ALGEBRAIC EQUATIONS IN STATE CONDITION

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Abstract. In this paper, we will prove that a problem deciding whether there is an upper-triangular coordinate in which a character is not in the state of a Hilbert point is NP-hard. This problem is related to the GIT-semistability of a Hilbert point.

1. Introduction

Let $k$ be an algebraically closed field of characteristic zero and $rS = k[x_1, \ldots, x_r]$ be a polynomial ring of $r$ variables graded by degree. We will omit the superscript $r$ if there is no confusion. When non-negative integers $d$ and $b$ are fixed, there is a projective space

$$E_{d,b}^r = \mathbb{P}\left( \bigwedge^b rS_d \right),$$

which is a $GL_r(k)$-representation. Let $T_r$ be the maximal torus of $GL_r(k)$ which consists of diagonal matrices and $U_r$ be the set of all upper-triangular matrices with 1’s in the diagonal. There is a $G$-equivariant closed immersion

$$i_{r,P,d} : \text{Hilb}^P(\mathbb{P}^{r-1}_k) \rightarrow E_{d,Q(d)}^r$$

for $d \geq g_P$ where $g_P$ is the Gotzmann number associated to a Hilbert polynomial $P$, which is defined in [2]. Also $Q(d) = \binom{r+d-1}{d} - P(d)$.

For any point $v \in E_{d,b}^r$, the collection of states $\Xi_{G,v} = \{\Xi_{g,v}(T)|g \in GL_r(k)\}$ (defined in [5]) of $v$ determines whether $v$ is semistable or not, as stated in [7]. If $v$ is unstable, $\Xi_{G,v}$ determines the Hesselink strata of $\mathbb{P}\left( \bigwedge^b S_d \right)$ that contains $v$, which is stated in [3]. For an arbitrary character $\chi$ of $T$, $Z_{v,\chi} = \{g \in GL_r(k)|\chi \notin \Xi_{g,v}\}$ is a Zariski-closed subset of $GL_r(k)$. In this paper, we will construct a solvability check problem $(SC)$ which is equivalent to deciding if an arbitrary system of algebraic equations is solvable $(\text{SysA1})$ by specializing the defining equation of some $Z_{v,\chi}$ to the defining equation of $U_r \cap Z_{v,\chi}$ in $U_r$.

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It’s a well-known fact that to decide whether an arbitrary system of algebraic equation is solvable is an NP-hard problem (\[6\]). We will show that this problem can be reduced to the problem asking whether there is a \(g \in U_r\) such that \(\chi \notin \Xi_{g,w}\), in polynomial time. This means that such a problem is NP-hard. This problem is related to the GIT-semistability of a Hilbert point. By solving finitely many such problems, we can decide whether a Hilbert point is semistable or not.

2. Definitions and notation

First of all, we need to define the notion of generalization of a system of algebraic equation.

**Definition.** Suppose \(I\) is an ideal of \(S = k[x_1, \ldots, x_r]\). An ideal \(J\) of a finitely generated \(k\) algebra \(R\) is a generalization of \(I\) under \(\pi\) if there is a surjective ring homomorphism \(\pi : R \to S\) and a minimal generator \(\{z_1, \ldots, z_r\}\) of \(R\) satisfying the following:

- For any \(1 \leq i \leq r\), \(\pi(z_i) \in k \cup \{x_1, \ldots, x_r\}\).
- \(\pi(J) = I\).

\(I\) is a specialization of \(J\) if \(J\) is a generalization of \(I\).

For example, \(I = \langle x^2 + y^2 \rangle \subset k[x, y]\) is a specialization of \(J = \langle z(x^2 + y^2), zw \rangle \subset k[x, y, z, w]\) under the map \(\pi : k[x, y, z, w] \to k[x, y]\) which satisfies \(\pi(x) = x\), \(\pi(y) = y\), \(\pi(z) = 1\) and \(\pi(w) = 0\).

We define some notation. Let \(<_{\text{lex}}\) be a lexicographic monomial order satisfying \(x_{i+1} <_{\text{lex}} x_i\) and let \(A^r_{d,b} = \Lambda^b \, r \, S_d\). Let \(r \, M_d\) be the set of all monomials in \(r \, S_d\) and

\[W^r_{d,b} = \left\{ \bigwedge_{i=1}^{b} m_i \mid m_i \in r \, M_d, m_i >_{\text{lex}} m_{i+1} \right\}.\]

\(W^r_{d,b}\) is a basis of \(A^r_{d,b}\). Suppose \(v \in A^r_{d,b}\) and \(w \in W^r_{d,b}\). We define \(v_w\) to be the \(w\)-component of the vector \(v\). That is,

\[v = \sum_{w \in W^r_{d,b}} v_w w.\]

Let \([v] \in E^r_{d,b}\) be the line in \(A^r_{d,b}\) through \(v\) and the origin of \(A^r_{d,b}\). For any \(g \in \text{GL}_r(k)\), \(g_{ij} \in k\) is the component of \(g\) in the \(i\)’th row and \(j\)’th column. That is,

\[g = \begin{bmatrix} g_{11} & \cdots & g_{1j} & \cdots \\ \vdots & \ddots & \vdots & \cdots \\ g_{i1} & \cdots & g_{ij} & \cdots \\ \vdots & \cdots & \cdots & \ddots \end{bmatrix}.\]

Also, \(\text{GL}_r(k)\) action on \(r \, S\) is given by \(g \cdot x_i = \sum_{1 \leq j \leq r} g_{ij} x_j\). Note that this action is a left action on \(r \, S\). For any \(v \in A^r_{d,b}\) and \(w \in k[W^r_{d,b}]\), \((g \cdot v)_w\) means \((\text{id}_{r \, \text{GL}_r(k)}, O_{\text{GL}_r(k)}) \otimes k \, e_v \circ \phi)(w)\) when \(g\) is an indeterminate. Here \(\phi\) is the co-action map

\[\phi : k[W^r_{d,b}] \to \Gamma(GL_r(k), O_{GL_r(k)}) \otimes_k k[W^r_{d,b}] \cong k \{g_{ij}\}_{i,j=1}^{r} \otimes g \otimes k[W^r_{d,b}]\]

and \(e_v\) is the evaluation map \(e_v : k[W^r_{d,b}] \to k\) at \(v\). Let’s define \(\chi_i \in X(T_r)\) for all \(1 \leq i \leq r\) as follows:

\[\chi_i(D) = D_{ii}\]
where $D \in T_r$. Let $\xi_{i,b} = \frac{\partial}{\partial x_i} (1, \ldots, 1) \in X(T_r)_R = X(T_r) \otimes_\mathbb{Z} \mathbb{R}$. Here $X(T_r)$ is the group of characters of the algebraic torus $T_r$.

Let $L_r = \{g \in \text{GL}_r(k) | \text{g is lower-triangular.}\}$. Let’s define a specialization map $\theta_r : \Gamma(\text{GL}_r(k), \mathcal{O}_{\text{GL}_r(k)}) \rightarrow \Gamma(U_r, \mathcal{O}_{U_r})$ as follows:

$$\theta_r(z_{ij}) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i > j, \\ z_{ij}, & \text{if } i < j, \end{cases}$$

where $\Gamma(\text{GL}_r(k), \mathcal{O}_{\text{GL}_r(k)}) = k[[z_{ij}]_{i,j=1}^r]_{\det z}$ and $\Gamma(U_r, \mathcal{O}_{U_r}) = k[[z_{ij}]_{1 \leq i < j \leq r}]$.

For any $C \subset k[[z_{ij}]_{i,j=1}^r]_{\det z}$, let span$C$ be the $k$-subspace of $k[[z_{ij}]_{i,j=1}^r]_{\det z}$ spanned by $C$. Let $\Sigma_r$ be the permutation group on the set $\{1, 2, \ldots, r\}$, which is a subgroup of $\text{GL}_r(k)$. Let $\Delta_v$ be the convex hull of $\Xi_v$ in $X(T_r)_R$ for all $v \in E_{d,b}^r$.

3. Polynomial Coefficients in Some Special Cases

Suppose $v \in A_{d,b}^r$. In this section, we will compute $v_w$ for some special $w \in W_{d,b}^r$. Let’s compute it when $b = 1$ first.

**Lemma 3.1.** Suppose $r \geq 2$. Let $p \in rS_d = A_{d,1}^r$. For any $g \in \text{GL}_r(k)$,

$$(g.p)_{x_1^{d-j}x_2^j} = \sum_{i_1 + \ldots + i_r = j} \prod_{1 \leq a \leq r} g_{2a}^{i_a} \left. \frac{\partial^j p}{\partial x_1^{i_1} \ldots \partial x_r^{i_r}} \right|_{x_i = g_{i_1}}.$$ 

**Proof.** Without loss of generality, we can assume that $p$ is a monomial. When $p$ is a monomial, expanding $g.p$ proves the equality. \(\square\)

We can generalize Lemma 3.1 using the following lemma.

**Lemma 3.2.** Suppose $r \geq 2$. Let $p_1, p_2 \in rS_d = A_{d,1}^r$. For any $g \in \text{GL}_r(k)$,

$$(g.p_1 \land p_2)_{x_1^{d-j_1}x_2^{j_1} \land x_1^{d-j_2}x_2^{j_2}} = \left| (g.p_1)_{x_1^{d-j_1}x_2^{j_1}} (g.p_1)_{x_1^{d-j_2}x_2^{j_2}} ight| = \left| (g.p_2)_{x_1^{d-j_1}x_2^{j_1}} (g.p_2)_{x_1^{d-j_2}x_2^{j_2}} \right|$$

for all $1 \leq j_1 < j_2 \leq d$.

**Proof.** It can be derived from the definition. \(\square\)

In Lemma 3.1, we see that taking $(g.*)_m$ of $p$ separates each monomial with respect to the degrees of each variable of $p$ and $m$. Our construction would make use of this phenomenon. That is, we will control the degree of one variable, say $x_r^{d+1}$.

Fix $d$. Let $F$ be a sequence $\{F_i\}_{i=0}^{2l-1} \in (rS_d)^{2l} \subset (r+1S_d)^{2l}$. Let’s define $v_d^r(F) \in A_{2l+d,2}^r$ as follows:

$$v_d^r(F) = \left\{ \sum_{i=0}^{2l-1} x_{r+i}^{2l-i} F_i \right\} \land \left\{ \sum_{i=0}^{2l-1} x_{r+i}^{2l-i} F_i \right\}.$$

Note that $[v_d^r(F)] \in \text{Hilb}_F^P(P_k)$ where

$$P(t) = \left( \frac{r + t}{r} \right) - \left( \frac{r + t - 2l - d + 1}{r} \right) + \left( \frac{r + t - 2l - d - 1}{r - 2} \right).$$
Indeed, the graded ideal
\[ I_F = \left( \sum_{i=0}^{2l-1} x_{r+1}^i x_1^{2l-i} F_i, \sum_{i=0}^{2l-1} x_{r+1}^{i+1} x_1^{2l-i-1} F_i \right) \]
of \( r+1 \) satisfies the following properties.

**Lemma 3.3.** \( r+1 \) has the Hilbert polynomial
\[ P(t) = \left( \frac{r + t}{r} \right) - \left( \frac{r + t - 2l - d + 1}{r} \right) + \left( \frac{r + t - 2l - d - 1}{r - 2} \right). \]

Also, \( g_P = 2l + d \) so that \( i_{r+1, P, d+2l}(I_F) = [v^r_d(F)] \). If \( r \geq 2 \), then \( I_F \) is saturated.

**Proof.** \( I = I_F \) is isomorphic to \( \langle x_1, x_{r+1} \rangle(-2l - d + 1) \) as a graded \( r+1 \) module. Thus, \( \dim_k(I_{F})_{t+2l+d} \) is equal to the number of monomials in \( r+1 \) which is divisible by \( x_1 \) or \( x_{r+1} \), for every \( t \geq 0 \). This implies that \( I_F \) has the Hilbert polynomial
\[ Q(t) = \left( \frac{r + t - 2l - d + 1}{r} \right) - \left( \frac{r + t - 2l - d - 1}{r - 2} \right). \]

\( Q \) admits the Macaulay representation
\[ Q(t) = \left( \frac{r + t - 2l - d}{r} \right) + \left( \frac{r + t - 2l - d - 1}{r - 1} \right). \]

By the definition of \( n(Q) \) in [2, p. 65], \( g_P = 2l + d \). The regularity of \( I_F \) is equal to the regularity of \( \langle x_1, x_{r+1} \rangle(-2l - d + 1) \), which is equal to \( 2l + d \). Let \( J \) be the saturation of \( I_F \). The Hilbert polynomial of \( J \) is \( Q \). This implies that the regularity of \( J \) is at most \( g_P = 2l + d \). Therefore, \( \dim_k J_t = Q(t) = \dim_k I_t \) for all \( t \geq 2l + d \) by [2] (1.2) Satz, (2.9) Lemma]. Suppose \( r \geq 2 \). If there is a homogeneous \( q \in J \setminus I \), then \( q \in J_t \) for some \( t < 2l + d \). We derive an inequality \( 2 = \dim_k I_{2l+d} \geq \dim_k q_{2l+d} \geq r + 1 \geq 3 \), which is false. \( \square \)

We can analyze the polynomial coefficient of \( g.v^r_d(F) \) as follows:

**Lemma 3.4.** \( \{f_{a,r,l,F}\}_{a=0}^{l-1} \) is a basis for
\[ \text{span}\{\theta_{r+1}((g.v^r_d(F))_{x_1^{2l+d-a} x_{r+1}^a} \} \mid 0 \leq a \leq l - 1 \}
\]
where
\[ f_{a,r,l,F} = \sum_{0 \leq i < j \leq 2l-1} \tilde{F}_i \tilde{F}_j g_{1r+1}^{i+j-2l-1} \left[ \binom{i+j}{a} \binom{2l-a-1}{j} \right] + \sum_{i=0}^{2l-1} \tilde{F}_i^2 g_{1r+1}^{2l-i-1} \left[ \binom{i}{a} \binom{2l-a-1}{i} \right] \]
and
\[ \tilde{F}_i = F_i(1, g_{12}, \ldots, g_{1r}). \]

**Proof.** Using Lemma 3.1 and Lemma 3.2, we can compute that
\[ f_{a,r,l,F} - f_{a-1,r,l,F} = \theta_{r+1}((g.v^r_d(F))_{x_1^{2l+d-a} x_{r+1}^a} \} \]
for all \( 1 \leq a \leq l - 1 \) and
\[ f_{0,r,l,F} = \theta_{r+1}((g.v^r_d(F))_{x_1^{2l+d} x_{r+1}^d}). \]
\( \square \)
Lemma 3.5. $\pi^\psi = \{\pi_j^\psi\}_{j=0}^{l-1}$ is a basis for
\[\text{span}\{\theta_{r+1}(\theta_{r+1}(F_\psi)_{x_1}^{2i-d-a}x_1^{d-a}x_2^{2i-a})|0 \leq a \leq l-1\}\]
where
\[\pi_j^\psi = \frac{(2l-1)!}{(2l-1-j)!} \left[ \sum_{a=0}^{l-1-j} \binom{2l-1-j}{a} (-1)^a g_{1r+1}^{2l-1} + \tilde{\psi}_j g_{1r+1}^{1j}\right]\]
and
\[\tilde{\psi}_j = \psi_j(1, g_{12}, \ldots, g_{1r}).\]

Proof. By the definition of $f_{a,r,l,F_\psi}$,
\[a!\left(\frac{2l-1}{a}\right)^{-1} f_{a,r,l,F_\psi} = \frac{(2l-1)!}{(2l-1-a)!} g_{1r+1}^{2l-1} + \sum_{i=a}^{l-1} \frac{1}{(i-a)!} \tilde{\psi}_i g_{1r+1}^{1j}\]
for all $0 \leq a \leq l-1$. Now
\[\sum_{a=j}^{l-1} \frac{(-1)^a}{(a-j)!} \left[ a!\left(\frac{2l-1}{a}\right)^{-1} f_{a,r,l,F_\psi}\right] = (2l-1)! \sum_{a=j}^{l-1} \frac{(-1)^a}{(a-j)!} g_{1r+1}^{2l-1} + \sum_{a=j}^{l-1} \sum_{i=a}^{l-1} \frac{(-1)^a}{(a-j)!} \tilde{\psi}_i g_{1r+1}^{1j}\]
\[= \frac{(2l-1)!}{(2l-1-j)!} \sum_{a=0}^{l-1-j} \binom{2l-1-j}{a} (-1)^a g_{1r+1}^{2l-1}\]
\[+ \sum_{a=j}^{l-1} \sum_{i=a}^{l-1} \frac{1}{(i-j)!} \binom{i-j}{a} (-1)^a \tilde{\psi}_i g_{1r+1}^{1j} = \pi_j^\psi.\]
Clearly $\{\pi_j^\psi|0 \leq j \leq l-1\}$ is a linearly independent set. This proves the lemma.

$\pi^\psi$ has the following property. This property depends on the characteristic of $k$, which is zero in this paper.

Lemma 3.6. The coefficient of $g_{1r+1}^{2l-1}$ in $\pi_j^\psi$ is non-zero. That is,
\[\sum_{a=0}^{l-1-j} \binom{2l-1-j}{a} (-1)^a \neq 0\]
for any choice of integers $l$ and $j$ satisfying $l \geq 1$ and $0 \leq j \leq l-1$.

Proof. Note that
\[\binom{2l-1-j}{a} \leq \binom{2l-1-j}{a+1}\]
for all $a$ satisfying $0 \leq a \leq l-1-j$. 
If \( l - 1 - j \) is even, 
\[
\sum_{a=0}^{l-1-j} \binom{2l - 1 - j}{a} (-1)^a = 1 + \sum_{a=1}^{\frac{l-1-j}{2}} \binom{2l - 1 - j}{2a} - \binom{2l - 1 - j}{2a - 1} > 0.
\]
Similarly, if \( l - 1 - j \) is odd, we can show that 
\[
\sum_{a=0}^{l-1-j} \binom{2l - 1 - j}{a} (-1)^a < 0
\]
because the first term is always strictly smaller than the absolute value of the second term.

\[\square\]

4. NP-hardness of a problem judging the existence of an upper-triangular coordinate

Suppose \( l \geq 3, r \geq 2 \) and \( p = \{p_i\}_{i=0}^{l-3} \subseteq k[x_2, \ldots, x_r]^{l-2}. \) Assume that 
\[
d \geq \max\{\deg(p_i) | 0 \leq i \leq l - 3\}
\]
where \( \deg(p_i) \) means the non-homogeneous degree of \( p_i. \) Let’s construct \( \psi(p) = \{\psi_i(p)\}_{i=0}^{l-1}. \)

- Define the first two terms as follows:
  \[
  (1) \quad \psi_i(p) = -\frac{(2l - 1)!}{(2l - 1 - i)!} \left[ \sum_{a=0}^{l-1-i} \binom{2l - 1 - i}{a} (-1)^a \right] x_1^d
  \]
  for \( i \in \{0, 1\}. \)

- For \( 2 \leq i \leq l - 1, \) let
  \[
  (2) \quad \psi_i(p) = -\frac{(2l - 1)!}{(2l - 1 - i)!} \left[ \sum_{a=0}^{l-1-i} \binom{2l - 1 - i}{a} (-1)^a \right] x_1^d
  
  + x_1^d p_{i-2} \left( \frac{x_2}{x_1}, \ldots, \frac{x_r}{x_1} \right).
  
Now we are ready to prove the following.

**Theorem 4.1.** Let \( l \geq 3. \) There is \( g \in U_{r+1} \) satisfying \( \chi = \chi_1^{2d+2l} \chi_{r+1}^{2l} \notin \Xi_{[g, \psi(p), \chi]} \) if and only if the ideal \( J\) of \( k[x_2, \ldots, x_r] \) generated by \( \{p_i | 0 \leq i \leq l - 3\} \) has a solution over \( k. \)

**Proof.** By definition, \( Z[\psi(p), \chi] \cap U_{r+1} \) is the zero set of the ideal 
\[
I \subset \Gamma(U_{r+1}, O_{U_{r+1}}) = k[\{g_{ij}|1 \leq i < j \leq r+1\}]
\]
generated by 
\[
\{\theta_{r+1}((g, \psi(p))) x_1^{2l - a} x_{r+1}^a \wedge x_1^{2d+2l - a} | 0 \leq a \leq l - 1\}.
\]
By Lemma 3.3 \( J \) is generated by 
\[
\{\pi_i^{\psi(p)} | 0 \leq i \leq l - 1\}.
\]
It suffices to show that the zero set of \( I \) is non-empty if and only if the zero set of \( J \) is non-empty. If there is an element \( \{x_{ij}\}_{1 \leq i < j \leq r-1} \) in the zero set of \( I, \) then \( g_{1r+1} = 1 \) because \( \pi_i^{\psi(p)} = 0 \) for \( i \in \{0, 1\} \) if and only if \( g_{1r+1}^{2l-1} = 1 \) and \( g_{1r+1}^{2l-1} - g_{1r+1} = 0 \)
by Lemma 3.6. Note that \((x_{12}, \ldots, x_{1r})\) is a solution of the system of equations defined by
\[
\{ x_i^{\psi(p)} | g_{1r+1} = 1 \} = \{ p_i(g_{12}, \ldots, g_{1r}) | 0 \leq i \leq l - 3 \}
\]
so that \(J\) has non-empty zero set. If there is an element \(\{x_i\}_{i=2}^r\) in the zero set of \(J\), \(\{z_{ij}\}_{1 \leq i < j \leq r+1}\) is in the zero set of \(I\) if \(z_{ir} = x_i\) for all \(2 \leq i \leq l - 1\) and \(z_{1r+1} = 1\).

Theorem 4.1 implies the following.

**Corollary 4.2.** For any ideal \(I\) of a polynomial ring, there is a Hilbert point \(v \in \text{Hilb}^P(\mathbb{P}_k^r)\), a choice of closed immersion \(\text{Hilb}^P(\mathbb{P}_k^r) \rightarrow \mathbb{P}(\chi)\) and a character \(\chi \in X(T_{r+1})\) such that there is an ideal \(J\) of \(\Gamma(\text{GL}_{r+1}(k),\mathcal{O}_{\text{GL}_{r+1}(k)})\) such that \(Z_{v,\chi}\) is the zero locus of \(J\) and \(J\) is a generalization of \(I\).

Let’s consider some decision problems. Let \(\text{SysAl}\) be a problem asking if a system of algebraic equations over \(\mathbb{Q}\) has a solution over \(k\) and \(\text{HC}\) be a problem asking if a graph has a Hamiltonian cycle. Using the proof of Corollary 2.3.2 in [6] p. 21], we can prove that \(\text{HC}\) can be reduced to \(\text{SysAl}\) in polynomial time. By Theorem 10.23 of [4], \(\text{HC}\) is an NP-complete problem so that \(\text{SysAl}\) is an NP-hard problem. Let’s describe a solvability check problem \(\text{SC}\) as follows:

- **Given:** A rational Hilbert point \(v \in \text{Hilb}^P(\mathbb{P}_k^r)\), a choice of closed immersion \(\text{Hilb}^P(\mathbb{P}_k^r) \rightarrow \mathbb{P}(\chi)\) and a character \(\chi \in X(T_r)\).
- **Decide:** Is there a coordinate \(g \in U_r\) satisfying \(\chi \notin \Xi_{g,v}\)?

Here, \(v \in \text{Hilb}^P(\mathbb{P}_k^r)\) is rational if it represents a saturated homogeneous ideal of \(\mathbb{S}\) generated by rational polynomials. Theorem 4.1 shows that there is a polynomial time reduction from \(\text{SysAl}\) to \(\text{SC}\). That is,

**Corollary 4.3.** The problem \(\text{SC}\) is NP-hard.

There is an extended version of \(\text{SC}\), which would be called \(\text{ESC}\), described as follows:

- **Given:** A rational Hilbert point \(v \in \text{Hilb}^P(\mathbb{P}_k^r)\), a choice of closed immersion \(\text{Hilb}^P(\mathbb{P}_k^r) \rightarrow \mathbb{P}(\chi)\) and a finite set of characters \(C \subset X(T_r)\).
- **Decide:** Is there a coordinate \(g \in U_r\) satisfying \(C \cap \Xi_{g,v} = \emptyset\)?

\(\text{SC}\) can be reduced to \(\text{ESC}\) in polynomial time so that we can prove the following:

**Corollary 4.4.** The problem \(\text{ESC}\) is NP-hard.

On the other hand, we can use Buchberger’s algorithm in [11] to solve the problem \(\text{ESC}\) because the zero set of an ideal \(I \subset \mathbb{S}\) is non-empty if and only if \(1 \notin I\) if and only if the Gröbner basis of \(I\) with respect to the lexicographic (or graded reverse-lexicographic) monomial order contains 1.

Let’s construct an example. Fix natural numbers \(r\) and \(d\). Suppose \(l = 3\), \(p_0 \in k[x_2, \ldots, x_r]\) and \(\deg(p_0) \leq d\). In this case, \(p\) is a sequence of length 1 and the ideal generated by \(\{p_i | 0 \leq i \leq l - 3\}\) has empty zero locus if and only if \(p_0\) is a non-zero constant polynomial. Let
\[
F' = -6x_1^{d+5} + 15x_1^{d+4}x_{r+1} - 10x_1^{d+3}x_{r+1}^2 + x_1^d x_{r+1} x_{r+1} + \frac{x_1^{d+3}x_{r+1}^2}{2} p_0 \left( \frac{x_2}{x_1}, \ldots, \frac{x_r}{x_1} \right).
\]
By the definition, \( I_{F_{\omega,(p)}} = \langle x_1 F', x_r+1 F' \rangle \). This means that there is a \( g \in U_{r+1} \) such that \( \chi_1^{2d+6} x_{r+1} \notin E_{\omega, v^o(F_{\omega,(p)})} \) if and only if \( p_0 \) is the zero polynomial or \( \deg(p_0) \geq 1 \).

5. A RELATION BETWEEN THE PROBLEM ESC AND GIT-semistability

In this section, every GIT problem is related to the action of \( GL_r(k) \) on \( E_{d,b}^r \). It will be proved that we can decide whether a rational Hilbert point is GIT-semistable by solving finitely many ESC. As a consequence of \[7, \text{Criterion 3.3}\], we have the following lemma.

**Lemma 5.1.** A rational point \( v \in E_{d,b}^r \) is GIT-semistable if and only if \( \xi_{d,b}^r \notin \Delta_{g,v} \) for all \( g \in GL_v(k) \).

**Proof.** \( v \) is semistable if and only if it is semistable under the action of every maximal torus of \( GL_v(k) \) by \[8, \text{Theorem 2.1}\]. Since every two maximal tori are conjugate, \[7, \text{Criterion 3.3}\] proves the lemma. \( \blacksquare \)

A point in \( X(T)_{\mathbb{R}} \) is not in a polytope \( \Delta \) if and only if there is a separating hyperplane in \( X(T)_{\mathbb{R}} \). That is,

**Lemma 5.2.** For any \( g \in GL_v(k) \) and \( v \in E_{d,b}^r \), \( \xi_{d,b}^r \notin \Delta_{g,v} \) if and only if there is an \( \omega \in X(T)_{\mathbb{R}}^\vee \) such that

\[
\omega(\xi_{d,b}^r) < \min \omega(\Xi_{g,v} \otimes_{\mathbb{R}} 1).
\]

For some special choices of \( \omega \in X(T)_{\mathbb{R}}^\vee \) and \( v \), we can still guarantee \( \Xi \) for every \( g \in L_r \).

**Lemma 5.3.** Suppose there are \( v \in E_{d,b}^r \) and \( \omega \in X(T)_{\mathbb{R}}^\vee \) satisfying

\[
\omega(\xi_{d,b}^r) < \min \omega(\Xi_v \otimes_{\mathbb{R}} 1)
\]

and \( \omega(\chi_i) \leq \omega(\chi_{i+1}) \) for all \( 1 \leq i < r \). Then, for any \( l \in L_r \),

\[
\omega(\xi_{d,b}^r) < \min \omega(\Xi_{l,v} \otimes_{\mathbb{R}} 1).
\]

**Proof.** Suppose \( \eta \in \Xi_{i,v} \otimes_{\mathbb{R}} 1 \setminus \Xi_v \otimes_{\mathbb{R}} 1 \). It suffices to show that \( \omega(\eta) \geq \min \omega(\Xi_v \otimes_{\mathbb{R}} 1) \). By definition, there is an \( m \in W_{d,b}^r \) satisfying \( \eta \in \Xi_{l,m} \) and \( \Xi_m \subset \Xi_v \). By expanding \( l.m \), we can prove that

\[
\omega(\eta) \geq \min \omega(\Xi_m \otimes_{\mathbb{R}} 1)
\]

using the condition \( \omega(\chi_i) \leq \omega(\chi_{i+1}) \), \( \forall 1 \leq i \leq r - 1 \). Since \( \Xi_m \subset \Xi_v \), we can deduce that \( \min \omega(\Xi_m \otimes_{\mathbb{R}} 1) \geq \min \omega(\Xi_v \otimes_{\mathbb{R}} 1) \). Thus the claimed statement is true. \( \blacksquare \)

Now, we can restate the condition for \( v \) to be unstable.

**Theorem 5.4.** Suppose \( v \in E_{d,b}^r \). \( v \) is unstable if and only if there are \( u \in U_r \) and \( q \in \Sigma_v \) satisfying

\[
\xi_{d,b}^r \notin \Delta_{uq,v}.
\]

**Proof.** If part is obvious by Lemma 5.1. Suppose there is \( g \in GL_v(k) \) satisfying

\[
\xi_{d,b}^r \notin \Delta_{g,v}.
\]

By Lemma 5.2, there is \( \omega \in X(T)_{\mathbb{R}}^\vee \) satisfying

\[
\omega(\xi_{d,b}^r) < \min \omega(\Xi_v \otimes_{\mathbb{R}} 1).
\]
There is a $p \in \Sigma_r$ satisfying $\omega(\chi_{p(i)}) \leq \omega(\chi_{p(i+1)})$ for all $i$. Let’s define $\omega_p(\chi_i) = \omega(\chi_{p(i)})$. Then,

\[ \omega_p(\xi_{d,b}^r) = \omega(\xi_{d,b}) < \min \omega(\Xi_{g.v} \otimes_R 1) = \min \omega_p(\Xi_{p^{-1}g.v} \otimes_R 1). \]

Now there are $l \in L_r, u \in U_r$ and $q \in \Sigma_r$ satisfying $p^{-1}g = luq$ by the LU-decomposition of general non-singular matrix. $p^{-1}g.v$ and $\omega_p$ satisfies the condition of Lemma 5.3. Thus,

\[ \omega_p(\xi_{d,b}^r) < \min \omega_p(\Xi_{l^{-1}uq.v} \otimes_R 1) = \min \omega_p(\Xi_{uq.v} \otimes_R 1). \]

By Lemma 5.2, $\xi_{d,b}^r \notin \Delta_{uq.v}$, as desired. □

Using Theorem 5.4 and Lemma 5.2 we can solve ESC for each choice of $\omega \in X(T)_R^\vee$ and $q \in \Sigma_r$ to check if

\[ \{ \chi \in X(T) | \omega(\chi) \leq \omega(\xi_{d,Q(d)}) \} \cap \Xi_{uq.v} = \emptyset \]

for a rational $v \in \text{Hilb}^P(\mathbb{P}^{r-1}_k)$ and an integer $d \geq g_P$. Note that we have to consider finitely many $\omega'$s because $A^r_{d,b}$ has only finitely many weights with respect to the action of $T_r$. In this way, we can check if $v$ is semistable or not. This fact implies that there is an algorithm deciding if a rational $v \in \text{Hilb}^P(\mathbb{P}^{r-1}_k)$ is GIT-semistable or not.

REFERENCES


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