1}{2\alpha_n} \left\lfloor \frac{n}{3} \right\rfloor - 4 \right) \leq 2 \cdot 2^{p(n)}\alpha_n.\]

It is easy to see that this inequality implies that $\alpha_n \geq c\sqrt{n}$ for some constant $c > 0$ and sufficiently large $n$.

Now comes the case where $r_j = i + 2$ for some $j \in \{1, 2, 3, 4\}$. In this case, the distance between $F_n(v_j)$ and $v$, on one hand, is $> 2^{d-1}$, and, on the other hand, by (12) and (10), it is $\leq 3\alpha_n 2^{p(n)}$. This leads to $3\alpha_n 2^{p(n)} > 2^{d-1}$. Combining with (14), one obtains
\[3\alpha_n 2^{p(n)} > 2^{p(n) - 2} \cdot \left\lfloor \frac{n}{3} \right\rfloor.\]

Thus, $\alpha_n \geq \frac{1}{12} \cdot \left\lfloor \frac{n}{3} \right\rfloor$, yielding that in this case $c(D, \infty)(T_n) \geq \frac{1}{36} \cdot \left\lfloor \frac{n}{3} \right\rfloor$. This inequality is sufficient for our purposes. This completes the proof of Theorem 2.7. \qed

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