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# POSITIVSTELLENSÄTZE FOR NONCOMMUTATIVE RATIONAL EXPRESSIONS

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ABSTRACT. We derive some Positivstellensätze for noncommutative rational expressions from the Positivstellensätze for noncommutative polynomials. Specifically, we show that if a noncommutative rational expression is positive on a polynomially convex set, then there is an algebraic certificate witnessing that fact. As in the case of noncommutative polynomials, our results are nicer when we additionally assume positivity on a convex set, that is, we obtain a so-called "perfect Positivstellensatz" on convex sets.

## 1. Introduction

We consider the positivity of noncommutative rational functions on polynomially convex sets. The theory on positive noncommutative polynomials has been well studied [3, 4, 6], essentially inspired by the operator theoretic methods from the theory of positive (commutative) polynomials on polynomially convex sets originating in the work [9, 10]. We note that going from the polynomial to the rational case is less clear than in the noncommutative case because we cannot "clear denominators", as it were.

A **noncommutative polynomial** (over $\mathbb{C}$) in $d$-variables is an element of the free associative algebra over $\mathbb{C}$ in the noncommuting letters $x_1, \ldots, x_d$. For example $1000x_1x_2x_1 - x_2^2$ and $x_1^2 + x_1x_2$ are noncommutative polynomials in two variables. A **matricial noncommutative polynomial** is a matrix with noncommutative polynomial entries. For example,

$$\begin{bmatrix} 7i & 1000x_1x_2x_1 - x_2^2 \\ x_1^2 + x_1x_2 & 0 \end{bmatrix}$$

is a matricial noncommutative polynomial. We define an involution $*$ on matricial noncommutative polynomials to be the involution which treats each $x_i$ as a self-adjoint variable. For example,

$$\begin{bmatrix} 7i & 1000x_1x_2x_1 - x_2^2 \\ x_1^2 + x_1x_2 & 0 \end{bmatrix}^* = \begin{bmatrix} -7i & x_1^2 + x_2x_1 \\ 1000x_1x_2x_1 - x_2^2 & 0 \end{bmatrix}.$$

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We say a collection $\mathcal{P}$ of square matricial noncommutative polynomials is **Archimedian** if $\mathcal{P}$ contains elements of the form $C_i - x_i^2$ for some real numbers $C_i$ and each element of $\mathcal{P}$ is self-adjoint.

Let $\mathcal{H}$ be the infinite dimensional separable Hilbert space. For a bounded and self-adjoint operator $T$, we say $T \geq 0$ if $T$ is positive semidefinite, we say $T > 0$ if $T$ is strictly positive definite in the sense that $\langle Th, h\rangle > 0$ for all nonzero vectors $h \in \mathcal{H}$. We define

$$\mathcal{D}_{\mathcal{P}} = \{X \in B(\mathcal{H})^d | p(X) \geq 0, \forall p \in \mathcal{P}, X_i = X_i^*\}.$$

Previously, Helton and McCullough showed the following Positivstellensatz for matricial noncommutative polynomials.

**Theorem 1.1** (Helton, McCullough [6]). *Let $\mathcal{P}$ be an Archimedian collection of matricial noncommutative polynomials. Let $q$ be a square matricial noncommutative polynomial. If $q > 0$ on $\mathcal{D}_{\mathcal{P}}$, then*

$$q = \sum_{\textit{finite}} s_i^* s_i + \sum_{\textit{finite}} r_j^* p_j r_j$$

*where $s_i, r_j$ are all matricial noncommutative polynomials and $p_j \in \mathcal{P}$.*

## 2. The rational Positivstellensatz

A **noncommutative rational expression** is a syntactically correct expression involving $+, (,), ^{-1}$ the letters $x_1, \ldots, x_d$ and scalar numbers. We say two nondegenerate expressions are **equivalent** if they agree on the intersection of their domains. (Nondegeneracy means that the expression is defined for at least one input, or equivalently that the domain is a dense set with interior. That is, examples such as $0^{-1}$ are disallowed.) Examples of noncommutative rational expressions include

$$1, x_1 x_1^{-1}, 1 + x_2(8x_1^3 x_2 x_1 + 8)^{-1}.$$

We note that the first two are equivalent.

A **matricial noncommutative rational expression** is a matrix with noncommutative rational expression entries.

We show the following theorem.

**Theorem 2.1.** *Let $\mathcal{P}$ be an Archimedian collection of noncommutative polynomials. Let $q$ be a square matricial noncommutative rational expression defined on all of $\mathcal{D}_{\mathcal{P}}$. If the noncommutative rational expression $q > 0$ on $\mathcal{D}_{\mathcal{P}}$, then*

$$q \equiv \sum_{\textit{finite}} s_i^* s_i + \sum_{\textit{finite}} r_j^* p_j r_j \tag{2.1}$$

*where $s_i, r_j$ are all matricial noncommutative rational expressions defined on $\mathcal{D}_{\mathcal{P}}$ and $p_j \in \mathcal{P}$.*

*Proof.* We let $g_j(x)$ be such that the term $g_j(x)^{-1}$ occurs in $q$. The proof will go by strong induction on the number of such terms. Define

$$\mathcal{O} = \mathcal{P} \cup \{\pm[1 - u_j g_j(x)]^*[1 - u_j g_j(x)], \pm[1 - g_j(x)u_j]^*[1 - g_j(x)u_j]\} \cup \{D_j - u_j^* u_j\}$$

where $D_j$ are positive real scalars chosen to be large enough so that $D_j - [g_j(x)^{-1}]^* g_j(x)^{-1}$ is positive on $\mathcal{D}_{\mathcal{P}}$.

We now define a self-adjoint noncommutative polynomial $\hat{q}(x,u)$ so that $\hat{q}(x,g) = q(x)$. Now $\hat{q}$ is a noncommutative polynomial in terms of $x_i$ and $u_j$. Moreover, in terms of the $x_i$ and $u_j$, we see that $q(x,u)$ is positive on $\mathcal{D}_{\mathcal{O}}$, so by Theorem 1.1,

$$\hat{q} = \sum s_i^* s_i + \sum r_j^* o_j r_j$$

for some $o_j \in \mathcal{O}$. We now analyze each term of the form $t_j = r_j^* o_j r_j$. We need to show that $t_j(x,g)$ is of the form (2.1). If $o_j \in \mathcal{P}$, we are fine. If $o_j = \pm[1 - u_j g_j(x)]^*[1 - u_j g_j(x)]$, we are also fine, since $t_j(x,g) = 0$, and similarly for the reversed case. If $o_j = D_j - [u_j]^* u_j$ we note that

$$o_j(x,g) = D_j - [g_j(x)^{-1}]^* g_j(x)^{-1} = [g_j(x)^{-1}]^*[D_j g_j(x)^* g_j(x) - 1] g_j(x)^{-1},$$

and since $D_j g_j(x)^* g_j(x) - 1 > 0$ on $\mathcal{D}_{\mathcal{P}}$, by induction it is of the form (2.1), so we are done. □

We note that the same proof can be adapted for the hereditary case in [6]. Moreover, we note that this implies the Agler model theory for rational functions on polynomially convex sets established previously in [1, 2].

## 3. The convex perfect rational Positivstellensatz

It is important to note that in Theorem 1.1 and Theorem 2.1, the complexity of the sum of squares representation is unbounded and we needed strict inequality. Specifically, in (2.1), the number of terms in each sum and the degree of each $s_i$ and $r_j$ are not bounded in the statement of the theorem. However, Helton, Klep and McCullough [4] showed that bounds do exist when we additionally assume that $\mathcal{D}_{\mathcal{P}}$ is convex and contains 0 and moreover that $\mathcal{P}$ consists of a single monic linear pencil, $L$, a self-adjoint linear matrix polynomial such that $L(0)$ is the identity. We note that for any finite set $\mathcal{P}$ of noncommutative polynomials such that $\mathcal{D}_{\mathcal{P}}$ is convex and contains 0, there exists such an $L$ [7].

Our goal is to prove the following:

**Theorem 3.1.** *Let $L$ be a monic linear pencil. Suppose $\mathcal{D}_{\{L\}}$ is convex. Let $r$ be a square matricial noncommutative rational expression defined on all of $\mathcal{D}_{\{L\}}$. The noncommutative rational expression $r \geq 0$ on all of $\mathcal{D}_{\{L\}}$ if and only if*

$$r \equiv \sum_{\textit{finite}} s_i^* s_i + \sum_{\textit{finite}} r_j^* L r_j \tag{3.1}$$

*where $s_i, r_j$ are all matricial noncommutative rational expressions defined on all of $\mathcal{D}_{\{L\}}$.*

*Proof.* Given an expression $r(x)$, we consider the expression $\tilde{r}(x,u)$ where each $g_j(x)^{-1}$ occurring in $r$ has been replaced by $u_j$ as in the proof of Theorem 2.1.

First we consider the minimal set $\mathcal{C}_r$ of rational expressions such that:

(1) $ab \in \mathcal{C}_r \Rightarrow b \in \mathcal{C}_r$,
(2) $(a+b)c \in \mathcal{C}_r \Rightarrow ac \in \mathcal{C}_r, bc \in \mathcal{C}_r$,
(3) $a+b \in \mathcal{C} \Rightarrow a \in \mathcal{C}_r, b \in \mathcal{C}_r$,
(4) $a^{-1}b \in \mathcal{C}_r \Rightarrow aa^{-1}b \in \mathcal{C}_r$.

From $\mathcal{C}_r$, form a set $\tilde{\mathcal{C}_r}$ by replacing each occurence of $g_j(x)^{-1}$ in elements of $\mathcal{C}_r$ with a new symbol $u_j$. We define the set of $\mathcal{M}_r$ to be

$$\mathcal{M}_r = \{g_j(x)u_j b - b | g_j(x)u_j b \in \tilde{\mathcal{C}_r}\}.$$

Define

$$\mathscr{Z}_r = \{(X, U, v) | m(X, U)v = 0, m \in \mathcal{M}_r, L(X) \geq 0\}.$$

We note that for $(X, U, v) \in \mathscr{Z}_r$ and $\tilde{a}(x, u) \in \tilde{\mathcal{C}}_r$, one can show we have that $\tilde{a}(X, U)v = \tilde{a}(x, g(X)^{-1})v$ via a recursive argument. We see that $\tilde{r}(x, u)$ satisfies

$$\langle r(X)v, v\rangle = \langle \tilde{r}(X, U)v, v\rangle \geq 0,$$

on $\mathscr{Z}_r$ since $\tilde{r}(X, U)v = r(X)v$ on $\mathscr{Z}_r$ by construction. Now, we apply the Helton-Klep-Nelson convex Positivstellensatz [5, Theorem 1.9], where the variety is given by $\mathscr{Z}_r$ and the convex set is $\{(X, U) | L(X) \geq 0\}$, to get that:

$$\tilde{r}(x, u) = \sum \tilde{s}_i^* \tilde{s}_i + \sum \tilde{r}_j^* L \tilde{r}_j + \sum \iota_k^* m_k + m_k^* \iota_k$$

where each $\iota_k$ is in the real radical of the ideal generated by the elements of $\mathcal{M}_r$. That is, each $\iota_k(X, U)v$ vanishes on $\mathscr{Z}_r$. So, substituting $g_j(x)^{-1}$ for $u_j$ we get that

$$r(x) \equiv \sum s_i^* s_i + \sum r_j^* L r_j.$$

□

We note that we could have proved a bit more: that on the variety $\mathscr{Z}_r$ that $\tilde{r}$ is positive and given by a sum of squares. This would essentially correspond to the so-called Moore-Penrose evaluation in [8]. Moreover, we note that the main result on positive rational functions, the noncommutative analogue of Artin's solution to Hilbert's 17th problem, that regular positive rational expressions are sums of squares [8], follows from our present theorem by taking an empty monic linear pencil; in fact, we obtain a slightly better matricial version of that result. Specifically, by taking the monic linear pencil $L(X) = I$. Now we note that $D_{\{L\}}$ consists of all tuples of self-adjoint operators in $\mathcal{B}(\mathcal{H})^d$. Applying Theorem 3.1 gives that any square matricial rational expression $r$ such that $r \geq 0$ on $D_{\{L\}}$ is a sum of squares of the form

$$r \equiv \sum_{\text{finite}} s_i^* s_i + \sum_{\text{finite}} r_j^* L r_j \equiv \sum_{\text{finite}} s_i^* s_i + \sum_{\text{finite}} r_j^* r_j.$$

We note that, without loss of generality, we can assume the $r_j$ sum is empty. That is, any square matricial rational expression $r$ which is defined for all self-adjoint inputs and takes positive semidefinite values everywhere must satisfy

$$r \equiv \sum_{\text{finite}} s_i^* s_i,$$

and is thus a sum of squares of rational functions, as in Artin's solution to Hilbert's 17th problem. Moreover, one has size bounds inherited from the Helton-Klep-Nelson convex Positivstellensatz [5], that is, checking that a noncommutative rational expression is effective using the algorithms given in [5].

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