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HERMITIAN u-INVARIANTS OVER FUNCTION FIELDS OF p-ADIC CURVES

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Abstract. Let p be an odd prime. Let F be the function field of a p-adic curve. Let A be a central simple algebra of period 2 over F with an involution σ. There are known upper bounds for the u-invariant of hermitian forms over (A,σ). In this article we compute the exact values of the u-invariant of hermitian forms over (A,σ).

1. Introduction

Let A be a central simple algebra over a field K. Let σ be an involution on A. Let $k = K^\sigma = \{x \in K \mid \sigma(x) = x\}$. Suppose $\operatorname{char} k \neq 2$. Suppose $\varepsilon \in \{1,-1\}$. If V is a finitely generated right A-module and $h : V \times V \to A$ is an ε-hermitian space over (A,σ), the *rank* of h is defined to be $\operatorname{Rank}(h) = \dfrac{\dim_K(V)}{\deg(A)\operatorname{ind}(A)}$. Let $\operatorname{Herm}^\varepsilon(A,\sigma)$ denote the category of ε-hermitian spaces over (A,σ). The hermitian u-invariant [Mah05, 2.1] of (A,σ,ε) is defined to be:

$$u(A,\sigma,\varepsilon) = \sup\{n \mid \text{there exists an anisotropic } h \in \operatorname{Herm}^\varepsilon(A,\sigma), \operatorname{Rank}(h) = n\}.$$

Suppose that σ and τ are involutions on A. Mahmoudi has proved that [Mah05, 2.2] if σ and τ are of the same type, then $u(A,\sigma,\varepsilon) = u(A,\tau,\varepsilon)$; if σ is orthogonal and τ is symplectic, then $u(A,\sigma,\varepsilon) = u(A,\tau,-\varepsilon)$; if σ is unitary, then $u(A,\sigma,1) = u(A,\sigma,-1)$. Thus we have only three types of hermitian u-invariants [Mah05, 2.3], and we denote:

$$u(A,\sigma,\varepsilon) = \begin{cases} u^+(A), & \text{if } \varepsilon = 1 \text{ and } \sigma \text{ is orthogonal,} \\ & \text{or, } \varepsilon = -1 \text{ and } \sigma \text{ is symplectic;} \\ u^-(A), & \text{if } \varepsilon = -1 \text{ and } \sigma \text{ is orthogonal,} \\ & \text{or, } \varepsilon = 1 \text{ and } \sigma \text{ is symplectic;} \\ u^0(A), & \text{if } \sigma \text{ is unitary,} \end{cases}$$

where u^+ is called the *orthogonal* hermitian u-invariant, u^- is called the *symplectic* hermitian u-invariant and u^0 is called the *unitary* hermitian u-invariant.

In section 3, we provide upper bounds for hermitian u-invariants of division algebras with Springer's property over $\mathscr{A}_i(2)$-fields. For definitions of $\mathscr{A}_i(2)$-fields and Springer's property, see the beginning of section 3.

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Theorem 1.1. *Let D be a division algebra over a field K with an involution σ. Suppose $k = K^\sigma$, $\operatorname{char} k \neq 2$, $\varepsilon \in \{1, -1\}$ and $d = \deg(D)$. Suppose k is an $\mathscr{A}_i(2)$-field and D satisfies the Springer's property.*
(i) If σ is of the first kind, then $u^+(D) \leq (1 + \frac{1}{d})2^{i-1}$ and $u^-(D) \leq (1 - \frac{1}{d})2^{i-1}$.
(ii) If σ is of the second kind, then $u^0(D) \leq 2^{i-1}$.

Let p be an odd prime number. Let F be the function field of a smooth projective geometrically integral curve over a p-adic field. The field F is also called a *semi-global* field. Let D be a central division F-algebra with an involution σ of the first kind. Suppose $D \neq F$. As a consequence of an inequality of Mahmoudi [Mah05, 3.6] with $u(F) = 8$ ([PS10] or [HB10] and [Lee13]), $u^+(D) \leq 27$ and $u^-(D) \leq 10$. Parihar and Suresh [PS13] have proved that $u^+(D) \leq 14$ and $u^-(D) \leq 8$. We obtain exact values of hermitian u-invariants:

Theorem 1.2. *Let F be the function field of a curve over a p-adic field with $p \neq 2$. Let D be a central division algebra over F. Let L/F be a quadratic extension.*
(1) If D is quaternion, then $u^+(D) = 6$ and $u^-(D) = 2$.
(2) If D is quaternion and $D \otimes_F L$ is division, then $u^0(D \otimes_F L) = 4$.
(3) If D is biquaternion, then $u^+(D) = 5$ and $u^-(D) = 3$.

Let A be a central simple algebra over a field k. Suppose $\operatorname{char} k \neq 2$ and $\operatorname{per}(A) = 2$. Then, by a special case [Mer81] of the Merkur′ev-Suslin theorem [MS82], A is Brauer equivalent to $H_1 \otimes \cdots \otimes H_n$ for some quaternion algebras $H_1, \cdots, H_n$ over k. Let K/k be a quadratic extension. In [PS13], upper bounds for $u^+(A)$, $u^-(A)$, $u^0(A \otimes K)$ are given and they depend only on $u(k)$ and n. In section 5, we obtain sharper upper bounds for these hermitian u-invariants. In fact we prove the following

Theorem 1.3. *Let A be a central simple algebra over a field k. Suppose* $\operatorname{char} k \neq 2$ *and* $\operatorname{per}(A) = 2$. *Suppose A is Brauer equivalent to $H_1 \otimes \cdots \otimes H_n$ for n quaternion algebras $H_1, \cdots, H_n$ over k.*
(1) $u^+(A) \leq (\frac{4}{5} + \frac{1}{5}(\frac{9}{4})^n)u(k)$;
(2) $u^-(A) \leq (-\frac{1}{5} + \frac{1}{5}(\frac{9}{4})^n)u(k)$;
(3) $u^0(A \otimes_k K) \leq (\frac{1}{5} + \frac{3}{10}(\frac{9}{4})^n)u(k)$ for all quadratic extension K/k.

2. Preliminaries

Let K be a field. Let A be a central simple algebra over K with an involution σ. Let $k = K^\sigma$. We suppose $\operatorname{char}(k) \neq 2$ throughout the paper. Let V be a finitely generated right A-module and $\varepsilon \in \{1, -1\}$. A map $h : V \times V \to A$ is called *an ε-hermitian form* over (A, σ) if h is bi-additive; $h(xa, yb) = \sigma(a)h(x, y)b$ for all $a, b \in A$, $x, y \in V$; and $h(y, x) = \varepsilon\sigma(h(x, y))$ for all $x, y \in V$. We call h *an ε-hermitian space* if given $h(x, y) = 0$ for all $x \in V$; we have $y = 0$. We say that h is *isotropic* if there exists $x \in V$, $x \neq 0$ such that $h(x, x) = 0$; otherwise we say that h is *anisotropic*.

Lemma 2.1 (Morita invariance)**.** *Let K, A, σ, k be as before. Suppose $A \simeq M_m(D)$ for a central division algebra D over K. Suppose σ is an involution on A and $\varepsilon \in \{1, -1\}$. Then there exists an involution τ on D and $\varepsilon_0 \in \{1, -1\}$ such that $u(A, \sigma, \varepsilon) = u(D, \tau, \varepsilon\varepsilon_0)$.*

Furthermore, $u^+(A) = u^+(D)$, $u^-(A) = u^-(D)$ and $u^0(A) = u^0(D)$.

Proof. It is a consequence of [Knu91, ch. I, 9.3.5] and [KMRT98, 4.2]. □

From now on, we mostly focus on central division algebras.

Lemma 2.2. *Let D be a central division algebra over a field K with an involution σ. Let $k = K^\sigma$, $\operatorname{char} k \neq 2$. Suppose k is a non-archimedean local field.*
(1) If σ is of the first kind and $D \neq k$, then $u^+(D) = 3$, $u^-(D) = 1$.
(2) If σ is of the second kind, then $u^0(D) = 2$.

Proof. See [Tsu61, Thm. 1, Thm. 3] and [Sch85, ch. 10, 2.2]. □

We fix the following notation from 2.3 to 2.9. Let (k, v) be a complete discrete valued field with residue field $\overline{k}$, $\operatorname{char} \overline{k} \neq 2$. Let D be a finite-dimensional division k-algebra with center K with an involution σ such that $K^\sigma = k$. By [CF67, ch. II, 10.1], v extends to a valuation v' on K such that $v'(x) = \frac{1}{[K:k]} v(N_{K/k}(x))$ for all $x \in K^*$. By [Wad86], v' extends to a valuation w on D such that $w(x) = \frac{1}{\operatorname{ind}(D)} v'(\operatorname{Nrd}_{D/K}(x))$ for all $x \in D^*$. Since $\operatorname{Nrd}_{D/K}(x) = \operatorname{Nrd}_{D/K}(\sigma(x))$, we have $w(\sigma(x)) = w(x)$ for all $x \in D$. Let $R_w = \{x \in D \mid w(x) \geq 0\}$ and $\mathfrak{m}_w = \{x \in D \mid w(x) > 0\}$. Let $\overline{D} = R_w/\mathfrak{m}_w$ be the residue division algebra (see [Rei03, 13.2]) of (D, w) over $\overline{k}$ with involution $\overline{\sigma}$ such that $\overline{\sigma}(\overline{x}) = \overline{\sigma(x)}$ for all $x \in R_w$, where $\overline{x} = x + \mathfrak{m}_w$. Let $h\colon V \times V \to D$ be an ε-hermitian space over (D, σ). By [Knu91, Ch. I, 6.2], V is free with an orthogonal basis $\{e_1, \ldots, e_n\}$ such that $h(e_i, e_i) = a_i$ for some $a_i \in D$ with $\sigma(a_i) = \varepsilon a_i$ for all $1 \leq i \leq n$; and $h(e_i, e_j) = 0$ for all $1 \leq i \leq n$, $1 \leq j \leq n$ and $i \neq j$. We denote $h = \langle a_1, \cdots, a_n \rangle$. If $w(a_i) = 0$ for all $1 \leq i \leq n$, then $\overline{h} = \langle \overline{a}_1, \cdots, \overline{a}_n \rangle \in \operatorname{Herm}^\varepsilon(\overline{D}, \overline{\sigma})$. Let t_D be a parameter of (D, w). By [Lar99, 2.7], there exists $\pi_D \in D$ such that $w(\pi_D) \equiv w(t_D) \mod 2w(D^*)$ and $\sigma(\pi_D) = \varepsilon' \pi_D$ for some $\varepsilon' \in \{1, -1\}$. Larmour proved the following hermitian analogue of a theorem of Springer.

Proposition 2.3 ([Lar06, 3.4] or [Lar99, 3.27])**.** *Let k, v, D, K, σ, w, h, π_D and ε' be as above. There exist $h_1 \in \operatorname{Herm}^\varepsilon(D, \sigma)$, $h_2 \in \operatorname{Herm}^{\varepsilon\varepsilon'}(D, \operatorname{Int}(\pi_D) \circ \sigma)$, with $h \simeq h_1 \perp h_2 \pi_D$ and each diagonal entries of h_1 and h_2 have w-value 0. Further, the following are equivalent: (i) h is isotropic; (ii) h_1 or h_2 is isotropic; (iii) $\overline{h}_1$ or $\overline{h}_2$ is isotropic.*

Corollary 2.4. $u(D, \sigma, \varepsilon) = u(\overline{D}, \overline{\sigma}, \varepsilon) + u(\overline{D}, \overline{\operatorname{Int}(\pi_D) \circ \sigma}, \varepsilon\varepsilon')$.

Proof. Suppose $h \in \operatorname{Herm}^\varepsilon(D, \sigma)$ and $h \simeq h_1 \perp h_2 \pi_D$ as in Proposition 2.3. Since $\operatorname{Rank}(h) = \operatorname{Rank}(h_1) + \operatorname{Rank}(h_2) = \operatorname{Rank}(\overline{h_1}) + \operatorname{Rank}(\overline{h_2})$, if $\operatorname{Rank}(h) > u(\overline{D}, \overline{\sigma}, \varepsilon) + u(\overline{D}, \overline{\operatorname{Int}(\pi_D) \circ \sigma}, \varepsilon\varepsilon')$, then

$$\operatorname{Rank}(\overline{h_1}) > u(\overline{D}, \overline{\sigma}, \varepsilon)$$

or

$$\operatorname{Rank}(\overline{h_2}) > (\overline{D}, \overline{\operatorname{Int}(\pi_D) \circ \sigma}, \varepsilon\varepsilon').$$

Then $\overline{h}_1$ or $\overline{h}_2$ is isotropic. By Proposition 2.3, h is isotropic. Hence $u(D, \sigma, \varepsilon) \leq u(\overline{D}, \overline{\sigma}, \varepsilon) + u(\overline{D}, \overline{\operatorname{Int}(\pi_D) \circ \sigma}, \varepsilon\varepsilon')$.

Conversely, suppose $g_1 = \langle a_1, \cdots, a_m \rangle \in \operatorname{Herm}^\varepsilon(\overline{D}, \overline{\sigma})$ such that $\overline{\sigma}(a_i) = \varepsilon a_i$, $m = u(\overline{D}, \overline{\sigma}, \varepsilon)$ and g_1 is anisotropic. Since $a_i \neq 0$, there exists $b_i \in R_w$, $w(b_i) = 0$ such that $\overline{b_i} = a_i$. Let $c_i = \frac{1}{2}(b_i + \varepsilon\sigma(b_i))$. Then $\sigma(c_i) = \varepsilon c_i$ and $\overline{c_i} = a_i$. Let $h_1 = \langle c_1, \cdots, c_m \rangle \in \operatorname{Herm}^\varepsilon(D, \sigma)$. Then $\overline{h_1} = g_1$ and by [Lar06, 2.3], h_1 is anisotropic.

Suppose $g_2 = \langle a_{m+1}, \cdots, a_{m+n}\rangle \in \mathrm{Herm}^{\varepsilon\varepsilon'}(\overline{D}, \overline{\mathrm{Int}(\pi_D)\circ\sigma})$ such that g_2 is anisotropic. Similar to the previous paragraph, there exists $h_2 \in \mathrm{Herm}^{\varepsilon\varepsilon'}(D, \mathrm{Int}(\pi_D)\circ\sigma)$ such that $\overline{h_2} = g_2$ and h_2 is anisotropic.

By Proposition 2.3, $h = h_1 \perp h_2\pi_D$ is anisotropic and $\mathrm{Rank}(h) = m+n$. $u(D,\sigma,\varepsilon) \geq u(\overline{D},\overline{\sigma},\varepsilon) + u(\overline{D}, \overline{\mathrm{Int}(\pi_D)\circ\sigma}, \varepsilon\varepsilon')$. □

Lemma 2.5. *Suppose D is ramified at the discrete valuation of k, $Z(D) = k$ and $\mathrm{per}(D)|2$. Then there exist an involution σ on D of the first kind and elements $\alpha, \pi_D \in D$ satisfying the following conditions:*

(a) $\overline{\sigma}$ is an involution of the second kind;

(b) α^2 is a unit at the valuation v of k and $Z(\overline{D}) = \overline{k}(\overline{\alpha})$;

(c) π_D is a parameter of D, $\sigma(\pi_D) = \pm\pi_D$ and $\overline{\mathrm{Int}(\pi_D)\circ\sigma}$ is of the first kind.

Proof. Suppose D is ramified. Then D is Brauer equivalent to $D_0 \otimes (u,\pi)$ with D_0 a central division algebra over k unramified at v, $\pi \in k^*$ a parameter and $u \in k^* \setminus k^{*2}$ a unit at v. Furthermore, $\overline{D}$ Brauer is equivalent to $\overline{D_0} \otimes \overline{k}(\sqrt{u})$ and $Z(\overline{D}) = \overline{k}(\sqrt{u})$ [TW15, 8.77].

(a) By [CDT95, prop. 4], the non-trivial automorphism of $Z(\overline{D})/\overline{k}$ extends to an involution on $\overline{D}$ of the second kind and it can be lifted to an involution σ on D of the first kind.

(b) Since k is complete, by [CDT95, p. 53, Lem. 1], there exists $\alpha \in D$ such that $\alpha^2 \in Z(D)$, $\overline{\alpha} \in Z(\overline{D})$ corresponds to $\sqrt{u}$ in the isomorphism $Z(\overline{D}) = \overline{k}(\sqrt{u})$ and $\sigma(\alpha) = -\alpha$.

(c) By [JW90, prop. 1.7], there exists a parameter $t_D \in D$ such that $\overline{\mathrm{Int}(t_D)}$ is the non-trivial $Z(\overline{D})/\overline{k}$-automorphism. Since $\overline{\sigma}$ is of the second kind and $\overline{\mathrm{Int}(t_D)}$ induces the non-trivial automorphims of $Z(\overline{D})$, $\overline{\mathrm{Int}(t_D)\circ\sigma}$ is of the first kind. Since σ is an involution, $w(t_D) = w(\sigma(t_D))$ and hence $\overline{\sigma(t_D)t_D^{-1}} \neq 0 \in \overline{D}$.

Case 1. Suppose that $\overline{\sigma(t_D)t_D^{-1}} = 1$. Let $\pi_D = t_D + \sigma(t_D)$. Then $\sigma(\pi_D) = \pi_D$. Since $\pi_D t_D^{-1} = 1 + \sigma(t_D)t_D^{-1}$, $\overline{\pi_D t_D^{-1}} = 1 + \overline{\sigma(t_D)t_D^{-1}} = 1 + 1 = 2 \neq 0$. Hence $w(\pi_D) = w(t_D)$. Since $\overline{\pi_D t_D^{-1}} = 2$, $\overline{\mathrm{Int}(\pi_D)\circ\sigma} = \overline{\mathrm{Int}(t_D)\circ\sigma}$ and hence $\overline{\mathrm{Int}(\pi_D)\circ\sigma}$ is of the first kind. Thus π_D satisfies (c).

Case 2. Suppose that $\overline{\sigma(t_D)t_D^{-1}} \neq 1$. Let $\pi_D = \alpha t_D - \sigma(\alpha t_D)$. Then $\sigma(\pi_D) = -\pi_D$. We have $\pi_D t_D^{-1} = \alpha - \sigma(t_D)\sigma(\alpha)t_D^{-1}$. Since $\overline{\sigma(\alpha)} = -\overline{\alpha}$ and $\overline{t_D\alpha t_D^{-1}} = -\overline{\alpha}$, we have

$$\begin{aligned}
\overline{\pi_D t_D^{-1}} &= \overline{\alpha} - \overline{\sigma(t_D)\sigma(\alpha)t_D^{-1}} \\
&= \overline{\alpha} + \overline{\sigma(t_D)t_D^{-1}t_D\alpha t_D^{-1}} \\
&= \overline{\alpha} + \overline{\sigma(t_D)t_D^{-1}}(-\overline{\alpha}) \\
&= (1 - \overline{\sigma(t_D)t_D^{-1}})\overline{\alpha} \neq 0.
\end{aligned}$$

Hence $w(\pi_D) = w(t_D)$. Since $\overline{\sigma(\alpha)} = -\overline{\alpha}$, $\alpha^2 \in k$ and $\overline{t_D \alpha t_D^{-1}} = -\overline{\alpha}$, we have $\overline{\sigma(t_D)\alpha\sigma(t_D)^{-1}} = -\overline{\alpha}$ and

$$\begin{aligned}
& \overline{(\pi_D \alpha \pi_D^{-1} + \alpha)\pi_D t_D^{-1}} \\
= \; & \overline{\pi_D \alpha t_D^{-1}} + \overline{\alpha \pi_D t_D^{-1}} \\
= \; & \overline{(\alpha t_D - \sigma(t_D)\sigma(\alpha))\alpha t_D^{-1}} + \overline{\alpha(\alpha t_D - \sigma(t_D)\sigma(\alpha))t_D^{-1}} \\
= \; & \overline{\alpha t_D \alpha t_D^{-1}} - \overline{\sigma(t_D)\sigma(\alpha)\alpha t_D^{-1}} + \overline{\alpha^2} + \overline{\alpha\sigma(t_D)\alpha t_D^{-1}} \\
= \; & -\overline{\alpha^2} + \overline{\sigma(t_D)\alpha^2 t_D^{-1}} + \overline{\alpha^2} + \overline{\alpha(\sigma(t_D)\alpha\sigma(t_D)^{-1})\sigma(t_D)t_D^{-1}} \\
= \; & \overline{\alpha^2\sigma(t_D)t_D^{-1}} - \overline{\alpha^2\sigma(t_D)t_D^{-1}} = 0.
\end{aligned}$$

Since $\overline{\pi_D t_D^{-1}} \neq 0$, $\overline{\pi_D \alpha \pi_D^{-1} + \alpha} = 0$ and hence $\overline{(\mathrm{Int}(\pi_D) \circ \sigma)}(\overline{\alpha}) = \overline{\alpha}$. Thus π_D satisfies (c).

To summarize, σ, α and π_D satisfy the required properties. $\square$

Corollary 2.6. *Suppose* $Z(D) = k$ *and* $\mathrm{per}(D) = 2$.
(1) If D *is unramified at the discrete valuation of* k*, then*

$$u^+(D) = 2u^+(\overline{D}) \quad \textit{and} \quad u^-(D) = 2u^-(\overline{D}).$$

(2) If D *is ramified at the discrete valuation of* k*, then*

$$u^+(D) = u^0(\overline{D}) + u^+(\overline{D}) \quad \textit{and} \quad u^-(D) = u^0(\overline{D}) + u^-(\overline{D}).$$

Proof. Suppose D is unramified. Then we can take $\pi_D = \pi$, where π is a parameter of k. Since $\sigma(\pi) = \pi$, we have $\varepsilon' = 1$ and $\mathrm{Int}(\pi_D) \circ \sigma = \sigma$. Hence, by Corollary 2.4, we have the required result.

Suppose D is ramified. Then choose σ and π_D as in Lemma 2.5. Then, by Corollary 2.4, we have the required result. $\square$

Let K/k be a quadratic extension. Let D be a central division algebra over k with an involution σ of the first kind. Then $\sigma \otimes \iota$ is an involution on $D \otimes_k K$ of the second kind with ι being the non-trivial automorphism of K/k.

Remark 2.7. Suppose $D \otimes K$ is division. Then there are three possibilities of ramification:
(1) K/k is unramified and $D \otimes K/K$ is unramified;
(2) K/k is unramified and $D \otimes K/K$ is ramified;
(3) K/k is ramified and $D \otimes K/K$ is unramified.
We show that "K/k is ramified and $D \otimes K/K$ is ramified" cannot happen.

In fact, if K/k is ramified, then $K = k(\sqrt{\pi})$ for some parameter $\pi \in k$ and $\overline{K} = \overline{k}$. If D/k is unramified, then $D \otimes K/K$ is unramified. Suppose D/k is ramified. Then D is Brauer equivalent to $D_0 \otimes (u, \pi)$ for some D_0 unramified on k and $u \in k$ a unit at the valuation of k [TW15, 8.77]. Thus $D \otimes K$ is Brauer equivalent to $D_0 \otimes K$ and hence $D \otimes K/K$ is unramified.

Consequently, (2) and (3) can be shortened to
(2) $D \otimes K/K$ *is ramified. (3)* K/k *is ramified.*

Remark 2.8. Suppose we are in case (2) of Remark 2.7. Suppose $K = k(\sqrt{\lambda})$ and D is Brauer equivalent to $D_0 \otimes (u, \pi)$ for some D_0 unramified on k and $u \in k$ a unit at the valuation of k. Then $\overline{K} = \overline{k}(\sqrt{\overline{\lambda}})$, $Z(\overline{D}) = \overline{k}(\sqrt{\overline{u}})$ and $Z(\overline{D \otimes K}) = \overline{k}(\sqrt{\overline{u}}, \sqrt{\overline{\lambda}})$. Here u and λ are in different square classes of k, otherwise $(u, \pi)_K$

is split and hence $D \otimes K$ is unramified over K. Since $\overline{D \otimes K} = \overline{D} \otimes \overline{K} = \overline{D} \otimes \overline{k}(\sqrt{\overline{u}}, \sqrt{\overline{\lambda}})$ and $\overline{D}$ has an involution of the first kind, $\overline{D \otimes K}$ has three possible types of involutions of the second kind with fixed fields $\overline{k_1} = \overline{k}(\sqrt{\overline{u}})$, $\overline{k_2} = \overline{k}(\sqrt{\overline{\lambda}})$ and $\overline{k_3} = \overline{k}(\sqrt{\overline{u\lambda}})$ respectively. The corresponding $u^0(\overline{D \otimes K})$ are written as $u^0(\overline{D \otimes K}/\overline{k}_1)$, $u^0(\overline{D \otimes K}/\overline{k}_2)$ and $u^0(\overline{D \otimes K}/\overline{k}_3)$.

Corollary 2.9. *Let K/k be a quadratic extension and let ι be the non-trivial automorphism of K/k. Let D be a central division algebra over k with an involution σ of the first kind such that $D \otimes_k K$ is division.*

(1) If $D \otimes K$ is unramified at the discrete valuation of K and K/k is unramified, then $u^0(D \otimes K) = 2u^0(\overline{D} \otimes \overline{K})$.

(2) If $D \otimes K/K$ is ramified, then $u^0(D \otimes K) = u^0(\overline{D} \otimes \overline{K}/\overline{k}_1) + u^0(\overline{D} \otimes \overline{K}/\overline{k}_3)$.

(3) If K/k is ramified, then $u^0(D \otimes K) = u^+(\overline{D}) + u^-(\overline{D})$.

Proof. (1) Suppose D is unramified and K/k is unramified. Then $\overline{D \otimes K} = \overline{D} \otimes \overline{K}$ and $\overline{K}/\overline{k}$ is a quadratic extension. Let π be a parameter of k. Take $\pi_D = \pi$. Then $\sigma(\pi_D) = \pi_D$ and $\overline{\mathrm{Int}(\pi_D) \circ (\sigma \otimes \iota)} = \overline{\sigma \otimes \iota}$. By Corollary 2.4, $u^0(D \otimes K) = 2u^0(\overline{D} \otimes \overline{K})$.

(2) Suppose D is ramified. Suppose σ and π_D are as in Lemma 2.5. Then the fixed field of $\overline{\sigma \otimes \iota}$ is $\overline{k}_3$ and the fixed field of $\overline{\mathrm{Int}(\pi_D) \circ (\sigma \otimes \iota)}$ is $\overline{k}_1$ (where $\overline{k}_1$ and $\overline{k}_3$ are as in Remark 2.8). Thus, by Corollary 2.4, we have $u^0(D \otimes K) = u^0(\overline{D} \otimes \overline{K}/\overline{k}_1) + u^0(\overline{D} \otimes \overline{K}/\overline{k}_3)$.

(3) Suppose K/k is ramified. Let σ_0 be an involution of the first kind on D and $\sigma \simeq \sigma_0 \otimes \gamma$, where γ is the canonical involution of (u, π). We have $\overline{D \otimes K} = \overline{D}$ and $\overline{\sigma_0} \otimes \overline{\iota} = \overline{\sigma}$. Let $\pi_D = \sqrt{\pi} \in K \subset D \otimes K$. Then $\overline{\mathrm{Int}(\pi_D) \circ (\sigma_0 \otimes \iota)} = \overline{\sigma}$. Thus, by Corollary 2.4, $u(D \otimes K, \sigma, \varepsilon) = u(\overline{D}, \overline{\sigma_0}, \varepsilon) + u(\overline{D}, \overline{\sigma_0}, -\varepsilon)$. Hence $u^0(D \otimes K) = u^+(\overline{D}) + u^-(\overline{D})$. □

We end this section with the following well-known lemma.

Lemma 2.10. *Let k be a discrete valued field with residue field $\overline{k}$ and completion $\widehat{k}$. Suppose $\mathrm{char}(\overline{k}) \neq 2$. Let D be a division algebra over k with center K. Let σ be an involution on D such that $K^\sigma = k$. If $D \otimes \widehat{k}$ is division, then $u(D, \sigma, \varepsilon) \geq u(D \otimes \widehat{k}, \widehat{\sigma}, \varepsilon)$, where $\widehat{\sigma} = \sigma \otimes \mathrm{Id}_{\widehat{k}}$.*

Proof. Let v be the discrete valuation on k and $\pi \in k$ be a parameter. Since $D \otimes \widehat{k}$ is division, v extends to a valuation w on D. Let $\varepsilon = \pm 1$ and $\mathrm{Sym}^\varepsilon(D, \sigma) = \{x \in D \mid \sigma(x) = \varepsilon x\}$. Let $e_1, \cdots, e_r$ be a k-basis of $\mathrm{Sym}^\varepsilon(D, \sigma)$. Let $a \in \mathrm{Sym}^\varepsilon(D, \sigma) \otimes \widehat{k}$ and write $a = a_1 e_1 + \cdots + a_r e_r$ with $a_i \in \widehat{k}$. Let $b_i \in k$ be such that $a_i \equiv b_i$ modulo $\pi^{ew(a)+1}$ and $b = b_1 e_1 + \cdots + b_r e_r \in \mathrm{Sym}^\varepsilon(D, \sigma)$, where e is the ramification index $[w(D^*) : v(k^*)]$. Then $w(a) = w(b)$ and $\overline{ab^{-1}} = 1 \in \overline{D \otimes \widehat{k}}$.

Let $s = ab^{-1} \in D \otimes \widehat{k}$. Then $w(s) = 0$ and $\overline{s} = 1$

$$\begin{array}{llllll} a = sb & \Longrightarrow & \widehat{\sigma}(a) = \sigma(b)\widehat{\sigma}(s) & \Longrightarrow & \varepsilon a = \varepsilon b \widehat{\sigma}(s) \\ & \Longrightarrow & a = b\widehat{\sigma}(s) & \Longrightarrow & sb = b\widehat{\sigma}(s) \\ & \Longrightarrow & s = (\mathrm{Int}(b) \circ \widehat{\sigma})(s) & \Longrightarrow & (\mathrm{Int}(b) \circ \widehat{\sigma})|_{k(s)} = \mathrm{Id}_{k(s)}. \end{array}$$

Since $k(s)$ is complete, by Hensel's lemma, there exists $c \in k(s)$ such that $c^2 = s$ and $\overline{c}^2 = \overline{s} = 1 \in \overline{k(s)}$. Then

$$\begin{aligned}(\mathrm{Int}(b) \circ \widehat{\sigma})(c) = c \quad &\Longrightarrow \quad b\widehat{\sigma}(c) = cb \\ &\Longrightarrow \quad a = sb = ccb = cb\widehat{\sigma}(c) \\ &\Longrightarrow \quad \langle a \rangle \simeq \langle b \rangle \otimes \widehat{k}.\end{aligned}$$

Let h be an ε-hermitian form over $(D \otimes \widehat{k}, \sigma)$. Since $D \otimes \widehat{k}$ is division, $h = \langle \alpha_1, \cdots, \alpha_n \rangle$ for some $\alpha_i \in \mathrm{Sym}^{\varepsilon}(D, \sigma) \otimes \widehat{k}$. For each α_i, let $\beta_i \in \mathrm{Sym}^{\varepsilon}(D, \sigma)$ be such that $\langle \alpha_i \rangle \simeq \langle \beta_i \rangle \otimes \widehat{k}$ and $h_0 = \langle \beta_1, \cdots, \beta_n \rangle$. Then h_0 is an ε-hermitian form over (D, σ) and $h_0 \otimes \widehat{k} \simeq h$. If h is anisotropic over $\widehat{k}$, then h_0 is anisotropic. In particular, $u(D, \sigma, \varepsilon) \geq u(D \otimes \widehat{k}, \sigma \otimes \mathrm{Id}, \varepsilon)$. □

3. Division algebras over $\mathscr{A}_i(2)$-fields

A field k is called an *$\mathscr{A}_i(m)$-field* [Lee13, 2.1] if every system of r homogeneous forms of degree m in more than rm^i variables over k has a non-trivial simultaneous zero over a field extension L/k such that $\gcd(m, [L : k]) = 1$.

Let A be a central simple algebra over a field k. We say that A satisfies the *Springer's property* if for any involution σ on A of the first kind, $\varepsilon \in \{1, -1\}$ and for any odd degree extension L/k, if h is an anisotropic ε-hermitian space over (A, σ), then $h \otimes L$ is anisotropic. In [PSS01], Parimala, Sridharan and Suresh have shown that Springer's property holds for hermitian or skew-hermitian spaces over quaternion algebras with involution of the first kind. In [Wu15], the author has shown that Springer's property holds for hermitian or skew-hermitian spaces over central simple algebras with involution of the first kind over function fields of p-adic curves.

Now we prove Theorem 1.1(i).

Proof. Let σ be an orthogonal involution on D. Let $\mathrm{Sym}^{\varepsilon}(D, \sigma) = \{x \in D \mid \sigma(x) = \varepsilon x\}$ and $r = \dim_k(\mathrm{Sym}^{\varepsilon}(D, \sigma))$. Then $r = d(d+\varepsilon)/2$ [KMRT98, 2.6]. Let $e_1, \cdots, e_r$ be a k-basis of $\mathrm{Sym}^{\varepsilon}(D, \sigma)$. Let h be an ε-hermitian form over (D, σ) of rank $n > (1 + \frac{\varepsilon}{d})2^{i-1}$. Then for $x \in D^n$, we have

$$h(x, x) = q_1(x, x)e_1 + \cdots + q_r(x, x)e_r,$$

with each q_i a quadratic form over k in d^2n variables [Mah05, proof of prop. 3.6].

Since k is an $\mathscr{A}_i(2)$-field and $d^2n > d(d+\varepsilon)2^{i-1} = r2^i$, there exists an odd degree extension L/k such that $\{q_1, \cdots, q_r\}$ have a simultaneous non-trivial zero over L. Then h_L is isotropic over D_L. By Springer's property, h is isotropic over D. Hence $u(D, \sigma, \varepsilon) \leq (1 + \frac{\varepsilon}{d})2^{i-1}$.

Similarly, if σ is a symplectic involution on D, then $r = d(d - \varepsilon)/2$ and hence $u(D, \sigma, \varepsilon) \leq (1 - \frac{\varepsilon}{d})2^{i-1}$. □

Next, we prove Theorem 1.1(ii).

Proof. Let σ be an involution on D of the second kind. Let $\mathrm{Sym}(D) = \{x \in D \mid \sigma(x) = x\}$. Then $\mathrm{Sym}(D)$ is vector space over k and $\dim_k \mathrm{Sym}(D) = d^2$, where $d^2 = \dim_K(D)$. Let $e_1, \cdots, e_{d^2}$ be a k-basis of $\mathrm{Sym}(D)$. Let h be a hermitian form over (D,σ) of rank $n > 2^{i-1}$. Then, for $x \in D^n$, $h(x,x) \in \mathrm{Sym}(D)$ and we have

$$h(x,x) = q_1(x,x)e_1 + \cdots + q_{d^2}(x,x)e_{d^2},$$

with each q_i a quadratic form over k in $2d^2n$ variables.

Since k is an $\mathscr{A}_i(2)$-field and $2d^2n > 2d^2 2^{i-1} = d^2 2^i$, there exists an odd degree extension L/k such that $\{q_1, \cdots, q_{d^2}\}$ have a simultaneous non-trivial zero over L. In particular, h_L is isotropic over D_L. By Springer's property, h is isotropic over D. Hence $u^0(D) \leq 2^{i-1}$. □

Corollary 3.1. *If D is a quaternion division algebra over an $\mathscr{A}_i(2)$-field k and σ is of the first kind, then $u^+(D) \leq 3 \cdot 2^{i-2}$ and $u^-(D) \leq 2^{i-2}$.*

Proof. Since, by [PSS01, 3.5], (D,σ,ε) satisfies Springer's property, the corollary follows from Theorem 1.1(i). □

Corollary 3.2. *If D is a quaternion division algebra over a global function field k, then $u^+(D) = 3$, $u^-(D) = 1$, and $u^0(D) = 2$.*

Proof. By the Chevalley-Warning theorem [Che35, War35], every finite field is a C_1-field. By the Tsen-Lang theorem [Lan52], every global function field is a C_2-field. Since every C_2-field is an $\mathscr{A}_2(2)$-field [Lee13, between 2.1 and 2.2], by Theorem 1.1(i), $u^+(D) \leq 3$ and $u^-(D) \leq 1$ and by Theorem 1.1(ii), $u^0(D) \leq 2$. The equality follows from Lemmas 2.10 and 2.2. □

Corollary 3.3. *Let F be the function field of an integral variety X over a p-adic field with $p \neq 2$. Let D be a quaternion algebra over F. If $\dim(X) = n$, then $u^+(D) \leq 3 \cdot 2^n$ and $u^-(D) \leq 2^n$.*

Proof. Since D is a quaternion algebra, by [PSS01, 3.5], D satisfies the Springer's property. Since $\dim(X) = n$, by [HB10] and [Lee13], F is an $\mathscr{A}_{n+2}(2)$-field. Hence the corollary follows from Corollary 3.1. □

Corollary 3.4. *Let F be the function field of a p-adic curve. Let D be a division algebra over F with an involution of the first kind.*
(1) If D is a quaternion division algebra, then $u^+(D) \leq 6$ and $u^-(D) \leq 2$.
(2) If D is a biquaternion division algebra, then $u^+(D) \leq 5$ and $u^-(D) \leq 3$.

Proof. By [Sal97, 3.4; Sal98], $\deg(D) = d = 2$ or 4. If $d = 2$, then D is a quaternion algebra and the result follows from Corollary 3.3. Suppose $d = 4$. By [Wu15, 1.5], D satisfies Springer's property. Since F is an $\mathscr{A}_3(2)$-field, the result follows from Theorem 1.1(i). □

Corollary 3.5. *Let F be the function field of a p-adic curve. Let L/F be a quadratic extension. Let D be a division algebra over F with an involution of the first kind. Then $u^0(D \otimes_F L) \leq 4$.*

Proof. By [Wu15, 1.5], D satisfies Springer's property. Since F is an $\mathscr{A}_3(2)$-field, the result follows from Theorem 1.1(ii). □

4. Division algebras over semi-global fields

Let p be an odd prime number. Let F be the function field of a curve over a p-adic field. Let D be a division algebra over F with an involution σ. In this section, we show that the bounds in Corollary 3.4 for the u-invariants of the hermitian of forms over central simple algebras over F are in fact exact values. We also compute

$u^0(D)$ if D is a quaternion division algebra with an involution of the second kind over F.

Lemma 4.1. *Let k be a complete discrete valued field with residue field $\overline{k}$. Suppose $\overline{k}$ is a non-archimedean local field or a global function field with $\mathrm{char}(\overline{k}) \neq 2$. Let D be a division algebra over k with an involution of the first kind and K/k a quadratic extension.*
(1) If D is a quaternion division algebra, then $u^+(D) = 6$ and $u^-(D) = 2$.
(2) If D is a biquaternion algebra, then $u^+(D) = 5$ and $u^-(D) = 3$.
(3) If $D \otimes_k K$ is a division algebra, then $u^0(D \otimes_k K) = 4$.

Proof. (1) Suppose D is an unramified quaternion algebra. Then $\overline{D}$ is a quaternion algebra. Since $\overline{k}$ is either a local field or a global function field, by Lemma 2.2 and Corollary 3.2, we have $u^+(\overline{D}) = 3$, $u^-(\overline{D}) = 1$ and $u^-(\overline{D}) = 2$. Thus, by Corollary 2.6(1), $u^+(D) = 2 * 3 = 6$ and $u^-(D) = 2 * 1 = 2$.

Suppose D is a ramified quaternion algebra. Then $\overline{D}$ is a quadratic extension of $\overline{k}$ and by Lemma 2.2 and Corollary 2.6(2), $u^+(D) = 2+4 = 6$ and $u^-(D) = 2+0 = 2$.

(2) Suppose D is a biquaternion algebra. Since k is a complete discrete valued field with $\overline{k}$ a global field or local field, D is ramified by a theorem of Albert [Lam05, III, 4.8] and a theorem of Springer [Lam05, VI, 1.9]. Thus $\overline{D}$ is a quaternion algebra and hence by Lemma 2.2 and Corollary 2.6(2), $u^+(D) = 2 + 3 = 5$ and $u^-(D) = 2 + 1 = 3$.

(3) Suppose $D \otimes K$ is a division algebra. Then, by Corollary 2.9, we have either $u^0(D \otimes K) = 2u^0(\overline{D \otimes K})$ or $u^0(D \otimes K) = u^0(\overline{D} \otimes \overline{K}/\overline{k_1}) + u^0(\overline{D} \otimes \overline{K}/\overline{k_3})$ or $u^0(D \otimes K) = u^+(\overline{D_0}) + u^-(\overline{D_0})$ for some central division algebra D_0 unramified over k with $\deg(D) = \deg(D_0)$.

In the case of Corollary 2.9(1), $u^0(D \otimes K) = 2u^0(\overline{D \otimes K}) = 2 * 2 = 4$.

In the case of Corollary 2.9(2), $u^0(D \otimes K) = u^0(\overline{D} \otimes \overline{K}/\overline{k_1}) + u^0(\overline{D} \otimes \overline{K}/\overline{k_3})$. Since $\overline{k}$ is a p-adic field or a global field, so are $\overline{k_1}$ and $\overline{k_3}$. We have $u(\overline{k_1}) = u(\overline{k_3}) = 4$. Since $\overline{D} \otimes \overline{K}$ is a quadratic extension of $\overline{k_1}$, we have $u^0(\overline{D} \otimes \overline{K}/\overline{k_1}) = \frac{1}{2}u(\overline{k_1}) = 2$. Similarly, $u^0(\overline{D} \otimes \overline{K}/\overline{k_3}) = \frac{1}{2}u(\overline{k_3}) = 2$. Thus, we also have $u^0(D \otimes K) = 2 + 2 = 4$.

In the case of Corollary 2.9(3), $u^0(D \otimes K) = u^+(\overline{D_0}) + u^-(\overline{D_0}) = 3 + 1 = 4$. □

Now we prove our main result Theorem 1.2.

Proof. Since D is a division algebra, by [RS13, 2.6], there exists a divisorial discrete valuation v of F such that $D \otimes F_v$ is division. Since v is a divisorial discrete valuation, the residue field at v is either a p-adic field or a global function field.

(1) and (3) follow from Corollary 3.4, Lemma 4.1(1)(2) and Lemma 2.10.

(2) By [RS13, 2.6], there exists a divisorial discrete valuation v of F such that $D \otimes L \otimes F_v$ is division. Thus, the result follows from Corollary 3.5, Lemma 4.1(3) and Lemma 2.10. □

5. Tensor product of quaternions over arbitrary fields

In this section, we revisit and prove Theorem 1.3. We begin with the following.

Lemma 5.1. *For $n \geq 1$, let $a_n = \frac{4}{5} + \frac{1}{5}(\frac{9}{4})^n$, $b_n = -\frac{1}{5} + \frac{1}{5}(\frac{9}{4})^n$ and $c_n = \frac{1}{5} + \frac{3}{10}(\frac{9}{4})^n$.*

Then

$$a_{n+1} = \frac{3}{4}a_n + c_n,\ b_{n+1} = \frac{3}{2}b_n + \frac{1}{2}c_n,\ c_n = \frac{1}{2}a_n + b_n,\ \frac{3}{2}a_n \geq c_n \geq \frac{3}{2}b_n,$$

for all $n \geq 1$.

Proof. It follows from definitions of a_n, b_n and c_n above. □

Now we prove Theorem 1.3.

Proof. By Lemma 2.1, we may assume that $A = H_1 \otimes \cdots \otimes H_n$. Let $\sigma = \tau_1 \otimes \cdots \otimes \tau_n$, where τ_i is the canonical involutions of H_i for $1 \leq i \leq n$. For $n \geq 1$, let $a_n = \frac{4}{5} + \frac{1}{5}(\frac{9}{4})^n$, $b_n = -\frac{1}{5} + \frac{1}{5}(\frac{9}{4})^n$ and $c_n = \frac{1}{5} + \frac{3}{10}(\frac{9}{4})^n$.

We proceed by induction. For $n = 1$, by [Mah05, 3.4] and [Lee84, 2.10] we have $u^+(H_1) \leq a_1 u(k)$, by [Sch85, ch. 10, 1.7], we have $u^-(H_1) \leq b_1 u(k)$ and by [PS13, 4.4], we have $u^0(H_1) \leq c_1 u(k)$.

Suppose $u^+(H_1 \otimes_k \cdots \otimes_k H_n) \leq a_n u(k)$, $u^-(H_1 \otimes_k \cdots \otimes_k H_n) \leq b_n u(k)$ and $u^0(H_1 \otimes_k \cdots \otimes_k H_n) \leq c_n u(k)$.

Let $H_1, \cdots, H_{n+1}$ be quaternion algebas over k, τ_i the canonical involution of H_i and $\sigma = \tau_1 \otimes \cdots \otimes \tau_{n+1}$ on $A = H_1 \otimes \cdots \otimes H_{n+1}$. Since H_{n+1} is a quaternion algebra and τ_{n+1} is the canonical involution, there exist $\lambda_{n+1}, \mu_{n+1} \in H^*_{n+1}$ such that $\tau_{n+1}(\lambda_{n+1}) = -\lambda_{n+1}$, $\tau_{n+1}(\mu_{n+1}) = -\mu_{n+1}$, $\lambda_{n+1}\mu_{n+1} = -\mu_{n+1}\lambda_{n+1}$ and $k(\lambda_{n+1})/k$ is a quadratic extension. Let $\lambda = 1 \otimes \cdots \otimes 1 \otimes \lambda_{n+1} \in A$, $\mu = 1 \otimes \cdots \otimes 1 \otimes \mu_{n+1} \in A$ and $\tilde{A}$ be the centralizer of $k(\lambda)$ in A. Then $\tilde{A} = H_1 \otimes \cdots \otimes H_n \otimes k(\lambda)$. Let $\sigma_1 = \sigma|_{\tilde{A}}$ and $\sigma_2 = \mathrm{Int}(\mu^{-1}) \circ \sigma_1$. By [Mah05, 3.1, 3.2], we have σ_1 is unitary, σ_2 and σ are of the same type and

$$\begin{aligned} u(A, \sigma, \varepsilon) \quad \leq \quad & \min\{u(\tilde{A}, \sigma_1, \varepsilon) + \tfrac{1}{2}u(\tilde{A} \otimes k(\lambda), \sigma_2, -\varepsilon), \\ & \tfrac{1}{2}u(\tilde{A} \otimes k(\lambda), \sigma_1, \varepsilon) + u(\tilde{A} \otimes k(\lambda), \sigma_2, -\varepsilon)\}. \end{aligned}$$

Since σ_1 is unitary and $\tilde{A} = H_1 \otimes_k \cdots \otimes_k H_n \otimes k(\lambda)$, by the induction hypothesis, we have $u(\tilde{A}, \sigma_1, \varepsilon) \leq c_n u(k)$. By [PS13, 4.2],

$$\begin{aligned} u(\tilde{A}, \sigma_2, -\varepsilon) &= u(H_1 \otimes_k \cdots \otimes_k H_n \otimes k(\lambda), \sigma_2, -\varepsilon) \\ &\leq \frac{3}{2}u(H_1 \otimes_k \cdots \otimes_k H_n, \tau_1 \otimes \cdots \otimes \tau_n, -\varepsilon). \end{aligned}$$

Since both σ and $\tau_1 \otimes \cdots \otimes \tau_n$ are of the first kind and of different types, we have

$$u^+(H_1 \otimes_k \cdots \otimes_k H_{n+1}) \leq \min\{\frac{1}{2}(\frac{3}{2}a_n) + c_n, \frac{3}{2}a_n + \frac{1}{2}c_n\}u(k) = \frac{3}{4}a_n + c_n = a_{n+1}u(k),$$

$$u^-(H_1 \otimes_k \cdots \otimes_k H_{n+1}) \leq \min\{\frac{1}{2}(\frac{3}{2}b_n) + c_n, \frac{3}{2}b_n + \frac{1}{2}c_n\}u(k) = \frac{3}{2}b_n + \frac{1}{2}c_n = b_{n+1}u(k).$$

Finally by [PS13, 4.3], $u^0(H_1 \otimes_k \cdots \otimes_k H_{n+1} \otimes_k K) \leq \min\{\frac{1}{2}a_{n+1} + b_{n+1}, a_{n+1} + \frac{1}{2}b_{n+1}\}u(k) = \frac{1}{2}a_{n+1} + b_{n+1} = c_{n+1}u(k)$. Here Lemma 5.1 was used in all three calculations. □

Remark. When $n = 2$, $a_2 = \frac{29}{16}$ is the same as that of [PS13, 4.5], $b_2 = \frac{13}{16}$ is smaller than the bound $\frac{17}{16}$ of [PS13, 4.6, 4.7]. When k is a semi-global field, $u^-(D) \leq \lfloor \frac{13}{2} \rfloor = 6$ is smaller than the bound 8 of [PS13, 4.8].

When $n \geq 3$, a_n is smaller than the bound $\frac{3^{2n-6}}{4^n} \cdot 213$ of [PS13, 4.10, 4.11].

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