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GRAPHS OF HYPERBOLIC GROUPS AND A LIMIT SET INTERSECTION THEOREM

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ABSTRACT. We define the notion of *limit set intersection property* for a collection of subgroups of a hyperbolic group; namely, for a hyperbolic group G and a collection of subgroups $\mathcal{S}$ we say that $\mathcal{S}$ satisfies the *limit set intersection property* if for all $H, K \in \mathcal{S}$ we have $\Lambda(H) \cap \Lambda(K) = \Lambda(H \cap K)$. Given a hyperbolic group admitting a decomposition into a finite graph of hyperbolic groups structure with QI embedded condition, we show that the set of conjugates of all the vertex and edge groups satisfies the limit set intersection property.

1. INTRODUCTION

Limit set intersection theorems first appear in the work of Susskind and Swarup ([SS92]) in the context of geometrically finite Kleinian groups. Later on Anderson ([Anda], [Andb]) undertook a detailed study of this for general Kleinian groups. In the context of (Gromov) hyperbolic groups this is true for quasiconvex subgroups (see [GMRS97], Lemma 2.6). Recently W. Yang has looked at the case of relatively quasiconvex subgroups of relatively hyperbolic groups. See [Yan12]. However, this theorem is false for general subgroups of hyperbolic groups. No characterizations other than quasiconvexity are known for a pair of subgroups H, K of a hyperbolic group G which guarantee that $\Lambda(H) \cap \Lambda(K) = \Lambda(H \cap K)$. This motivates us to look for subgroups other than quasiconvex subgroups which satisfy the limit set intersection property. Our starting point is the following celebrated theorem of Bestvina and Feighn. (Graphs of groups are briefly recalled in section 3.)

Theorem 1.1 ([BF92], [BF96])**.** *Suppose $(\mathcal{G}, Y)$ is a finite graph of hyperbolic groups with QI embedded condition and the hallways flare condition. Then the fundamental group, say G, of this graph of groups is hyperbolic.*

As we will see below, there are examples of hyperbolic groups admitting such a decomposition into graphs of hyperbolic groups where the vertex or edge groups are not quasiconvex. However, we still have the following:

Theorem A. *Suppose a hyperbolic group G admits a decomposition into a finite graph $(\mathcal{G}, Y)$ of hyperbolic groups with QI embedded condition. Then the set of all conjugates of the vertex and edge groups of G satisfies the limit set intersection property.*

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This appears as Theorem 4.1 later on. (It follows from a result of S.M. Gersten (see Corollary 6.7 in [Ger98]) that if a hyperbolic group G admits a decomposition into a finite graph $(\mathcal{G}, Y)$ of hyperbolic groups with QI embedded condition, then it also satisfies the *hallways flare* condition of Bestvina-Feighn.) We note that a special case of our theorem was already known by a result of Ilya Kapovich ([Kap01]). There it was proved that given a k-acylindrical graph of hyperbolic groups $(\mathcal{G}, Y)$ with quasi-isometrically embedded condition and with fundamental group G, which turns out to be hyperbolic by Theorem 1.1, the vertex groups are quasiconvex subgroups of G. Hence, the conjugates of all the vertex groups satisfy the limit set intersection property in this case. However, in general graphs of hyperbolic groups satisfying the conditions of Theorem 1.1 the vertex groups need not be quasiconvex. The easiest way to construct such examples is perhaps by taking the suitable ascending HNN extensions of free groups.

Example 1.2. Suppose $F = F(x_1, x_2, ..., x_n)$ is a free group of rank $n > 1$. Let $\phi : F \to F$ be an injective homomorphism. Consider the ascending HNN extension $G = \langle x_1, x_2, ..., x_n, t : t^{-1}x_it = \phi(x_i), i = 1, ..., n\rangle$. If the elements $\phi(x_i)$ are chosen carefully so that the presentation for G satisfies the $C'(1/6)$ small cancellation condition, then G becomes hyperbolic. In the resulting group G the vertex group F has infinite height, and thus from the main result of [GMRS97] it follows that it is not quasiconvex in G. See also Proposition 5.4 of [Kap00a] for a related construction. However, in all these cases the set of conjugates of F has the limit set intersection property.

It is important to note that in the above example when ϕ is an automorphism of F then $F \trianglelefteq G$, and thus in such a case our theorem is trivially true. However, when ϕ is not surjective then $\Lambda(F)$ need not be the whole of $\partial(G)$ in general. For an interesting example of this type one is referred to [Kap00b].

A word about the proof of Theorem A. The proof of Theorem A crucially uses the main result and also the proof techniques of Mitra from [Mit98] (see Theorem 4.2 below). Given a conjugate of a vertex group, say xG_vx^{-1}, one first notes that $\Lambda(xG_v) = \Lambda(xG_vx^{-1})$ since the Hausdorff distance of xG_v and xG_vx^{-1} is finite (see Lemma 2.9). Next given any conjugates of vertex groups, say $x_1G_{v_1}x_1^{-1}$ and $x_2G_{v_2}x_2^{-1}$, and a point $\xi \in \Lambda(x_1G_{v_1}x_1^{-1}) \cap \Lambda(x_2G_{v_2}x_2^{-1}) = \Lambda(x_1G_{v_1}) \cap \Lambda(x_2G_{v_2})$, by Mitra's theorem (Theorem 4.2) on the existence of Cannon-Thurston maps for the inclusion of the vertex groups into the group G, there are geodesic rays $\gamma_i \subset x_iG_{v_i}$ such that γ_i limits to ξ in G, $i = 1, 2$. Next analyzing carefully the *ladder* construction of Mitra in [Mit98] we show that γ_2 may be replaced by a geodesic ray $\gamma_3 \subset x_2G_{v_2}$ such that γ_3 also limits to ξ and the Hausdorff distance of γ_1 and γ_3 is finite. This is the main technical part of the paper culminating in Proposition 4.9. The rest of the proof is like that of Lemma 2.6 of [GMRS97].

2. Boundary of Gromov hyperbolic spaces and limit sets of subspaces

We assume that the reader is familiar with the basics of (Gromov) hyperbolic metric spaces and the coarse language. We shall however recall some basic definitions and results that will be explicitly used in the sections to follow. For details one is referred to [Gro85] or [BH99].

Notation and convention. *In this section we shall assume that all the hyperbolic metric spaces are proper geodesic metric spaces.* We use QI to mean both *quasi-isometry* and *quasi-isometric* depending on the context. Hausdorff distance of two subsets A, B of a metric space Z is denoted by $Hd(A, B)$. For any subset A of a metric space Y and any $D \geq 0$, $N_D(A)$ will denote the D-neighborhood of A in Y. *We assume that all our groups are finitely generated.* All our graphs are connected metric graphs where each edge has length 1 unless otherwise specified. For a group G and $S \subset G$ we will denote by $\Gamma(G, S)$ the Caley graphs of G with respect to S. See [BH99], Chapter I.1, for more on these notions.

Definition 2.1.

(1) Suppose G is a group generated by a finite set $S \subset G$ and let $\gamma \subset \Gamma(G, S)$ be a path joining two vertices $u, v \in \Gamma(G, S)$. Let $u_0 = u, u_1, u_2, ..., u_n = v$ be the consecutive vertices on γ. Let $u_{i+1} = u_i x_i$, $x_i \in S \cup S^{-1}$ for $0 \leq i \leq n-1$. Then we shall say that *the word* $w = x_0 x_1 ... x_{n-1}$ *labels the path* γ.

(2) Also, given $w \in \mathbb{F}(S)$, the free group on S, its image in G under the natural map $\mathbb{F}(S) \to G$ will be called *the element of* G *represented by* w.

Definition 2.2 (See [BH99])**.**

(1) Let X be a hyperbolic metric space and $x \in X$ be a base point. Then the (Gromov) boundary ∂X of X is the equivalence classes of geodesic rays α such that $\alpha(0) = x$ where two geodesic rays α, β are said to be *equivalent* if $Hd(\alpha, \beta) < \infty$.

The equivalence class of a geodesic ray α is denoted by $\alpha(\infty)$.

(2) If $\{x_n\}$ is an unbounded sequence of points in X, we say that $\{x_n\}$ converges to some boundary point $\xi \in \partial X$ if the following holds: Let α_n be any geodesic joining x to x_n. Then any subsequence of $\{\alpha_n\}$ contains a subsequence uniformly converging on compact sets to a geodesic ray α such that $\alpha(\infty) = \xi$. In this case, we say that ξ is the *limit* of $\{x_n\}$ and write $\lim_{n\to\infty} x_n = \xi$.

(3) The *limit set of a subset* Y of X is the set $\{\xi \in \partial X : \exists \{y_n\} \subset Y \text{ with } \lim_{n\to\infty} y_n = \xi\}$. We denote this set by $\Lambda(Y)$.

The following lemma is a basic exercise in hyperbolic geometry, and so we mention it without proof. It basically uses the thin triangle property of hyperbolic metric spaces. (See [BH99], Chapter III.H, Exercise 3.11.)

Lemma 2.3. *Suppose* $\{x_n\}, \{y_n\}$ *are two sequences in a hyperbolic metric space* X *both converging to some points of* ∂X. *If* $\{d(x_n, y_n)\}$ *is bounded, then* $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.

Lemma 2.4.

(1) *There is a natural topology on the boundary* ∂X *of any proper hyperbolic metric space* X *with respect to which* ∂X *becomes a compact space.*

(2) *If* $f : X \to Y$ *is a quasi-isometric embedding of proper hyperbolic metric spaces, then* f *induces a topological embedding* $\partial f : \partial Y \to \partial X$.

If f *is a quasi-isometry, then* ∂f *is a homeomorphism.*

We refer the reader to Proposition 3.7 and Theorem 3.9 in Chapter III.H of [BH99] for a proof of Lemma 2.4.

Definition 2.5.

(1) A map $f : Y \to X$ between two metric spaces is said to be a *proper embedding* if for all $M > 0$ there is $N > 0$ such that $d_X(f(x), f(y)) \leq M$ implies $d_Y(x, y) \leq N$ for all $x, y \in Y$.

A family of proper embeddings between metric spaces $f_i : X_i \to Y_i$, $i \in I$, where I is an indexing set, is said to be *uniformly proper* if for all $M > 0$ there is an $N > 0$ such that for all $i \in I$ and $x, y \in X_i$, $d_{Y_i}(f_i(x), f_i(y)) \leq N$ implies that $d_{X_i}(x, y) \leq M$.

(2) If $f : Y \to X$ is a proper embedding of hyperbolic metric spaces, then we say that a *Cannon-Thurston* (**CT**) *map exists for* f if f gives rise to a continuous map $\partial f : \partial Y \to \partial X$.

This means that given a sequence of points $\{y_n\}$ in Y converging to $\xi \in \partial Y$, the sequence $\{f(y_n)\}$ converges to a point of ∂X and the resulting map $\partial f : \partial Y \to \partial X$ is continuous. Note that our terminology is slightly different from Mitra([Mit98]). The following lemma is immediate.

Lemma 2.6. *Suppose X, Y are hyperbolic metric spaces and $f : Y \to X$ is a proper embedding. If the CT map exists for f, then we have $\Lambda(f(Y)) = \partial f(\partial Y)$.*

We mention the following lemma with brief remarks about proofs, since it states some standard facts from hyperbolic geometry.

Lemma 2.7. *Suppose Z is a δ-hyperbolic metric space and $\{x_n\}$ and $\{y_n\}$ are two sequences in Z such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \xi \in \partial X$. For each n let α_n, β_n be two geodesics in X joining x_1 to x_n and y_1 to y_n respectively.*

(1) *Then there are subsequences $\{n_k\}$ of natural numbers such that the sequences of geodesics $\{\alpha_{n_k}\}$ and $\{\beta_{n_k}\}$ converge uniformly on compact sets to two geodesic rays α, β joining x_1, y_1 respectively to ξ.*

(2) *Moreover, there is* (i) *a constant D depending only on δ and $d(x_1, y_1)$ and* (ii) *sequences of points $p_{n_k} \in \alpha_{n_k}$ and $q_{n_k} \in \beta_{n_k}$ such that $d(p_{n_k}, q_{n_k}) \leq D$, and $\lim_{k\to\infty} p_{n_k} = \lim_{k\to\infty} q_{n_k} = \xi$.*

(3) *The conclusion* (2) *remains valid if we replace α_n, β_n by K-quasigeodesics for some $K \geq 1$. In other words if x_n, y_n are joined to x_1, y_1 by K-quasigeodesics α_n, β_n respectively, then there is* (i) *a constant D depending on δ, $d(x_1, y_1)$ and K,* (ii) *a subsequence $\{n_k\}$ of natural numbers, and* (iii) *sequences of points $p_{n_k} \in \alpha_{n_k}$, $q_{n_k} \in \beta_{n_k}$ such that $d(p_{n_k}, q_{n_k}) \leq D$ and $\lim_{k\to\infty} p_{n_k} = \lim_{k\to\infty} q_{n_k} = \xi$.*

For a proof of (1),(2) see Lemma 3.3 and Lemma 3.13, and for (3) see Theorem 1.7 (stability of quasigeodesics) in Chapter III.H of [BH99]. More precisely, for proving (3) we may choose geodesic segments $\alpha_n^{'}$, $\beta_n^{'}$ connecting the endpoints of the quasigeodesics α_n and β_n respectively and then apply (1) for these geodesics to extract subsequences $\{\alpha_{n_k}^{'}\}$ and $\{\beta_{n_k}^{'}\}$ of $\{\alpha_n^{'}\}$ and $\{\beta_n^{'}\}$, respectively, both converging uniformly on compact sets. Then we can find two sequences of points $p_{n_k}^{'} \in \alpha_{n_k}^{'}$, $q_{n_k}^{'} \in \beta_{n_k}^{'}$ satisfying (2). Finally, by stability of quasi-geodesics for all k there are $p_{n_k} \in \alpha_{n_k}$, $q_{n_k} \in \beta_{n_k}$ such that $d(p_{n_k}, p_{n_k}^{'})$ and $d(q_{n_k}, q_{n_k}^{'})$ are uniformly small. This will prove (3).

Definition 2.8 (**Limit set intersection property**)**.** Suppose G is a Gromov hyperbolic group. Let $\mathcal{S}$ be any collection of subgroups of G. We say that $\mathcal{S}$ has the *limit set intersection property* if for all $H, K \in \mathcal{S}$ we have $\Lambda(H) \cap \Lambda(K) = \Lambda(H \cap K)$.

We state two elementary results on limit sets for future use.

Lemma 2.9. *Suppose G is a hyperbolic group and H is any* subset *of G. Then for all $x \in G$ we have:*

(1) $\Lambda(xH) = \Lambda(xHx^{-1})$.
(2) $\Lambda(xH) = x\Lambda(H)$.

Proof. (1) follows from Lemma 2.3. For (2) one notes that G acts naturally on a Cayley graph X of G by isometries and thus by homeomorphisms on $\partial X = \partial G$ by Lemma 2.4. □

3. Graphs of groups

We presume that the reader is familiar with the Bass-Serre theory. However, we briefly recall some of the concepts that we shall need. For details one is referred to section 5.3 of J.P. Serre's book *Trees* ([Ser00]). Although we always work with nonoriented metric graphs like Cayley graphs, we need oriented graphs possibly with multiple edges between adjacent vertices and loops to describe graphs of groups. Hence the following definition is quoted from [Ser00].

Definition 3.1. A graph Y is a pair (V, E) together with two maps

$$E \to V \times V, \quad e \mapsto (o(e), t(e)) \text{ and}$$
$$E \to E, \quad e \mapsto \bar{e}$$

such that $o(\bar{e}) = t(e), t(\bar{e}) = o(e)$ and $\bar{\bar{e}} = e$ for all $e \in E$.

For an edge e we refer to $o(e)$ as the *origin* and $t(e)$ as the *terminus* of e; the edge $\bar{e}$ is the same edge e with opposite orientation. We write $V(Y)$ for V and $E(Y)$ for E. We refer to $V(Y)$ as the *set of vertices* of Y and $E(Y)$ as the *set of edges* of Y. We shall denote by $|e|$ the edge e without any orientation.

Definition 3.2. A **graph of groups** $(\mathcal{G}, Y)$ consists of the following data:

(1) a (finite) connected graph Y as defined above,
(2) for all $v \in V(Y)$ (and edge $e \in E(Y)$) there is a group G_v (respectively G_e) together with two injective homomorphisms $\phi_{e,0(e)} : G_e \to G_{o(e)}$ and $\phi_{e,t(e)} : G_e \to G_{t(e)}$ for all $e \in E(Y)$ such that the following conditions hold:
    (i) $G_e = G_{\bar{e}}$,
    (ii) $\phi_{e,o(e)} = \phi_{\bar{e},t(\bar{e})}$ and $\phi_{e,t(e)} = \phi_{\bar{e},o(\bar{e})}$.

We shall refer to the maps $\phi_{e,v}$ as the *canonical maps* of the graph of groups. We shall refer to the groups G_v and G_e, and $v \in V(Y)$ and $e \in E(Y)$ as *vertex groups* and *edge groups* respectively. For topological motivations of graphs of groups and the following definition of the fundamental group of a graph of groups one is referred to [SW79] or [Hat01].

Definition 3.3 (Fundamental group of a graph of groups). Suppose $(\mathcal{G}, Y)$ is a graph of groups where Y is a (finite) connected oriented graph. Let $T \subset Y$ be a maximal tree. Then the fundamental group $G = \pi_1(\mathcal{G}, Y, T)$ of $(\mathcal{G}, Y)$ is defined in terms of generators and relators as follows:

The generators of G are the elements of the disjoint union of the generating sets of the vertex groups G_v, $v \in V(Y)$ and the set $E(Y)$ of *oriented* edges of Y.

The relators are of four types: (1) Those coming from the vertex groups, (2) $\bar{e} = e^{-1}$ for all edges e, (3) $e = 1$ for $|e| \in T$, and (4) $e\phi_{e,t(e)}(a)e^{-1} = \phi_{e,o(e)}(a)$ for all oriented edges e and $a \in G_e$.

Bass-Serre tree of a graph of groups. Suppose $(\mathcal{G}, Y)$ is a graph of groups and let T be a maximal tree in Y as in the above definition. Let $G = \pi_1(\mathcal{G}, Y, T)$ be the fundamental group of the graph of groups. The Bass-Serre tree, denoted by $\mathcal{T}$ from now on, is the tree with vertex set $\bigsqcup_{v\in V(Y)} G/G_v$ and edge set $\bigsqcup_{e\in E(Y)} G/G^e_e$ where $G^e_e = \phi_{e,t(e)}(G_e) < G_{t(e)}$. The edge relations are given by

$$t(gG^e_e) = geG_{t(e)},\ o(gG^e_e) = gG_{o(e)} ...(*).$$

Note that when $|e| \in T$ we have $e = 1$ in G.

Tree of metric spaces from a graph of groups. Given a graph of groups $(\mathcal{G}, Y)$ and a maximal tree $T \subset Y$ one can form in a natural way a graph, say X, on which the fundamental group $G = \pi_1(\mathcal{G}, Y, T)$ acts by isometries properly and cocompactly and which admits a simplicial (hence also Lipschitz) G-equivariant map $X \to \mathcal{T}$. The construction of X can be described as follows.

We assume that Y is a finite connected graph and that all the vertex groups and the edge groups are finitely generated. We fix a finite generating set S_v for each one of the vertex groups G_v; similarly for each edge group G_e we fix a finite generating set S_e and assume that $\phi_{e,t(e)}(S_e) \subset S_{t(e)}$ for all $e \in E(Y)$. Let $S = \bigcup_{v\in V(Y)} S_v \cup (E(Y) \setminus E(T))$ be a generating set of G where in $E(Y) \setminus E(T)$ we shall include only nonoriented edges of Y not in T. We define X from the disjoint union of the following graphs by introducing some extra edges as follows:

(1) **Vertex spaces.** For all $\tilde{v} = gG_v \in V(\mathcal{T})$, where $v \in Y$ and $g \in G$ we let $X_{\tilde{v}}$ denote the subgraph of $\Gamma(G, S)$ with vertex set the coset gG_v; two vertices $gx, gy \in X_{\tilde{v}}$ are connected by an edge iff $x^{-1}y \in S_v$. We shall refer to these subspaces of X as *vertex spaces*.
(2) **Edge spaces.** Similarly for any edge $\tilde{e} = gG^e_e$ of $\mathcal{T}$, let $X_{\tilde{e}}$ denote the subgraph of $\Gamma(G, S)$ with vertex set geG^e_e where two vertices gex, gey are connected by an edge iff $x^{-1}y \in \phi_{e,t(e)}(S_e)$. We shall refer to these subspaces of X as *edge spaces*.
(3) The extra edges connect the edge spaces with the vertex spaces as follows: For all edges $\tilde{e} = gG^e_e$ of $\mathcal{T}$ connecting the vertices $\tilde{u} = gG_{o(e)}$ and $\tilde{v} = geG_{t(e)}$ of $\mathcal{T}$ and $x \in G^e_e$, join $gex \in X_{\tilde{e}} = geG^e_e$ to $gex \in X_{\tilde{v}} = geG_{t(e)}$ and $gexe^{-1} \in X_{\tilde{u}} = gG_{o(e)}$ by edges of length 1/2 each. We define $f_{\tilde{e},\tilde{v}} : X_{\tilde{e}} \to X_{\tilde{v}}$ and $f_{\tilde{e},\tilde{u}} : X_{\tilde{e}} \to X_{\tilde{u}}$ by setting $f_{\tilde{e},\tilde{v}}(gex) = gex$ and $f_{\tilde{e},\tilde{u}}(gex) = gexe^{-1}$.

We have a natural simplicial map $\pi : X \to \mathcal{T}$ (more precisely to the first barycentric subdivision of $\mathcal{T}$). This map is the coarse analog of the **tree of metric spaces** introduced by [BF92] (see also [Mit98]). By abuse of terminology we shall refer to this also as a tree of metric spaces or a tree of metric graphs. We recall some notation and definitions from [Mit98] and collect some basic properties.

(1) We note that $X_u = \pi^{-1}(u)$ and $X_e = \pi^{-1}(e)$ for all $u \in V(\mathcal{T})$ and $e \in E(\mathcal{T})$. For all $u \in V(\mathcal{T})$ the intrinsic path metric of X_u will be denoted by d_u. Similarly, we use d_e for the intrinsic path metric on X_e. It follows that with these intrinsic metrics the metric spaces X_e, X_u are isometric to the Cayley graphs $\Gamma(G_e, S_e)$ and $\Gamma(G_u, S_u)$ respectively. Therefore, if all the

vertex and edge groups are Gromov hyperbolic, then the vertex and edge spaces of X are uniformly hyperbolic metric spaces.

(2) **Quasi-isometric lifts of geodesics.** Suppose $u, v \in \mathcal{T}$ and let $[u, v]$ denote the geodesic in $\mathcal{T}$ joining them. A *K-QI section* of π over $[u, v]$ or a *K-QI lift* of $[u, v]$ (in X) is a set theoretic section $s : [u, v] \to X$ of π which is also a K-QI embedding. In general, we are only interested in defining these sections over the vertices in $[u, v]$.

(3) **Hallways flare condition.** We will say that $\pi : X \to \mathcal{T}$ satisfies the *hallways flare* condition if for all $K \geq 1$ there are numbers $\lambda_K > 1, M_K \geq 1, n_K \geq 1$ such that given a geodesic $\alpha : [-n_K, n_K] \to \mathcal{T}$ and two K-QI lifts α_1, α_2 of α, if $d_{\alpha(0)}(\alpha_1(0), \alpha_2(0)) \geq M_K$, then

$$max\{d_{\alpha(n_K)}(\alpha_1(n_K), \alpha_2(n_K)), d_{\alpha(-n_K)}(\alpha_1(-n_K), \alpha_2(-n_K))\} \geq \lambda_K d_{\alpha(0)}(\alpha_1(0), \alpha_2(0)).$$

(4) **Graphs of groups with QI embedded conditions.** Suppose $(\mathcal{G}, Y)$ is a graph of groups such that each vertex and edge group is finitely generated. We say that it satisfies the QI embedded condition if all the inclusion maps of the edge groups into the vertex groups are quasi-isometric embeddings with respect to any choice of finite generating sets for the vertex and edge groups.

It is clear that if $(\mathcal{G}, Y)$ is a graph of groups with QI embedded condition, then all the maps $f_{e,u} : X_e \to X_u$ are uniform QI embeddings. If $\pi : X \to \mathcal{T}$ is the tree of metric spaces obtained as above from a graph of groups $(\mathcal{G}, Y)$ which satisfies the QI embedded condition and the hallways flare condition, then we shall refer to the corresponding QI embedding constant and the functions n_K, λ_K, M_K as the *parameters* of the tree of metric spaces.

Lemma 3.4. *There is a naturally defined proper and cocompact action of G on X such that the map $\pi : X \to \mathcal{T}$ is G-equivariant.*

Proof. We note that X is obtained from the disjoint union of the cosets of the vertex and edge groups of $(\mathcal{G}, Y)$. The group G has a natural action on this disjoint union. It is also easy to check that under this action adjacent vertices of X go to adjacent vertices. Thus we have a simplicial G-action on X. Clearly the natural map $\pi : X \to \mathcal{T}$ is G-equivariant. To show that the action is proper it is enough to show that the vertex stabilizers are uniformly finite. However, if a point $x \in gG_v$ is fixed by an element $h \in G$, then h fixes $gG_v \in V(\mathcal{T})$. However, stabilizers of gG_v are simply gG_vg^{-1}, and the action of gG_vg^{-1} on $gG_v \subset X$ is fixed point free. Hence, the G-action on X is fixed point free.

That the G-action is cocompact on X follows from the fact that the G-actions on $V(\mathcal{T})$ and $E(\mathcal{T})$ are cofinite. □

Fix a vertex $v_0 \in Y$ and the vertex $G_{v_0} \in V(\mathcal{T})$. Look at the corresponding vertex space $G_{v_0} \subset X$ and let x_0 denote $1 \in G_{v_0}$. Let $\Theta : G \to X$ denote the orbit map $g \mapsto gx_0$. By the Milnor-Schwarz lemma this orbit map is a quasi-isometry since the G-action is proper and cocompact by the above lemma.

Lemma 3.5. *There is a constant D_0 such that for all vertex spaces $gG_v \subset X$ we have $Hd(\Theta(gG_v), gG_v) \leq D_0$.*

It follows that for any $g.x \in gG_v \subset X$ we have $g.x \in \Theta^{-1}(B(gx, D_0))$.

Proof. For proving the lemma let γ_v be a geodesic in X joining x_0 to the identity element of G_v. Then for all $x \in G_v$, $gx\gamma_v$ is a path joining $\Theta(gx) = gxx_0$ and

$gx \in gG_v$. Hence one can choose D_0 to be the maximum of the lengths of γ_v, $v \in V(Y)$. □

The following corollary is an immediate consequence of the above two lemmas.

Corollary 3.6. *The vertex spaces and edge spaces of X are uniformly properly embedded in X.*

Notation. We shall use $i_w : X_w \to X$ to denote the canonical inclusion of the vertex and edge spaces of X into X. Let $\tilde{v} = gG_v \in V(\mathcal{T})$. It follows from the above corollary that Θ induces a coarsely well-defined quasi-isometry from $gG_v \subset G$ to $X_{\tilde{v}}$. Namely, we can send any $x \in gG_v$ to a point y of $X_{\tilde{v}}$ such that $d_X(\Theta(x), y) \le D_0$, where D_0 is as in the above corollary. We shall denote this by $\Theta_{g,v} : gG_v \to X_{\tilde{v}}$.

4. The main theorem

For the rest of the paper we shall assume that G is a hyperbolic group which admits a graph of groups decomposition $(\mathcal{G}, Y)$ with the QI embedded condition where all the vertex and edge groups are hyperbolic. Let $\mathcal{T}$ be the Bass-Serre tree of this graph of groups.

We aim to show that in G the family of subgroups $\{gG_vg^{-1} : v \in V(Y), g \in G\} \cup \{gG_eg^{-1} : e \in E(Y), g \in G\}$ satisfies the limit set intersection property. However, clearly this set of subgroups is the same as $\{G_v : v \in V(\mathcal{T})\} \cup \{G_e : e \in E(\mathcal{T})\}$.

Theorem 4.1. *Suppose a hyperbolic group G admits a decomposition into a graph of hyperbolic groups $(\mathcal{G}, Y)$ with the quasi-isometrically embedded condition and suppose $\mathcal{T}$ is the corresponding Bass-Serre tree. Then for all $w_1, w_2 \in V(\mathcal{T}) \cup E(\mathcal{T})$ we have $\Lambda(G_{w_1}) \cap \Lambda(G_{w_2}) = \Lambda(G_{w_1} \cap G_{w_2})$.*

Note. Since each group in $\{G_e : e \in E(\mathcal{T})\}$ is obtainable as the intersection of two groups in $\{G_v : v \in V(\mathcal{T})\}$, it is enough to prove the theorem for any pair of vertex groups. This is exactly what we do below.

The idea of the proof is to pass to the tree of space $\pi : X \to \mathcal{T}$ using the orbit map $\Theta : G \to X$ defined in the previous section and then use the techniques of [Mit98]. The following theorem is an important ingredient of the proof.

Theorem 4.2 ([Mit98])**.** *The inclusion maps $i_w : X_w \to X$ admit CT maps $\partial i_w : \partial X_w \to \partial X$ for all $w \in V(\mathcal{T})$.*

Recall that if $u, v \in V(\mathcal{T})$ are connected by an edge e, then there are natural maps $f_{e,u} : X_e \to X_u$ and $f_{e,v} : X_e \to X_v$. We know that these maps are uniform QI embeddings. We assume that they are all K-QI embeddings for some $K > 1$. They induce embeddings $\partial f_{e,u} : \partial X_e \to \partial X_u$ and $\partial f_{e,v} : \partial X_e \to \partial X_v$ by Lemma 2.4. Therefore, we get partially defined maps from ∂X_u to ∂X_v with domain $Im(\partial f_{e,u})$. Let us denote this by $\psi_{u,v} : \partial X_u \to \partial X_v$. By definition for all $x \in X_e$ we have $\psi_{u,v}(f_{e,u}(x)) = f_{e,v}(x)$.

Definition 4.3.

(1) If $\xi \in \partial X_u$ is in the domain of $\psi_{u,v}$ and $\psi_{u,v}(\xi) = \eta$, then we say that η is a *flow* of ξ and that ξ *can be flowed to* ∂X_v.

(2) Suppose $w_0 \neq w_n \in V(\mathcal{T})$ and $w_0, w_1, \ldots, w_n$ are consecutive vertices of the geodesic $[w_0, w_n] \subset \mathcal{T}$. We say that a point $\xi \in \partial X_{w_0}$ *can be flowed to* ∂X_{w_n} if there are $\xi_i \in \partial X_{w_i}$, $0 \leq i \leq n$, where $\xi_0 = \xi$ such that $\xi_{i+1} = \psi_{w_i, w_{i+1}}(\xi_i)$, $0 \leq i \leq n-1$. In this case, ξ_n is called the *flow* of ξ_0 in X_{w_n}.

Since the maps $\psi_{u,v}$ are injective on their domains for all $u \neq v \in V(\mathcal{T})$ and $\xi \in \partial X_u$, the flow of ξ in ∂X_v is unique if it exists.

Lemma 4.4. *Suppose $w_1, w_2 \in V(\mathcal{T})$ and $\xi_i \in \partial X_{w_i}$, $i = 1, 2$, such that ξ_2 is the flow of ξ_1. Let α_i be a geodesic in the vertex space X_{w_i} such that $\alpha_i(\infty) = \xi_i$, $i = 1, 2$. Then $Hd(\alpha_1, \alpha_2) < \infty$.*

Proof. It is enough to check it when w_1, w_2 are adjacent vertices. Suppose e is the edge connecting w_1, w_2. The lemma follows from the stability of quasigeodesics in the hyperbolic space X_{w_i} and the fact that every point of X_e is at distance $1/2$ from X_{w_i}, $i = 1, 2$. □

Corollary 4.5. *Under the CT maps $\partial i_{w_j} : \partial X_{w_j} \to \partial X$, $j = 1, 2$, the points ξ_1, ξ_2 go to the same point of ∂X, i.e., $\partial i_{w_1}(\xi_1) = \partial i_{w_2}(\xi_2)$.*

Lemma 4.6. *Let $w_1, w_2 \in \mathcal{T}$ and suppose they are joined by an edge e. Suppose $\xi_1 \in \partial X_{w_1}$ cannot be flowed to ∂X_{w_2}. Let $\alpha \subset X_{w_1}$ be a geodesic ray such that $\alpha(\infty) = \xi_1$. Then for all $D > 0$ the set $N_D(\alpha) \cap f_{e,w_1}(X_e)$ is bounded.*

Proof. If $N_D(\alpha) \cap f_{e,w_1}(X_e)$ is not bounded for some $D > 0$, then ξ_1 is in the limit set of $f_{e,w_1}(X_e)$ and so ξ_1 can be flowed to ∂X_{w_2} by Lemma 2.6. This contradiction proves the lemma. □

Now we briefly recall the *ladder construction* of Mitra, which was crucial for the proof of the main theorem of [Mit98]. We shall need it for the proof of Theorem 4.1.

Mitra's ladder $B(\lambda)$. Fix $D_0, D_1 > 0$. Let $v \in V(\mathcal{T})$ and λ be a finite geodesic segment of X_v. We shall define the set $B(\lambda)$ to be a union of vertex space geodesics $\lambda_w \subset X_w$ where w is in a subtree T_1 of $\mathcal{T}$ containing v. The construction is inductive. Inductively one constructs the n-sphere $S_{T_1}(v, n)$ of T_1 centered at v and the corresponding λ_w's, $w \in S(v, n)$.

$S_{T_1}(v, 1)$: There are only finitely many edges e incident on v such that $N_{D_0}(\lambda) \cap f_{e,v}(X_e) \neq \emptyset$. Then $S(v, 1)$ is the set of terminal points of all the edges e that start at v such that the diameter of $N_{D_0}(\lambda) \cap f_{e,v}(X_e)$ is at least D_1. In this case, for each edge e connecting v to say $v_1 \in S(v, 1)$, we choose two points, say $x, y \in N_{D_0}(\lambda) \cap f_{e,v}(X_e)$, such that $d_v(x, y)$ is maximum. Then we choose $x_1, y_1 \in X_{v_1}$ such that $d(x, x_1) = 1$ and $d(y, y_1) = 1$ and defines λ_{v_1} to be a geodesic in X_{v_1} joining x_1, y_1.

$S_{T_1}(v, n+1)$ from $S_{T_1}(v, n)$: Suppose $w_1 \in S(v, n)$. Then a vertex w_2 adjacent to w_1 with $d_T(v, w_2) = n + 1$ belongs to $S(v, n+1)$ if the diameter of $N_{D_0}(\lambda_{w_1}) \cap f_{e,w_1}(X_e)$ is at least D_1 in X_{w_1}, where e is the edge connecting w_1, w_2. To define λ_{w_2} one chooses two points $x, y \in N_{D_0}(\lambda_{w_1}) \cap f_{e,w_1}(X_e)$ such that $d_{w_1}(x, y)$ is maximum, and $x_1, y_1 \in X_{w_2}$ such that $d(x, x_1) = 1$ and $d(y, y_1) = 1$ and define λ_{w_2} to be a geodesic in X_{w_2} joining x_1, y_1.

Theorem 4.7 (Mitra [Mit98]). *There are constants $D_0 > 0, D_1 > 0$, and $C > 0$ depending on the defining parameters of the tree of metric spaces $\pi : X \to \mathcal{T}$ such that the following holds:*

For any $v \in V(\mathcal{T})$ and any geodesic segment $\lambda \subset X_v$ the corresponding ladder $B(\lambda)$ is a C-quasiconvex subset of X.

To prove this theorem, Mitra defines a coarse Lipschitz retraction map $P : X \to B(\lambda)$, which we now recall. For the proof of how this works one is referred to [Mit98]. *However, we shall subsequently assume that appropriate choices of D_0, D_1 are made in our context so that all the ladders are uniformly quasiconvex subsets of X.*

Coarsely Lipschitz retraction on the ladders. Suppose $\lambda \subset X_v$ is a geodesic. Let $T_1 = \pi(B(\lambda))$. For each $w \in T_1$, $\lambda_w = X_w \cap B(\lambda)$ is a geodesic in X_w. We know that there is a coarsely well-defined nearest point projection $P_w : X_w \to \lambda_w$. (See Proposition 3.11 in Chapter III.Γ of [BH99].) Now for each $x \in X_w$, $w \in T_1$ define $P(x) = P_w(x)$. If $x \in X_w$ and $w \notin T_1$, then connect w to T_1 by a geodesic in $\mathcal{T}$. Since $\mathcal{T}$ is a tree there is a unique such geodesic. Let $w_1 \in T_1$ be the endpoint of this geodesic and let e be the edge on this geodesic incident on w_1 going out of T_1. Mitra proved that in this case the projection of $f_{e,w_1}(X_e)$ on λ_{w_1} is uniformly small. It follows by careful choice of D_0, D_1. (See Lemma 3.1 in [Mit98].) Choose a point x_{w_1} on this projection. Define $P(x) = x_{w_1}$.

Theorem 4.8 (Theorem 3.8, [Mit98]). *The map $P : X \to B(\lambda)$ is a coarsely Lipschitz retraction.*

In other words, it is a retraction and there are constants A, B such that $d(P(x), P(y)) \le Ad(x, y) + B$ for all $x, y \in X$.

Using the above theorems of Mitra we shall now prove the converse of Corollary 4.5. This is the last ingredient for the proof of Theorem 4.1.

Proposition 4.9. *Suppose $v \neq w \in \mathcal{T}$ and there are points $\xi_v \in \partial X_v$ and $\xi_w \in \partial X_w$ which map to the same point $\xi \in \partial X$ under the CT maps $\partial X_v \to \partial X$ and $\partial X_w \to \partial X$ respectively. Then ξ_v can be flowed to ∂X_w.*

Proof. Using Lemma 4.4 we can assume that the point v is such that ξ_v cannot further be flowed along vw and similarly ξ_w cannot be flowed in the direction of wv. Let $\alpha : [0, \infty) \to X_v$ and $\beta : [0, \infty) \to X_w$ be geodesic rays in X_v, X_w respectively such that $\alpha(\infty) = \xi_v$ and $\beta(\infty) = \xi_w$. Let e_v, e_w be the first edges from the points v, w along the direction of vw and wv respectively. Then $N_{D_0}(\alpha) \cap f_{e_v,v}(X_{e_v}) \subset X_v$ and $N_{D_0}(\beta) \cap f_{e_w,w}(X_{e_w}) \subset X_w$ are both bounded sets by Lemma 4.6 (where D_0 could be chosen as one in Theorem 4.7).

For all $n \in \mathbb{N}$ let $\alpha_n := \alpha|_{[0,n]}$ and $\beta_n := \beta|_{[0,n]}$ respectively. The ladders $B(\alpha_n), B(\beta_n)$ are uniformly quasiconvex subsets of X by Theorem 4.8. Hence there are uniform ambient quasigeodesics of X in these ladders joining $\alpha(0), \alpha(n)$ and $\beta(0), \beta(n)$ respectively. Choose one such for each one of them and let us call them γ_n and $\gamma_n^{'}$ respectively. Now, since α and β limit on the same point $\xi \in \partial X$, by Lemma 2.7(3) there is a uniform constant D such that for a subsequence $\{n_k\}$ of natural numbers there are points $x_{n_k} \in \gamma_{n_k}$ and $y_{n_k} \in \gamma_{n_k}^{'}$ such that $d(x_{n_k}, y_{n_k}) \le D$ and $\lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} y_{n_k} = \xi \in \partial X$.

Let v_1, w_1 be the vertices on the geodesic $[v, w] \subset \mathcal{T}$ adjacent to v, w respectively. Let $A_{v_k} := B(\alpha_{n_k}) \cap X_{v_1}$ and $A_{w_k} := B(\beta_{n_k}) \cap X_{w_1}$ respectively.

If we remove the edge space X_{e_v} from X, then the remaining space has two components: one containing X_v and the other containing X_w. Call them Y_1, Y_2 respectively. We note that since the diameter of A_{v_k} is uniformly bounded, if at all nonempty, the portion of γ_{n_k} contained in Y_2, if it travels at all into Y_2, is uniformly bounded. This implies that the portion of γ_{n_k} joining $\alpha(0)$ and A_{v_k} is uniformly small if $A_{v_k} \neq \emptyset$.

Hence, there are infinitely many $k \in \mathbb{N}$ such that $x_{n_k} \in Y_1$ and $y_{n_k} \in Y_2$. Since we are dealing with a tree of spaces and $d(x_{n_k}, y_{n_k}) \leq D$ for all $k \in \mathbb{N}$, this implies there are points $z_k \in f_{e_v}(X_{e_v})$ such that $d(x_{n_k}, z_k) \leq D$ for all $k \in \mathbb{N}$. Thus ξ_v can be flowed to ∂X_{v_1} by Lemma 2.6. This contradiction proves the proposition. □

Proof of Theorem 4.1. Suppose $w_i = g_i G_{v_i}$, $i = 1, 2$, for some $g_1, g_2 \in G$ and $v_1, v_2 \in V(Y)$. This implies that $G_{w_i} = g_i G_{v_i} g_i^{-1}$, $i = 1, 2$. Also, $\Lambda(g_i G_{v_i} g_i^{-1}) = \Lambda(g_i G_{v_i})$ by Lemma 2.9(1). Hence we need to show that $\Lambda(g_1 G_{v_1}) \cap \Lambda(g_2 G_{v_2}) = \Lambda(g_1 G_{v_1} g_1^{-1} \cap g_2 G_{v_2} g_2^{-1})$. Using Lemma 2.9(2), therefore, it is enough to show that

$$\Lambda(G_{v_1}) \cap \Lambda(gG_{v_2}) = \Lambda(G_{v_1} \cap gG_{v_2}g^{-1}) \text{ for all } v_1, v_2 \in V(Y), g \in G.$$

Clearly, $\Lambda(G_{v_1} \cap gG_{v_2}g^{-1}) \subset \Lambda(G_{v_1}) \cap \Lambda(gG_{v_2})$. Thus we need to show that $\Lambda(G_{v_1}) \cap \Lambda(gG_{v_2}) \subset \Lambda(G_{v_1} \cap gG_{v_2}g^{-1})$.

Given an element $\xi \in \Lambda(G_{v_1}) \cap \Lambda(gG_{v_2})$ there are $\xi_1 \in \partial G_{v_1}$ and $\xi_2 \in \partial(gG_{v_2})$, both of which map to ξ under the CT maps $\partial G_{v_1} \to \partial G$ and $\partial(gG_{v_2}) \to \partial G$ by Lemma 2.6. Now, we have a quasi-isometry $\Theta : \Gamma(G, S) \to X$. By Lemma 3.5 each coset of any vertex group in G is mapped uniformly Hausdorff close to the same coset in X, i.e., the corresponding vertex space. Hence Θ induces uniform quasi-isometries $\Theta_{g,v}$ from $gG_v \subset \Gamma(G, S)$ to $gG_v \subset X$ for all $g \in G$, $v \in Y$. For avoiding confusion let us denote the subset $G_{v_1} \subset X$ by X_{w_1} and $gG_{v_2} \subset X$ by X_{w_2}. It follows that $\partial\Theta_{1,v_1}(\xi_1) \in \partial X_{w_1}$ and $\partial\Theta_{g,v_2}(\xi_2) \in \partial X_{w_2}$ are mapped to the same element of ∂X under the CT maps $\partial X_{w_i} \to \partial X$, $i = 1, 2$. Hence by Proposition 4.9 $\partial\Theta_{1,v_1}(\xi_1)$ can be flowed to, say, $\xi_2^{'} \in \partial X_{w_2}$. By Corollary 4.5 the image of $\xi_2^{'}$ and $\Theta_{g,v_2}(\xi_2)$ under the CT map $\partial X_{w_2} \to \partial X$ are the same. Hence, we can replace ξ_2 by $(\partial\Theta_{g,v_2})^{-1}(\xi_2)$ and assume that $\Theta_{1,v_1}(\xi_1)$ flows to $\Theta_{g,v_2}(\xi_2)$.

Then by Lemma 4.4 for any geodesic rays $\alpha_i \subset X_{w_i}$ with $\alpha_i(\infty) = \partial\Theta_{x_i,v_i}(\xi_i)$ for $i = 1, 2$, where $x_1 = 1$ and $x_2 = g$, we have $Hd(\alpha_1, \alpha_2) < \infty$. Pulling back these geodesics by Θ_{1,v_1} and Θ_{g,v_2} we get uniform quasigeodesic rays, say $\beta_1 \subset G_{v_1}$ and $\beta_2 \subset gG_{v_2}$, such that $\beta_i(\infty) = \xi_i$, $i = 1, 2$, and $Hd(\beta_1, \beta_2) < \infty$.

Now let $p_i = \beta_1(i)$ and $q_i \in \beta_2$, $i \in \mathbb{N}$, be such that $d(p_i, q_i) \leq D$ where $Hd(\beta_1, \beta_2) = D$. Join p_i to q_i by a geodesic in $\Gamma(G, S)$. Suppose w_i is the word labeling this geodesic. Since there are only finitely many possibilities for such words, there is a constant subsequence $\{w_{n_k}\}$ of $\{w_n\}$. Let $h_k = p_{n_1}^{-1} p_{n_k}$ and $h_k^{'} = q_{n_1}^{-1} q_{n_k}$. Let x be the group element represented by w_{n_k}. Then we have $p_{n_1}.h_k.x = p_{n_1}.x.h_k^{'}$ or $h_k^{'} = x h_k x^{-1}$. Since $h_k^{'}$ connects two elements of gG_{v_2}, it is in G_{v_2}. Hence $h_k \in G_{v_1} \cap xG_{v_2}x^{-1}$. Thus $p_{n_1} h_k p_{n_1}^{-1} \in p_{n_1} G_{v_1} p_{n_1}^{-1} \cap p_{n_1} x G_{v_2} (p_{n_1} x)^{-1} = G_{v_1} \cap gG_{v_2} g^{-1}$. Finally, since $d(p_{n_1} h_k p_{n_1}^{-1}, p_{n_1} h_k) = d(p_{n_1} h_k p_{n_1}^{-1}, p_{n_k}) = d(1, p_{n_1})$ for all $k \in \mathbb{N}$, $\lim_{n\to\infty} p_{n_1} h_k p_{n_1}^{-1} = \lim_{n\to\infty} p_{n_k} = \xi_1$. This completes the proof. □

The following corollary has been pointed out by Mahan Mj. We use the same notation as in the main theorem.

Corollary 4.10. *If $H_i \subset G_{w_i}$, $i = 1, 2$, are two quasiconvex subgroups, then $\Lambda(H_1) \cap \Lambda(H_2) = \Lambda(H_1 \cap H_2)$.*

Proof. Assume that $w_i = g_i G_{v_i}$, $i = 1, 2$. Then $G_{w_i} = g_i G_{v_i} g_i^{-1}$. Let $K_i = g_i^{-1} H_i g_i < G_{v_i}$. We may construct a new finite graph starting from Y by adding two vertices u_1, u_2 where u_i is connected to v_i by an edge e_i, $i = 1, 2$. Let us call this graph Y_1. Define a new graph of groups $(\mathcal{G}_1, Y_1)$ by keeping the definition the same on Y and setting $G_{u_i} = G_{e_i} = K_i$, $i = 1, 2$, and defining $\phi_{e_i,u_i} = 1_{K_i}$ and ϕ_{e_i,v_i} to be the inclusion maps $K_i \subset G_{v_i}$ for $i = 1, 2$. This produces a new graph of groups with the QI embedded condition and with fundamental group isomorphic to G. Suppose the Bass-Serre tree of the new graph of groups is $\mathcal{T}_1$.

Now we can apply Theorem 4.1 to $G_{w_i'}$, $i = 1, 2$, where $w_i' = g_i K_i \in V(\mathcal{T}_1)$, $i = 1, 2$, to finish the proof. □

Example 4.11. We now give an example where intersection of limit sets is not equal to the limit set of the intersection. Suppose G is a hyperbolic group with an infinite normal subgroup H such that G/H is not torsion. Let $g \in G$ be such that its image in G/H is an element of infinite order. Let $K = \langle g \rangle$. Then $H \cap K = (1)$ whence $\Lambda(H \cap K) = \emptyset$. However, H being an infinite normal subgroup of G we have $\Lambda(H) = \partial G$. Thus $\Lambda(H) \cap \Lambda(K) = \Lambda(K) \neq \emptyset$.

We end with some questions.

Question 4.12.

(1) If a hyperbolic group G admits a decomposition into a graph of hyperbolic groups with QI embedded condition and G_v is a vertex group, then how to describe $\Lambda(G_v) \subset \partial G$?

(2) It has been pointed out to the author by Prof. Ilya Kapovich that the first interesting case where this question should be considered is a hyperbolic strictly ascending HNN extension, say G, of a finitely generated nonabelian free group F as described in Example 1.2. For instance, (i) is $\Lambda(F)$ a dendrite? (ii) what can we say about the intersections of the $\Lambda(gF)$'s for various $g \in G$?

It would also be interesting to describe ∂G in these cases.

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