1. **Introduction**

The (first part of the) following problem was suggested by von Neumann (see pp. 60-61, Appendix 3 in [8]).

**Problem 1.** Let $a, b, c \in B(H)$. If $a$ is normal and if $ac = ca$, does it follow that $a^*c = ca^*$? More generally, if $a$ and $b$ are normal and if $ac = cb$, does it follow that $a^*c = cb^*$?

If the operators $a$ and $c$ belong to a finite factor $\mathcal{M}$, then the first part of the problem was resolved (in the affirmative) by von Neumann himself. In full generality, a problem was resolved by Fuglede [4].

Furthermore, von Neumann mentioned that a “formal” analogue of Problem 1 for unbounded operators can be *non-rigorously* answered in the negative due to the fact that a product of 2 unbounded operators does not always exists. A partial affirmative answer was given by Putnam (see Theorem 1.6.2 in [9]). He proved that if $cb \subset ac$, then $cb^* \subset ac^*$ provided that $c$ is *bounded*.

In what follows, we propose a rigorous analogue of Problem 1 for unbounded operators affiliated with a von Neumann algebra $\mathcal{M}$. We start with a proper framework.

The set of all operators affiliated to a von Neumann algebra $\mathcal{M}$ does not necessarily form an algebra. At the same time, the class of unital $\ast$-algebras which consist of operators affiliated with $\mathcal{M}$ is vast. In particular, it contains all algebras of measurable operators [12] and those of $\tau$-measurable operators [7].

According to [15], in this class, there is a unique maximal element called $LS(\mathcal{M})$. We call $LS(\mathcal{M})$ the algebra of all locally measurable operators affiliated with $\mathcal{M}$. An equivalent constructive definition of $LS(\mathcal{M})$ is given in Section 2.

We now properly restate Problem 1 for unbounded operators affiliated with $\mathcal{M}$.  

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**FUGLEDE-PUTNAM THEOREM FOR LOCALLY MEASURABLE OPERATORS**

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**Abstract.** We extend the Fuglede-Putnam theorem from the algebra $B(H)$ of all bounded operators on the Hilbert space $H$ to the algebra of all locally measurable operators affiliated with a von Neumann algebra.

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1 The operations in these algebras are strong sum, strong product, the scalar multiplication and the usual adjoint of operators. For precise definitions, see Section 2.

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Problem 2. Let $\mathcal{M}$ be a von Neumann algebra and let $a, b, c \in LS(\mathcal{M})$. If $a$ and $b$ are normal and if $ac = cb$, does it follow that $a^*c = cb^*$?

Theorem 5 in [2] delivers the positive answer to Problem 2 for the case when $a$, $b$ and $c$ are measurable operators affiliated with a von Neumann algebra $\mathcal{M}$ of type I (see also [1]). In the case of an arbitrary finite von Neumann algebra $\mathcal{M}$, Problem 2 is resolved in the affirmative in [5] (see Corollary 3.6 there).

We answer Problem 2 in the affirmative in full generality. Our methods are stronger than those of [1], [2], [4], [5], [9] and are of independent interest.

The following theorem is the main result of the paper.

**Theorem 3.** Let $\mathcal{M}$ be an arbitrary von Neumann algebra and let $a, b, c$ be locally measurable operators affiliated with $\mathcal{M}$. If $a$ and $b$ are normal and if $ac = cb$, then $a^*c = cb^*$.

The corollary below extends the classical spectral theorem for normal operators affiliated with a von Neumann algebra (see also [1]). In the case of an arbitrary finite von Neumann algebra $\mathcal{M}$, Problem 2 is resolved in the affirmative in full generality. Our methods are stronger than those of [1], [2], [4], [5], [9] and are of independent interest.

**Theorem 4.** Let $\mathcal{M}$ be an arbitrary von Neumann algebra and let $a, b$ be locally measurable operators affiliated with $\mathcal{M}$. If $a$ is normal and $ab = ba$, then $eb = be$ for every spectral projection $e$ of the operator $a$. If $a$ and $b$ are normal, then the following conditions are equivalent:

(a) $ab = ba$;
(b) $ef = fe$ for every spectral projection $e$ of the operator $a$ and for every spectral projection $f$ of the operator $b$;
(c) $\phi(a)\psi(b) = \psi(b)\phi(a)$ for every Borel complex function $\phi$ and $\psi$ on $\mathbb{C}$, which are bounded on compact subsets.

2. Preliminaries

Let $H$ be a Hilbert space, let $B(H)$ be the $*$-algebra of all bounded linear operators on $H$, and let $I$ be the identity operator on $H$. Given a von Neumann algebra $\mathcal{M}$ acting on $H$, denote by $Z(\mathcal{M})$ the centre of $\mathcal{M}$ and by $\mathcal{P}(\mathcal{M}) = \{p \in \mathcal{M} : p = p^2 = p^*\}$ the lattice of all projections in $\mathcal{M}$. Let $\mathcal{P}_{fin}(\mathcal{M})$ be the set of all finite projections in $\mathcal{M}$.

A linear operator $a : \mathcal{D}(a) \to H$, where the domain $\mathcal{D}(a)$ of $a$ is a linear subspace of $H$, is said to be affiliated with $\mathcal{M}$ if $ba \subseteq ab$ for all $b$ from the commutant $\mathcal{M}'$ of algebra $\mathcal{M}$.

A densely-defined closed linear operator $a$ (possibly unbounded) affiliated with $\mathcal{M}$ is said to be measurable with respect to $\mathcal{M}$ if there exists a sequence $\{p_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathcal{M})$ such that $p_n \uparrow 1$, $p_n(H) \subset \mathcal{D}(a)$ and $p_n^* = 1 - p_n \in \mathcal{P}_{fin}(\mathcal{M})$ for every $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers. Let us denote by $S(\mathcal{M})$ the set of all measurable operators.

Let $a, b \in S(\mathcal{M})$. It is well known that $a + b$, $ab$ and $a^*$ are densely-defined and preclosed operators. Moreover, the closures $a + b$ (strong sum), $\overline{ab}$ (strong product) and $a^*$ are also measurable, and equipped with these operations $S(\mathcal{M})$ is a unital $*$-algebra over the field $\mathbb{C}$ of complex numbers [12]. It is clear that $\mathcal{M}$ is a $*$-subalgebra of $S(\mathcal{M})$.

A densely-defined linear operator $a$ affiliated with $\mathcal{M}$ is called locally measurable with respect to $\mathcal{M}$ if there is a sequence $\{z_n\}_{n=1}^{\infty}$ of central projections in $\mathcal{M}$ such that $z_n \uparrow 1$, $z_n(H) \subset \mathcal{D}(a)$ and $az_n \in S(\mathcal{M})$ for all $n \in \mathbb{N}$. 
The set $LS(\mathcal{M})$ of all locally measurable operators is a unital $*$-algebra over the field $\mathbb{C}$ with respect to the same algebraic operations as in $S(\mathcal{M})$ [1], and $S(\mathcal{M})$ is a $*$-subalgebra of $LS(\mathcal{M})$. It is clear that if $\mathcal{M}$ is finite, the algebras $S(\mathcal{M})$ and $LS(\mathcal{M})$ coincide. If von Neumann algebra $\mathcal{M}$ is of type III and $\dim(\mathcal{Z}(\mathcal{M})) = \infty$, then $S(\mathcal{M}) = \mathcal{M}$, but $LS(\mathcal{M}) \neq \mathcal{M}$.

For every subset $E \subseteq LS(\mathcal{M})$, the sets of all self-adjoint (resp., positive) operators in $E$ will be denoted by $E_h$ (resp. $E_+$.). The partial order in $LS(\mathcal{M})$ is defined by its cone $LS_+(\mathcal{M})$ and is denoted by $\leq$.

Let $a$ be a closed operator with dense domain $\mathcal{D}(a)$ in $H$ and let $a = u|a|$ be the polar decomposition of the operator $a$, where $|a| = (a^*a)^{\frac{1}{2}}$ and $u$ is a partial isometry in $B(H)$ such that $u^*u$ (respectively, $uu^*$) is the right (left) support $r(a)$ (respectively, $l(a)$) of $a$. It is known that $a = |a|^*u$ and $a \in LS(\mathcal{M})$ (respectively, $a \in S(\mathcal{M})$) if and only if $|a| \in LS(\mathcal{M})$ (respectively, $|a| \in S(\mathcal{M})$) and $u \in \mathcal{M}$ [8 §2.2, 2.3]. If $a$ is a self-adjoint operator affiliated with $\mathcal{M}$, then the spectral family of projections $e_{\lambda}(a) = e_{[-\infty,\lambda]}(a)$, $\lambda \in \mathbb{R}$, for $a$ belongs to $\mathcal{M}$ [6 §2.1]. A locally measurable operator $a$ is measurable if and only if $e^+_{\lambda}(|a|) \in \mathcal{P}_{fin}(\mathcal{M})$ for some $\lambda > 0$ [8 §2.2].

In what follows, we use the notation $n(a) = 1 - r(a)$ for the projection onto the kernel of the operator $a$.

Assume now that $\mathcal{M}$ is a semifinite von Neumann algebra equipped with a faithful normal semifinite trace $\tau$. A densely-defined closed linear operator $a$ affiliated with $\mathcal{M}$ is called $\tau$-measurable if for each $\varepsilon > 0$ there exists $e \in \mathcal{P}(\mathcal{M})$ with $\tau(e^+) \leq \varepsilon$ such that $e(H) \subseteq \mathcal{D}(a)$. Let us denote by $S(\mathcal{M}, \tau)$ the set of all $\tau$-measurable operators. It is well known [7] that $S(\mathcal{M}, \tau)$ is a $*$-subalgebra of $S(\mathcal{M})$ and $\mathcal{M} \subseteq S(\mathcal{M}, \tau)$. It is clear that if $\mathcal{M}$ is a semifinite factor, the algebras $S(\mathcal{M}, \tau)$ and $S(\mathcal{M})$ coincide. Note also that for every $a \in S(\mathcal{M}, \tau)$ there exists $\lambda > 0$ such that $\tau(e^+_{\lambda}(|a|)) < \infty$ (see [7] and [6 §2.6]).

Measure topology is defined in $S(\mathcal{M}, \tau)$ by the family $V(\varepsilon, \delta), \varepsilon > 0, \delta > 0$, of neighborhoods of zero:

$$V(\varepsilon, \delta) = \{a \in S(\mathcal{M}, \tau) : \|ae\|_{\mathcal{M}} \leq \delta \text{ for some } e \in \mathcal{P}(\mathcal{M}) \text{ with } \tau(e^+) \leq \varepsilon\}.$$ 

Convergence of the sequence ${a_n} \subseteq S(\mathcal{M}, \tau)$ in measure topology is called convergence in measure. When equipped with measure topology, $S(\mathcal{M}, \tau)$ is a complete metrizable topological $*$-algebra (see [7]). For basic properties of the measure topology, see [7]. We remark only that $e_n \rightarrow 0$ in measure, $e_n \in \mathcal{P}(\mathcal{M})$, if and only if $\tau(e_n) \rightarrow 0$.

Let $\mathcal{M}$ be a von Neumann algebra equipped with a faithful normal semifinite trace $\tau$. We set

$$(L_1 \cap L_\infty)(\mathcal{M}, \tau) = \left\{x \in \mathcal{M} : \tau(|x|) < \infty\right\}.$$

The following property is standard.

**Property 5.** Let $\mathcal{M}$ be a semifinite von Neumann algebra and let $\tau$ be a faithful normal semifinite trace on $\mathcal{M}$. If $x, y \in (L_1 \cap L_\infty)(\mathcal{M}, \tau)$, then $\tau(xy) = \tau(yx)$.

3. **The Fuglede-Putnam theorem in $S(\mathcal{M}, \tau)$**

The proof of Theorem 3 in full generality is based on its special case for the $*$-algebra $S(\mathcal{M}, \tau)$.
Theorem 6. Let $\mathcal{M}$ be a semifinite von Neumann algebra and let $\tau$ be a faithful normal semifinite trace on $\mathcal{M}$. Let $a, b, c \in S(\mathcal{M}, \tau)$, and let $a$ and $b$ be normal. If $ac = cb$, then $a^*c = cb^*$.

Our strategy for proving Theorem 6 relies on a number of auxiliary lemmas. Whereas some of them look similar to those used in [5], the lemmas below appear to be stronger than their counterparts from [5].

Lemma 7. If $a \in \mathcal{M}$ is normal, $p \in \mathcal{P}(\mathcal{M})$ and $\tau(p) < \infty$ with $ap = pa$, then $ap = pa$.

Proof. Denote, for brevity,

$$a_1 = pap, \quad a_2 = pa(1 - p).$$

Due to the normality of $a$ and using the equality $ap = pa$, we have

$$a_1^*a_1 = (ap)^*(ap) = pa^*ap = paa^*p = (pa)(pa)^* = (a_1 + a_2)(a_1 + a_2)^* = a_1a_1^* + a_2a_2^*.$$ 

Since $\tau(p) < \infty$, it follows that $a_1, a_2 \in (L_1 \cap L_\infty)(\mathcal{M}, \tau)$. Taking the trace and using Property [5] we conclude that $\tau(a_2a_2^*) = 0$. Since $\tau$ is faithful, it follows that $a_2 = 0$. This completes the proof.

Lemma 8. Let $a, b \in \mathcal{M}$ be normal, $c \in S_+(\mathcal{M}, \tau)$ and $ac = cb$. Let $\lambda > 0$ be such that $\tau(e(\lambda, +\infty)(c)) < \infty$. Setting $p_1 = e_{[0, \lambda]}(c)$ and $p_2 = e_{(\lambda, +\infty)}(c)$, we obtain

$$p_1a = ap_i, \quad p_i b = bp_i, \quad i = 1, 2.$$ 

Proof. Set $a_{ij} = p_i ap_j$ and $b_{ij} = p_i bp_j$ for $i, j = 1, 2$.

Step 1. We claim that

$$\tau(a_{12}^*a_{12}) = \tau(a_{21}^*a_{21}), \quad \tau(b_{12}^*b_{12}) = \tau(b_{21}^*b_{21}).$$

Indeed, using the equality $a^*a = aa^*$ and Property [5] we have

$$\tau(a_{12}^*a_{12}) = \tau(pa^*a_2p_1p_2) = \tau(pa^*a_2p_1p_2) - \tau(p_2a^*p_2ap_2) = \tau(p_2a^*p_2) - \tau(p_2a^*p_2p_2) = \tau(p_2a^*p_2) = \tau(a_{21}^*a_{21}) = \tau(a_{21}^*a_{21}).$$

The proof of the second equality in the claim is identical.

Step 2. We claim that

$$\tau(a_{12}^*a_{12}) \leq \tau(b_{12}^*b_{12})$$

and

$$\tau(b_{21}^*b_{21}) \leq \tau(a_{21}^*a_{21}).$$

Since $p_2c^2p_2 \geq \lambda^2p_2$, it follows that

$$a_{12}a_{12}^* = p_1a \cdot p_2 \cdot a^*p_1 \leq \lambda^{-2} \cdot p_1a \cdot p_2c^2p_2 \cdot a^*p_1 = \lambda^{-2}(a_{12}c)(a_{12}c)^*.$$ 

By assumption,

$$a_{12}c = p_1ap_2 \cdot c = p_1 \cdot ac \cdot p_2 = p_1 \cdot cb \cdot p_2 = c \cdot p_1bp_2 = cb_{12}.$$ 

Thus,

$$a_{12}a_{12}^* \leq \lambda^{-2}(cb_{12})(cb_{12})^*.$$ 

Since $c_{12} \in \mathcal{M}$ and since $b_{12} \in (L_1 \cap L_\infty)(\mathcal{M}, \tau)$, it follows that

$$cb_{12} = c_{12}b_{12} \in (L_1 \cap L_\infty)(\mathcal{M}, \tau).$$
Hence (see Property 5),
\[ \tau(a_{12}^*a_{12}) = \tau(a_{12}a_{12}^*) \leq \lambda^{-2} \tau((cb_{12})(cb_{12})^*) = \lambda^{-2} \tau((cb_{12})^*(cb_{12})) < \infty. \]

Using now the inequality \( p_1c^2p_1 \leq \lambda^2 p_1 \), we have that
\[ (cb_{12})^*(cb_{12}) = b_{12}^*c^2b_{12} = b_{12}^* \cdot p_1c^2p_1 \cdot b_{12} \leq \lambda^2 \cdot b_{12}^*p_1b_{12} = \lambda^2 b_{12}^*b_{12} \]
and
\[ (3) \quad \tau(a_{12}^*a_{12}) \leq \tau(b_{12}^*b_{12}). \]

Let \( a' = b^* \) and \( b' = a^* \). Taking the adjoints in the equality \( ac = cb \), we obtain \( a'c = cb' \). In addition
\[ a_{12}^* \overset{\text{def}}{=} p_1a'p_2 = b_{21}^*, \quad b_{12}^* \overset{\text{def}}{=} p_1b'p_2 = a_{21}^*. \]

Applying (3) to the triple \((a', b', c)\), we obtain
\[ \tau(b_{21}^*b_{21}) = \tau(b_{21}b_{21}^*) = \tau((a_{12}^*)^*a_{12}) \leq \tau((b_{12}^*)^*b_{12}) = \tau(a_{21}a_{21}^*) = \tau(a_{21}^*a_{21}). \]

This proves the claim.

**Step 3.** Using Steps 1, 2, we obtain
\[ \tau(a_{12}^*a_{12}) \leq \tau(b_{21}^*b_{21}) = \tau(b_{21}b_{21}^*) \leq \tau(a_{21}^*a_{21}) = \tau(a_{12}^*a_{12}). \]

Thus,
\[ \tau(a_{12}^*a_{12}) = \tau(a_{21}^*a_{21}) = \tau(b_{12}^*b_{12}) = \tau(b_{21}^*b_{21}). \]

**Step 4.** We claim that \( ap_2 = p_2a \) and \( bp_2 = p_2b \).

By (1), we have
\[ a_{12}c = cb_{12}. \]

Now, using Property 5 we obtain
\[ \tau((a_{12}c)(a_{12}c)^*) = \tau((cb_{12})(cb_{12})^*) = \tau((cb_{12})^*(cb_{12})). \]

The definition of \( p_1 \) now yields
\[ (cb_{12})^*(cb_{12}) = b_{12}^*c^2b_{12} = b_{12}^* \cdot p_1c^2p_1 \cdot b_{12} \leq \lambda^2 b_{12}^* \cdot p_1 \cdot b_{12} = \lambda^2 b_{12}b_{12}. \]

It follows from Step 3 that
\[ \tau((a_{12}c)(a_{12}c)^*) \leq \lambda^2 \tau(a_{12}^*a_{12}). \]

In other words,
\[ \tau(a_{12} \cdot p_2c^2p_2 \cdot a_{12}^*) = \tau((a_{12}c)(a_{12}c)^*) \leq \lambda^2 \tau(a_{12}a_{12}^*) = \tau(a_{12} \cdot \lambda^2 p_2 \cdot a_{12}^*). \]

Hence,
\[ \tau(a_{12} \cdot p_2(c^2 - \lambda^2 1)p_2 \cdot a_{12}^*) \leq 0. \]

Since \( \tau \) is faithful and since
\[ a_{12} \cdot p_2(c^2 - \lambda^2 1)p_2 \cdot a_{12}^* \geq 0, \]

it follows that
\[ (4) \quad a_{12} \cdot p_2(c^2 - \lambda^2 1)p_2 \cdot a_{12}^* = 0. \]

For every \( \varepsilon > 0 \) and \( p_\varepsilon = e_{(\lambda+\varepsilon, +\infty)}(c) \) we have \( c^2p_\varepsilon = p_\varepsilon c^2p_\varepsilon \geq (\lambda + \varepsilon)^2 p_\varepsilon \).

Therefore,
\[ c^2p_2 \geq \lambda^2 p_2 + \varepsilon^2 p_\varepsilon, \quad p_2(c^2 - \lambda^2 1)p_2 \geq \varepsilon^2 p_\varepsilon. \]

We now infer from (4) that
\[ a_{12} \cdot p_\varepsilon \cdot a_{12}^* = 0. \]
Thus, we infer from Lemma 7 that $ap_2 = p_2a$. Similarly, $bp_2 = p_2b$. It follows immediately that

$$ap_1 = a - ap_2 = a - p_2a = p_1a, \quad bp_1 = b - bp_2 = b - p_2b = p_1b.$$  

$\square$

**Lemma 9.** Let $a, b \in \mathcal{M}$ be normal, $c \in S_+(\mathcal{M}, \tau)$ and $ac = cb$. Then $a^*c = cb^*$.

**Proof.** The assumption $c \in S_+(\mathcal{M}, \tau)$ guarantees that there exists $\lambda > 0$ such that $\tau(1 - e^{\lambda}(c)) < \infty$. Set $p_2 = e(\lambda, +\infty)(c)$, $p_1 = (1 - p_2)$ and $a_j = ap_j$, $b_j = bp_j$, $c_j = cp_j$, $j = 1, 2$. By Lemma 8, the operators $a$ and $b$ commute with projections $(1 - e^{\nu}(c))$ for all $\nu \geq \lambda$. Since finite linear combinations of projections $(1 - e^{\nu}(c))$, $\nu \geq \lambda$, converge to operator $c_2$ in the measure topology and since multiplication in $S(\mathcal{M}, \tau)$ is continuous in that topology, it follows that

$$(c_2a = ac_2 \text{ and, similarly, } c_2b = bc_2).$$

Appealing now to Lemma 8, we obtain

$$(c_2a = ac_2 = ac \cdot p_2 = cb \cdot p_2 = c \cdot bp_2 = c \cdot p_2b = c_2b) \quad \text{(5)}$$

Combining (5) and (5) now yields

$$a^*c_2 = (c_2a)^* = (c_2b)^* = c_2b^*.$$  

Taking (5) into account, we rewrite (5) as $bc_2 = c_2b$. Combining this with the assumption $ac = cb$, we infer $ac_1 = c_1b$. Taking into account that $c_1 \in \mathcal{M}$ and applying the classical Fuglede-Putnam theorem we derive that

$$a^*c_1 = c_1b^*.$$  

Thus,

$$a^*c = a^*c_1 + a^*c_2 = c_1b^* + c_2b^* = cb^*.$$  

$\square$

**Lemma 10.** Let $a, b \in \mathcal{M}$ be normal and let $c \in S(\mathcal{M}, \tau)$ be such that $ac = cb$. If $n(c^*) \geq n(c)$ or $n(c) \geq n(c^*)$, then $a^*c = cb^*$.

**Proof.** We only consider the first case (the second case can be reduced to the first one by considering the triple $(b^*, a^*, c^*)$ instead of the triple $(a, b, c)$).

Let $c = v|c|$ be a polar decomposition of $c$ so that $v^*v = r(c)$ and $vv^* = r(c^*)$. Let $w$ be a partial isometry such that $w^*w = n(c^*)$ and $ww^* \leq n(c)$. Define an isometry $u = v^* + w$ (that is, $u^*u = 1$). It is immediate that $u^*|c| = c$ and $uc = |c|$. Thus,

$$(uau^*) \cdot |c| = ua \cdot c = u \cdot ac = uc \cdot b = |c| \cdot b.$$  

Since $u^*u = 1$ and since $a$ is normal, it follows that $uau^*$ is also normal. Applying Lemma 9 to the triple $(uau^*, b, |c|)$, we obtain

$$(uau^*) \cdot |c| = |c| \cdot b^*.$$  

Therefore,

$$a^*c = a^* \cdot u^*|c| = u^* \cdot (uau^*) \cdot |c| = u^* \cdot |c| \cdot b^* = cb^*.$$  

This completes the proof.  

$\square$
We now give the proof of Theorem 6 in the case of arbitrary semifinite von Neumann algebra $\mathcal{M}$ with a faithful normal semifinite trace $\tau$.

Proof of Theorem 6. Let us suppose at first that $a, b \in \mathcal{M}$. By Theorem 2.1.3, there exist central projections $z_1, z_2 \in \mathcal{Z}(\mathcal{M})$ such that

$$z_1 + z_2 = 1, \quad n(c^*)z_1 \preceq n(c)z_1, \quad n(c)z_2 \preceq n(c^*)z_2.$$ 

It is immediate that

$$az_1 \cdot cz_1 = a \cdot z_1 c \cdot z_1 = a \cdot cz_1 \cdot z_1 = ac \cdot z_1^2 = cb \cdot z_1^2 = cz_1 \cdot bz_1;$$

$$az_2 \cdot cz_2 = a \cdot z_2 c \cdot z_2 = a \cdot cz_2 \cdot z_2 = ac \cdot z_2^2 = cb \cdot z_2^2 = cz_2 \cdot bz_2.$$ 

Clearly, $n(cz_k) = n(c)z_k$ and $n(c^*z_k) = n(c^*)z_k$, $k = 1, 2$, where the left hand side is taken in the algebra $\mathcal{Z}(\mathcal{M})$. Applying Lemma 10 to the triples $(a_1, b_1, c_1)$ and $(a_2, b_2, c_2)$, we obtain

$$a^*z_1 \cdot cz_1 = cz_1 \cdot b^*z_1, \quad a^*z_2 \cdot cz_2 = cz_2 \cdot b^*z_2.$$ 

Summing these equalities, we obtain that $a^*c = cb^*$. This proves the assertion for the case $a, b \in \mathcal{M}$.

Now let $a, b$ be arbitrary normal operators in $S(\mathcal{M}, \tau)$ and $ac = cb$. Let $q_n$ (respectively, $r_n$) be the spectral projection for $a$ (respectively, $b$) corresponding to the set $\{z : |z| \leq n\}$. It is clear that $\{q_n\}$ and $\{r_n\}$ are increasing sequences of projections with $\sup_{n \geq 1} q_n = 1$ and $\sup_{n \geq 1} r_n = 1$. In addition (see e.g. [10] Ch. 13, Theorems 13.24, 13.33),

$$aq_n = q_n a, \quad a^*q_n = q_n a^*, \quad br_n = r_nb, \quad b^*r_n = r_nb^*, \quad n \in \mathbb{N}.$$ 

Multiplying the equality $ac = cb$ by $q_n$ on the left and by $r_n$ on the right, we obtain

$$(q_n a) \cdot q_n cr_n = (q_n acr_n) \cdot (r_n b), \quad n \in \mathbb{N}.$$ 

Clearly, $q_n a \in \mathcal{M}$ and $r_nb \in \mathcal{M}$ are normal operators for every $n \in \mathbb{N}$. It follows from the preceding paragraph that

$$q_n \cdot a^*c \cdot r_n = (q_n a) \cdot (q_n acr_n) = (q_n acr_n) \cdot (r_n b)^* = q_n \cdot cb^* \cdot r_n.$$ 

Thus,

$$q_n (a^*c - cb^*) r_n = 0, \quad n \in \mathbb{N}.$$ 

Since $a, b \in S(\mathcal{M}, \tau)$, it follows that $\tau(1 - q_n) \to 0$, $\tau(1 - r_n) \to 0$ as $n \to \infty$. Thus $q_n \to 1$, $r_n \to 1$ in measure. Therefore, for every $x \in S(\mathcal{M}, \tau)$, we have $q_n x r_n \to x$ in measure as $n \to \infty$. Taking $x = a^*c - cb^*$, we complete the proof. □

4. The Fuglede-Putnam theorem in the $*$-algebra $LS(\mathcal{M})$

Lemma 11 below is the key tool used to extend the Fuglede-Putnam theorem from $\tau$-measurable operators to measurable ones.

Lemma 11. Let $\mathcal{M}$ be a semifinite von Neumann algebra and let $q \in \mathcal{P}(\mathcal{M})$ be a finite projection. Then there exists partition of unity $\{z_i\}_{i \in I} \subset \mathcal{P}(\mathcal{Z}(\mathcal{M}))$, such that every von Neumann algebra $z_i M$, $i \in I$, has a faithful normal semifinite trace $\tau_i$ with $\tau_i(z_i q) < \infty$. 
Proof. It is well known that a commutative von Neumann algebra $\mathcal{Z}(\mathcal{M})$ is $*$-isomorphic to the $*$-algebra $L^\infty(\Omega, \Sigma, \mu)$ of all essentially bounded measurable complex-valued functions defined on a measure space $(\Omega, \Sigma, \mu)$ with the measure $\mu$ satisfying the direct sum property (we identify functions that are equal almost everywhere) (see e.g. \cite[Ch. 7, §7.3]{B}). The direct sum property of a measure $\mu$ means that the Boolean algebra of all projections of the $*$-algebra $L^\infty(\Omega, \Sigma, \mu)$ is order complete, and for any non-zero $p \in \mathcal{P}(\mathcal{M})$ there exists a non-zero projection $r \leq p$ such that $\mu(r) < \infty$. The direct sum property of a measure $\mu$ is equivalent to the fact that the functional $\nu(f) := \int_\Omega f \, d\mu$ is a semifinite normal faithful trace on the algebra $L^\infty(\Omega, \Sigma, \mu)$. Therefore there exists partition of unity $\{r_j\}_{j \in J} \subset \mathcal{P}(L^\infty(\Omega, \Sigma, \mu))$, such that $\nu_j(f) = \nu(r_j f)$ is faithful normal finite trace on $r_j L^\infty(\Omega, \Sigma, \mu)$ for every $j \in J$.

Let $\varphi$ be a $*$-isomorphism from $\mathcal{Z}(\mathcal{M})$ onto the $*$-algebra $L^\infty(\Omega, \Sigma, \mu)$. Denote by $L^+(\Omega, \Sigma, m)$ the set of all measurable real-valued functions defined on $(\Omega, \Sigma, \mu)$ and taking values in the extended half-line $[0, \infty]$ (functions that are equal almost everywhere are identified).

By \cite[Ch. V, §2, Theorem 2.34 and Proposition 2.35]{B} there exists a faithful semifinite normal extended center valued trace $T$, 
\[ T: \mathcal{M}_+ \to L^+(\Omega, \Sigma, \mu), \]
such that $\mu(\{\omega \in \Omega : T(q)(\omega) = +\infty\}) = 0$. Thus characteristic functions $q_n = \chi_{A_n}$ corresponding to sets $A_n = \{\omega \in \Omega : n - 1 \leq T(q)(\omega) < n\}$, $n \in \mathbb{N}$, partition the unit element $\chi_\Omega$ of Boolean algebra $\mathcal{P}(L^\infty(\Omega, \Sigma, \mu))$. In addition 
\[ T(q \varphi^{-1}(q_n)) = \varphi^{-1}(q_n)T(q) \leq nq_n \]
for all $n \in \mathbb{N}$.

It is clear that $\{z_n^j = \varphi^{-1}(r_j q_n), \ j \in J, \ n \in \mathbb{N}\}$ is a partition of unity in $\mathcal{P}(\mathcal{Z}(\mathcal{M}))$. In addition, the functional $\tau_{j,n} : z_n^j \mathcal{M}_+ \to [0, \infty]$, given by the formula 
\[ \tau_{j,n}(x) = \nu_j(T(x)), \ x \in z_n^j \mathcal{M}_+, \]
is a faithful normal finite trace on $z_n^j \mathcal{M}$. In particular, 
\[ \tau_{j,n}(z_n^j q) = \nu_j(T(z_n^j q)) \leq n \nu_j(\varphi^{-1}(q_n)r_j) \leq n \nu_j(r_j) < \infty \]
for all $j \in J, \ n \in \mathbb{N}$.

Setting $i = (j, n)$ and $I = J \times \mathbb{N}$, we complete the proof.

\[ \square \]

**Lemma 12.** Let $\mathcal{M}$ be a von Neumann algebra and let $\{z_i\}_{i \in I} \subset \mathcal{Z}(\mathcal{M})$ be a partition of unity. If $x \in LS(\mathcal{M})$ is such that $xz_i = 0$ for every $i \in I$, then $x = 0$.

**Proof.** Since $z_i \leq n(x)$ for all $i \in I$, it follows that $1 = \sup_{i \in I} z_i \leq n(x)$. Thus $n(x) = 1$, i.e. $x = 0$.

\[ \square \]

The following lemma extends the result of Theorem 6 to the setting of measurable operators.

**Lemma 13.** Let $\mathcal{M}$ be a semifinite von Neumann algebra and let $a, b, c \in S(\mathcal{M})$. If $a$ and $b$ are normal and if $ac = cb$, then $a^*c = cb^*$.

**Proof.** Choose $n$ so large that projections $e_{|a|}(n, +\infty)$, $e_{|b|}(n, +\infty)$ and $e_{|c|}(n, +\infty)$ are finite. Let $q$ be a finite projection given by the formula 
\[ q = e_{|a|}(n, +\infty) \vee e_{|b|}(n, +\infty) \vee e_{|c|}(n, +\infty). \]
Let \( \{z_i\}_{i \in I} \) be the partition of unity constructed in Lemma 11. We have
\[
a z_i \cdot c z_i = c z_i \cdot b z_i, \quad i \in I.
\]
It follows from Lemma 11 that, for a given \( i \in I \),
\[
\tau_i(e_{|a|}(n, +\infty) z_i), \tau_i(e_{|b|}(n, +\infty) z_i), \tau_i(e_{|c|}(n, +\infty) z_i) < \infty.
\]
A standard argument yields
\[
e_{|a|}(n, +\infty) z_i = e_{|a z_i|}(n, +\infty),
\]
where the right hand side is taken in the algebra \( z_i \mathcal{M} \). It follows that \( a z_i, b z_i \) and \( c z_i \) are \( \tau_i \)-measurable operators for every \( i \in I \). Theorem 6 implies that
\[
a^* z_i \cdot c z_i = c z_i \cdot b^* z_i.
\]
The assertion follows now from Lemma 12. \( \square \)

Lemma 14 extends the Fuglede-Putnam theorem to the setting of locally measurable operators affiliated with a semifinite von Neumann algebra \( \mathcal{M} \).

**Lemma 14.** Let \( \mathcal{M} \) be a semifinite von Neumann algebra and let \( a, b, c \in LS(\mathcal{M}) \). If \( a \) and \( b \) are normal and if \( ac = cb \), then \( a^* c = cb^* \).

**Proof.** By the (constructive) definition of the algebra \( LS(\mathcal{M}) \), there exist central projections \( \{p_k\}_{k \geq 1}, \{q_i\}_{i \geq 1} \) and \( \{r_m\}_{m \geq 1} \) such that \( p_k \uparrow 1, q_i \uparrow 1 \) and \( r_m \uparrow 1 \) and such that
\[
ap_k, bq_i, cr_m \in S(\mathcal{M}), \quad k, l, m \geq 1.
\]
Denote the triple \((k, l, m)\) by \( n \) and set \( P_n = p_k q_l r_m \). Since
\[
ap_n \cdot c P_n = c P_n \cdot b P_n, \quad n \in \mathbb{N}^3,
\]
it follows from Lemma 13 that
\[
a^* P_n \cdot c P_n = c P_n \cdot b^* P_n, \quad n \in \mathbb{N}^3.
\]
In other words, (here, we let \( r_0 = 0 \)),
\[
(a^* c - cb^*)p_k q_l \cdot (r_m - r_{m-1}) = 0, \quad m \in \mathbb{N}.
\]
Since \( \{r_m - r_{m-1}\}_{m \geq 1} \) is a partition of unity which consists of central projections, it follows from Lemma 12 that
\[
(a^* c - cb^*)p_k q_l = 0, \quad k, l \in \mathbb{N}.
\]
Repeating the argument for \( l \) and, after that, for \( k \), we complete the proof. \( \square \)

The following assertion can be found in [1] (see Theorem 1 there). We provide a short proof for the convenience of the reader.

**Lemma 15.** Let \( \mathcal{M} \) be a purely infinite von Neumann algebra and let \( a, b, c \in LS(\mathcal{M}) \). If \( a \) and \( b \) are normal and if \( ac = cb \), then \( a^* c = cb^* \).

**Proof.** Recall that \( S(\mathcal{M}) = \mathcal{M} \). Choose central projections \( \{p_k\}_{k \geq 1}, \{q_i\}_{i \geq 1} \) and \( \{r_m\}_{m \geq 1} \) such that \( p_k \uparrow 1, q_i \uparrow 1 \) and \( r_m \uparrow 1 \) and such that
\[
ap_k, bq_i, cr_m \in \mathcal{M}, \quad k, l, m \geq 1.
\]
Denote the triple \((k, l, m)\) by \( n \) and let \( P_n = p_k q_l r_m \). We have
\[
ap_n \cdot c P_n = c P_n \cdot b P_n, \quad n \in \mathbb{N}^3.
\]
By the classical Fuglede-Putnam theorem, we have
\[ a^* P_n \cdot c P_n = c P_n \cdot b^* P_n, \quad n \in \mathbb{N}^3. \]
The same argument as in Lemma 14 yields the assertion. \( \square \)

**Proof of Theorem 3.** It is well known that for every von Neumann algebra \( \mathcal{M} \) there exist central projections \( z_1, z_2 \in \mathcal{Z}(\mathcal{M}) \) such that \( z_1 + z_2 = 1 \), \( \mathcal{M} z_1 \) is the semifinite von Neumann algebra and \( \mathcal{M} z_2 \) is the purely infinite von Neumann algebra (see, for example, [11, Ch. 2, §2.2]). We have
\[ a z_k \cdot c z_k = c z_k \cdot b z_k, \quad k = 1, 2. \]
Lemmas 14 and 15 imply that
\[ a^* z_k \cdot c z_k = c z_k \cdot b^* z_k, \quad k = 1, 2. \]
Summing these equalities, we complete the proof. \( \square \)

We need the following useful property of locally measurable operators.

**Lemma 16.** Let \( \mathcal{M} \) be a von Neumann algebra and let \( x \in LS(\mathcal{M}) \). Let \( \{p_n\}_{n \geq 1} \subset \mathcal{P}(\mathcal{M}) \) be such that \( p_n \uparrow 1 \). If \( p_n x p_n = 0 \) for every \( n \geq 1 \), then \( x = 0 \).

**Proof.** Fix \( m \in \mathbb{N} \). For every \( n \geq m \), we have
\[ p_m x p_n = p_m \cdot p_n x p_n = 0. \]
Thus, \( p_n \leq 1 - r(p_m x) \) for every \( n \geq 1 \). Since \( p_n \uparrow 1 \), it follows that \( r(p_m x) = 0 \) and, therefore, \( p_m x = 0 \).

Hence, \( x^* p_m = 0 \) for every \( m \geq 1 \). Thus, \( p_m \leq 1 - r(x^*) \) for every \( m \geq 1 \). Since \( p_m \uparrow 1 \), it follows that \( r(x^*) = 0 \) and, therefore, \( x = 0 \). \( \square \)

**Lemma 17.** Let \( \mathcal{M} \) be a von Neumann algebra and let \( a, b \in LS(\mathcal{M}) \). If \( a \) is normal and if \( ab = ba \), then \( eb = be \) for every spectral projection \( e \) of the operator \( a \).

**Proof.** Let \( b_1 = \Re(b) = \frac{b + b^*}{2} \) and \( b_2 = \Im(b) = \frac{b - b^*}{2i} \). By Theorem 3 we have that \( ab^* = b^* a \). Thus \( ab_j = b_j a \), \( j = 1, 2 \). Let a Borel function \( \phi \) be given by the formula \( \phi(t) = (t + i)^{-1}, t \in \mathbb{R} \), and let \( c_j = \phi(b_j), \quad j = 1, 2 \). Since \( b_j^* = b_j \) and since \( |\phi(t)| \leq 1, t \in \mathbb{R} \), it follows from the Spectral Theorem that \( c_j \in \mathcal{M}, \quad j = 1, 2 \). Since \( ab_j = b_j a \), it follows that
\[ a(b_j + i)^{-1} - (b_j + i)^{-1} a = (b_j + i)^{-1} \cdot (a(b_j + i) - a(b_j + i)) \cdot (b_j + i)^{-1} = 0, \]
that is, \( ac_j = c_j a \). Theorem 13.33 in [10] yields that \( ec_j = c_j e \), \( j = 1, 2 \), for every spectral projection \( e \) of the operator \( a \). Thus, \( eb_1 = b_1 e \) and \( eb_2 = b_2 e \). Summing these equalities, we obtain \( eb = be \). \( \square \)

**Proof of Corollary 4.** \( \mathbb{b} \Rightarrow \mathbb{b} \). Lemma 17 states that \( eb = be \) for every spectral projection \( e \) of the operator \( a \). Again applying Lemma 17 to the couple \( (b, e) \), we obtain that \( ef = fe \) for every spectral projection \( e \) of the operator \( a \) and for every spectral projection \( f \) of the operator \( b \).

\( \mathbb{b} \Rightarrow \mathbb{e} \). Let \( q_n \) (respectively, \( r_n \)) be the spectral projection for \( a \) (respectively, \( b \)) corresponding to the set \( D_n = \{ z : |z| \leq n \}, \quad n \in \mathbb{N} \). Denote \( \phi_n = \phi \cdot \chi_{D_n} \) and \( \psi_n = \psi \cdot \chi_{D_n} \). By the Spectral Theorem, we have
\[ q_n \cdot \phi(a) = \phi_n(a) \cdot q_n = \phi_n(aq_n), \quad r_n \cdot \psi(b) = \psi(b) \cdot r_m = \psi_m(b r_m). \]
Bounded operators $aq_n$ and $br_m$ are normal, and their spectral projections commute. By the Spectral Theorem for bounded operators, these operators commute and, therefore,

$$\phi_n(aq_n) \cdot \psi_m(br_m) = \psi_m(br_m) \cdot \phi_n(aq_n).$$

Thus,

$$q_n r_m \cdot \phi(a) \psi(b) \cdot q_n r_m = q_n r_m \cdot \phi_n(aq_n) \psi_m(br_m) \cdot q_n r_m$$

$$= q_n r_m \cdot \psi_m(br_m) \phi_n(aq_n) \cdot q_n r_m = q_n r_m \cdot \psi(b) \phi(a) \cdot q_n r_m.$$

Taking into account that $r_m \uparrow 1$ and using Lemma 16, we obtain

$$q_n \cdot \phi(a) \psi(b) \cdot q_n = q_n \cdot \psi(b) \phi(a) \cdot q_n.$$

Again appealing to Lemma 16 we obtain (c).

Taking $\phi(z) = z$ and $\psi(z) = z$ in (c), we obtain the implication (c) $\Rightarrow$ (a). □

References


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