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A LIMIT FORMULA FOR SEMIGROUPS DEFINED BY FOURIER-JACOBI SERIES

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ABSTRACT. I. J. Schoenberg showed the following result in his celebrated paper [Schoenberg, I. J., *Positive definite functions on spheres.* Duke Math. J. **9** (1942), 96-108]: let $\cdot$ and S^d denote the usual inner product and the unit sphere in $\mathbb{R}^{d+1}$, respectively. If $\mathcal{F}^d$ stands for the class of real continuous functions f with domain $[-1,1]$ defining positive definite kernels $(x,y) \in S^d \times S^d \to f(x \cdot y)$, then the class $\bigcap_{d\geq 1} \mathcal{F}^d$ coincides with the class of probability generating functions on $[-1,1]$. In this paper, we present an extension of this result to classes of continuous functions defined by Fourier-Jacobi expansions with nonnegative coefficients. In particular, we establish a version of the above result in the case in which the spheres S^d are replaced with compact two-point homogeneous spaces.

1. Introduction

Decades ago, I. J. Schoenberg ([16]) presented his famous characterization for positive definite functions on the unit sphere S^d in $\mathbb{R}^{d+1}$. If f is a real continuous function on $[-1,1]$ and $\cdot$ denotes the usual inner product of $\mathbb{R}^{d+1}$, his result can be described as follows: the kernel $(x,y) \in S^d \times S^d \to f(x \cdot y)$ is positive definite if, and only if, the function f has a Fourier-Gegenbauer series representation in the form

$$f(t) = \sum_{k=0}^{\infty} a_k^{(d-2)/2,(d-2)/2} P_k^{((d-2)/2,(d-2)/2)}(t), \quad t \in [-1,1],$$

in which all the coefficients $a_k^{(d-2)/2,(d-2)/2}$ are nonnegative and the series is convergent at $t=1$. Here, the symbol $P_k^{(\alpha,\beta)}$ stands for the usual Jacobi polynomial of degree k associated with the pair $(\alpha,\beta) \in (-1,\infty)^2$, as discussed in [1, 18]. As usual, the normalization for the Jacobi polynomials is

$$P_k^{(\alpha,\beta)}(1) = \left(\begin{array}{c} k+\alpha \\ k \end{array} \right) := \frac{\Gamma(k+\alpha+1)}{k!\Gamma(\alpha+1)},$$

in which Γ stands for the usual gamma function.

It is worth mentioning that, if X is a nonempty set, the positive definiteness of a symmetric kernel $K : X \times X \to \mathbb{R}$ refers to the fact that for any positive

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integer n and any distinct points $x_1, x_2, \ldots, x_n$ in X, the $n \times n$ matrix $[K(x_i, x_j)]$ is nonnegative definite.

The result just described is far-reaching and enters as a basic foundation in the analysis of a number of problems in many areas of mathematics: radial basis interpolation in approximation theory ([7]), covariance functions in statistics ([11]), special series expansions in Fourier analysis ([17]), matrix transforms and distance geometry ([14]), etc.

An interesting consequence of Schoenberg's result, also described in ([16]), provides the motivation for writing this paper. For $\alpha \geq \beta > -1/2$, let $\mathcal{F}^{\alpha,\beta}$ denote the family of all real continuous functions f on $[-1,1]$ having a series representation in the form

$$f(t) = \sum_{k=0}^{\infty} a_k^{\alpha,\beta} P_k^{(\alpha,\beta)}(t), \quad t \in [-1,1],$$

in which all the coefficients $a_k^{\alpha,\beta}$ are assumed nonnegative and the series is assumed to converge at $t = 1$. Since

$$|P_k^{(\alpha,\beta)}(t)| \leq P_k^{(\alpha,\beta)}(1), \quad t \in [-1,1], \quad k \geq 0,$$

holds under the same restriction on α and β, actually, one has uniform convergence of the series above on the whole interval $[-1,1]$. Hylleraas linearization formula proved in [12] and its enhancement obtained by Gasper in [9, 10] assert that $\mathcal{F}^{\alpha,\beta}$ is a semigroup under pointwise multiplication, as long as,

$$(\alpha+\beta+1)(\alpha+\beta+4)^2(\alpha+\beta+6) \geq (\alpha-\beta)^2[(\alpha+\beta+1)^2 - 7(\alpha+\beta+1) - 24].$$

In particular, it holds when $\alpha \geq \beta$ and $\alpha + \beta \geq -1$.

After observing the validity of the inclusion (see Formula (7.34) in [1, P. 63])

$$\mathcal{F}^{\gamma,\gamma} \subset \mathcal{F}^{\alpha,\alpha}, \quad \gamma > \alpha,$$

Schoenberg demonstrated that a function f belongs to $\bigcap_{d=1}^{\infty} \mathcal{F}^{(d-2)/2,(d-2)/2}$ if, and only if, it has a representation in the form

$$f(t) = \sum_{k=0}^{\infty} a_k t^k, \quad t \in [-1,1],$$

in which all the coefficients a_k are nonnegative and $\sum_{k=0}^{\infty} a_k < \infty$. Among other things, this very same result provided a characterization for the positive definite functions on the unit sphere in the real ℓ_2 ([16]) and established the converse of Problem 37 of Pólya and Szegö ([15, P. 107]) on matrix transformations. In probability theory, the result above corresponds to stating that $\bigcap_{d=1}^{\infty} \mathcal{F}^{(d-2)/2,(d-2)/2}$ is the class of probability generating functions on $[-1,1]$ ([5]).

Some other inclusions among the semigroup classes mentioned above hold. For instance,

$$\mathcal{F}^{\alpha,\beta-j} \subset \mathcal{F}^{\alpha,\beta}, \quad j \in \mathbb{Z}_+, \quad \beta - j > -1/2,$$

follows from Formula (7.32) in [1, P. 63] while

$$\mathcal{F}^{\gamma,\beta} \subset \mathcal{F}^{\alpha,\beta}, \quad \gamma > \alpha,$$

follows from Formula (7.33) in that same reference. Results proved in [19] imply some other interesting relations among theses classes of functions, which are also related to some early work of Gegenbauer. In particular, we may think of other decreasing sequences of semigroup classes and ask for a description of the intersection of them. For instance, the intersections $\bigcap_{d=1}^{\infty} \mathcal{F}^{(d-2)/2,-1/2}$, $\bigcap_{d=2}^{\infty} \mathcal{F}^{(d-2)/2,0}$ and $\bigcap_{d=4}^{\infty} \mathcal{F}^{(d-2)/2,1}$ are relevant in the analysis of positive definiteness and strict positive definiteness of kernels on compact two-point homogeneous spaces ([2,3,6]). Among other things, the general result to be proved in this paper will imply a description for the classes quoted above.

The paper proceeds as follows. In Section 2, we introduce notation and prove all the technical results on asymptotics for sequences of Jacobi polynomials needed in the paper. In Section 3, we prove the main result of the paper, that is, a description for the class $\bigcap_{m=1}^{\infty} \mathcal{F}^{\alpha_m,\beta_m}$, under specific assumptions on the sequences $\{\alpha_m\}$ and $\{\beta_m\}$. In Section 4, we apply the main result in order to describe the three intersection classes of functions mentioned in the previous paragraph.

2. Technical results

In this section, we will state and prove some specific limit formulas involving Jacobi and normalized Jacobi polynomials. They will enter in a decisive manner in the proof of the main result to be proved in Section 3.

Let us begin recalling one of the many generating formulas for Jacobi polynomials. By expanding the n-th derivative in Rodrigues formula

$$(1-t)^{\alpha}(1+t)^{\beta} P_n^{(\alpha,\beta)}(t) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dt^n}\left[(1-t)^{n+\alpha}(1+t)^{n+\beta}\right],$$

one obtains the explicit relation

$$P_n^{(\alpha,\beta)}(t) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}(t-1)^{n-k}(t+1)^k.$$

If $R_n^{(\alpha,\beta)}$ denotes the normalized Jacobi polynomial $P_n^{(\alpha,\beta)}(1)^{-1}P_n^{(\alpha,\beta)}$, it is promptly seen that

$$\begin{aligned} R_n^{(\alpha,\beta)}(t) &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}\binom{n+\alpha}{n}^{-1}(t-1)^{n-k}(t+1)^k \\ &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{n-k}\binom{n+\beta}{n-k}\binom{n-k+\alpha}{n-k}^{-1}(t-1)^{n-k}(t+1)^k. \end{aligned}$$

Just for the record, we observe that

$$\binom{n+\beta}{n-k}\binom{n-k+\alpha}{n-k}^{-1} = \frac{(\beta+n)(\beta+n-1)\dots(\beta+k+1)}{(\alpha+n-k)(\alpha+n-k-1)\dots(\alpha+1)}.$$

We now can prove the following limit formula for normalized Jacobi polynomials.

Theorem 2.1. *Let $\{\alpha_m\}$ and $\{\beta_m\}$ be sequences in $(-1,\infty)$ with $\{\alpha_m\}\to\infty$. If $\{\beta_m\alpha_m^{-1}\}\to c\in\mathbb{R}$, then the following asymptotic formula holds:*

$$\lim_{m\to\infty} R_n^{(\alpha_m,\beta_m)}(t) = \left[\frac{c(t-1)}{2}+\frac{t+1}{2}\right]^n, \quad t\in[-1,1], \quad n\in\mathbb{Z}_+.$$

Proof. If $\{\beta_m\alpha_m^{-1}\}\to c$, there are just two cases to be considered: if $c>0$, then $\{\beta_m\}\to\infty$. In particular, since

$$\binom{n+\beta_m}{n-k}\binom{n-k+\alpha_m}{n-k}^{-1} = \frac{\beta_m^{n-k}}{\alpha_m^{n-k}}\frac{(1+n/\beta_m)\dots(1+(k+1)/\beta_m)}{(1+(n-k)\alpha_m)\dots(1+1/\alpha_m)},$$

then

$$\lim_{m\to\infty}\binom{n+\beta_m}{n-k}\binom{n-k+\alpha_m}{n-k}^{-1}=c^{n-k}.$$

If $c=0$, then the same conclusion holds. Indeed, otherwise, by passing to a subsequence if necessary, we could assume the existence of $\epsilon>0$ so that

$$\frac{(\beta_m+n)(\beta_m+n-1)\dots(\beta_m+k+1)}{(\alpha_m+n-k)(\alpha_m+n-k-1)\dots(\alpha_m+1)}>\epsilon,\quad m=1,2,\dots.$$

Hence, we would have

$$\lim_{m\to\infty}(\beta_m+n)(\beta_m+n-1)\dots(\beta_m+k+1)=\infty,$$

and, consequently, $\{\beta_m\}\to\infty$. This would generate the contradiction

$$\begin{aligned}0 &= \lim_{m\to\infty}\left(\frac{\beta_m}{\alpha_m}\right)^{n-k}\lim_{m\to\infty}\frac{(1+n/\beta_m)\dots(1+(k+1)/\beta_m)}{(1+(n-k)\alpha_m)\dots(1+1/\alpha_m)}\\ &= \lim_{m\to\infty}\binom{n+\beta_m}{n-k}\binom{n-k+\alpha_m}{n-k}^{-1}>0.\end{aligned}$$

Returning to the expression for $R_n^{(\alpha,\beta)}$ previously deduced, we have that

$$\lim_{m\to\infty}R_n^{(\alpha_m,\beta_m)}(t)=\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{n-k}c^{n-k}(t-1)^{n-k}(t+1)^k,$$

and the assertion of the theorem follows. □

If $\alpha_m=\beta_m$, $m=1,2,\dots$, then the normalized Jacobi polynomials become normalized Gegenbauer polynomials, $c=1$, and we recover (4.4) in [16].

Next, restricting ourselves a little bit, we intend to show that the convergence in the previous theorem is uniform in m, whenever t is fixed (see Theorem 2.4 ahead). In order to achieve that, we will employ a Laplace-type integral representation for Jacobi polynomials, the one formulated in [18, P. 98].

Lemma 2.2. *For $\alpha>\beta>-1/2$ and $\theta\in[0,\pi]$, it holds*

$$\begin{aligned}R_n^{(\alpha,\beta)}(\cos\theta) &= A_\alpha^\beta\int_0^1\int_0^\pi\left[\cos^2\frac{\theta}{2}-r^2\sin^2\frac{\theta}{2}+ir\cos\phi\sin\theta\right]^n\\ &\qquad\times(1-r^2)^{\alpha-\beta-1}r^{2\beta+1}\sin^{2\beta}\phi\,d\phi\,dr,\end{aligned}$$

in which

$$\begin{aligned}A_\alpha^\beta &= \left[\int_0^1\int_0^\pi(1-r^2)^{\alpha-\beta-1}r^{2\beta+1}\sin^{2\beta}\phi\,d\phi\,dr\right]^{-1}\\ &= \frac{2\Gamma(\alpha+1)}{\pi^{1/2}\Gamma(\alpha-\beta)\Gamma(\beta+1/2)}.\end{aligned}$$

The next lemma is concerned with the estimation of an expression related to the integrand appearing in the statement of Lemma 2.2. Precisely, for $\theta,\phi\in[0,\pi]$ and $r\in(0,1)$, we will deal with

$$E_n(\theta,\phi,r):=\left[\cos^2\frac{\theta}{2}-r^2\sin^2\frac{\theta}{2}+ir\cos\phi\sin\theta\right]^n-\cos^{2n}\frac{\theta}{2},\quad n=0,1,\dots.$$

Some of the arguments in its proof will demand the usual beta function B given by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad \operatorname{Re} x, \operatorname{Re} y > 0,$$

and two of its alternative formulations:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = 2\int_0^{\pi/2} \sin^{2x-1}\phi \cos^{2y-1}\phi d\phi, \quad \operatorname{Re} x, \operatorname{Re} y > 0.$$

Lemma 2.3. *For $\alpha > \beta > -1/2$, define*

$$D_n^{\alpha,\beta}(\theta) := A_\alpha^\beta \int_{1/2}^1 \int_0^\pi E_n(\theta,\phi,r)(1-r^2)^{\alpha-\beta-1} r^{2\beta+1} \sin^{2\beta}\phi \, d\phi \, dr, \ \theta \in [0,\pi].$$

Then

$$|D_n^{\alpha,\beta}(\theta)| \le 8\frac{\beta+1}{\alpha+1}, \quad n = 0,1,\ldots.$$

Proof. The proof begins with a crude estimation of $E_n(\theta,\phi,r)$ in the range $(\phi,r) \in [0,\pi]\times[0,1]$, keeping θ fixed:

$$\begin{aligned} |E_n(\theta,\phi,r)| &\le \left|\cos^2\frac{\theta}{2} - r^2\sin^2\frac{\theta}{2} + ir\cos\phi\sin\theta\right|^n + \cos^{2n}\frac{\theta}{2} \\ &\le \left|\left(\cos\frac{\theta}{2} + ir\sin\frac{\theta}{2}\right)^2\right|^n + \cos^{2n}\frac{\theta}{2} \\ &= \left(\cos^2\frac{\theta}{2} + r^2\sin^2\frac{\theta}{2}\right)^n + \cos^{2n}\frac{\theta}{2}. \end{aligned}$$

In particular,

$$|E_n(\theta,\phi,r)| \le 2, \quad \phi \in [0,\pi], \quad r \in [0,1],$$

and, consequently,

$$|D_n^{\alpha,\beta}(\theta)| \le 2A_\alpha^\beta \int_{1/2}^1 \int_0^\pi (1-r^2)^{\alpha-\beta-1} r^{2\beta+1} \sin^{2\beta}\phi d\phi dr, \quad n = 0,1,\ldots.$$

A change of variables in the integral leads to

$$|D_n^{\alpha,\beta}(\theta)| \le A_\alpha^\beta \int_{1/4}^1 \int_0^\pi (1-u)^{\alpha-\beta-1} u^\beta \sin^{2\beta}\phi d\phi du, \quad n = 0,1,\ldots,$$

while additional adjustments produce the inequalities

$$\begin{aligned} |D_n^{\alpha,\beta}(\theta)| &\le 4A_\alpha^\beta \int_{1/4}^1 \int_0^\pi (1-u)^{\alpha-\beta-1} u^{\beta+1} \sin^{2\beta}\phi d\phi du \\ &\le 4A_\alpha^\beta \int_0^1 (1-u)^{\alpha-\beta-1} u^{\beta+1} du \int_0^\pi \sin^{2\beta}\phi d\phi, \quad n = 0,1,\ldots. \end{aligned}$$

It is now clear that

$$|D_n^{\alpha,\beta}(\theta)| \le 4A_\alpha^\beta B(\beta+2,\alpha-\beta)B(\beta+1/2,1/2), \quad n = 0,1,\ldots.$$

The estimate in the statement of the lemma follows after we introduce the value of the constant A_α^β given in Lemma 2.2 and simplify. ☐

It is worth mentioning that the upper bound for $|D_n^{\alpha,\beta}(\theta)|$ provided by the previous lemma does not depend upon n. We close the section proving the following key result.

Theorem 2.4. *Let θ be a fixed real number in $[0,\pi]$ and $\{\alpha_m\}$ and $\{\beta_m\}$ sequences in $(-1/2,\infty)$ with $\{\alpha_m\}\to\infty$ and $\{\beta_m\alpha_m^{-1}\}\to 0$. If $\epsilon>0$, then*

$$\left|R_n^{(\alpha_m,\beta_m)}(\cos\theta)-\left(\frac{1+\cos\theta}{2}\right)^n\right|<\epsilon,\quad n=0,1,\ldots,$$

provided m is large enough.

Proof. The inequality in the statement of the theorem is trivially true for $\theta=0$. We will proceed assuming that $\theta>0$. Let us write

$$C_n^{\alpha_m,\beta_m}(\theta):=R_n^{(\alpha_m,\beta_m)}(\cos\theta)-\left(\frac{1+\cos\theta}{2}\right)^n,$$

and

$$C_n^{\alpha_m,\beta_m}(\theta)=\left[C_n^{\alpha_m,\beta_m}(\theta)-D_n^{\alpha_m,\beta_m}(\theta)\right]+D_n^{\alpha_m,\beta_m}(\theta),$$

with $D_n^{\alpha_m,\beta_m}$ defined via the formula in the previous lemma. Since $\{\alpha_m\}\to\infty$ and $\{\beta_m\alpha_m^{-1}\}\to 0$, there exists an index m_0 so that $\alpha_m>\beta_m$ when $m\geq m_0$. The estimate in Lemma 2.3 implies that

$$\lim_{m\to\infty}|D_n^{\alpha_m,\beta_m}(\theta)|=0,\quad n=1,2,\ldots.$$

In order to complete the proof, let $\epsilon>0$ be given. We can find $n_0=n_0(\epsilon,\theta)>0$ so that

$$\left(\cos^2\frac{\theta}{2}+\frac{1}{4}\sin^2\frac{\theta}{2}\right)^n+\cos^{2n}\frac{\theta}{2}<\frac{\epsilon}{2},\quad n\geq n_0.$$

Since

$$\left(\cos^2\frac{\theta}{2}+r^2\sin^2\frac{\theta}{2}\right)^n\leq\left(\cos^2\frac{\theta}{2}+\frac{1}{4}\sin^2\frac{\theta}{2}\right)^n,\quad r\in(0,1/2),$$

the arguments in the proof of Lemma 2.3 lead to

$$\begin{aligned}&|C_n^{\alpha_m,\beta_m}(\theta)-D_n^{\alpha_m,\beta_m}(\theta)|\\&\qquad\leq\frac{\epsilon A_{\alpha_m}^{\beta_m}}{2}\int_0^{1/2}\int_0^{\pi}(1-r^2)^{\alpha_m-\beta_m-1}r^{2\beta_m+1}\sin^{2\beta_m}\phi\,d\phi\,dr,\end{aligned}$$

for $n\geq n_0$ and $m\geq m_0$. Recalling the definition of A_α^β in Lemma 2.2, we finally deduce that

$$|C_n^{\alpha_m,\beta_m}(\theta)-D_n^{\alpha_m,\beta_m}(\theta)|\leq\frac{\epsilon}{2},\quad n\geq n_0,\quad m\geq m_0.$$

By increasing m_0 if necessary, we can assume that

$$|D_n^{\alpha_m,\beta_m}(\theta)|<\frac{\epsilon}{2},\quad m\geq m_0,\quad n\geq n_0,$$

and, consequently, to conclude that

$$|C_n^{\alpha_m,\beta_m}(\theta)|<\epsilon,\quad m\geq m_0,\quad n\geq n_0.$$

However, since $\{\beta_m\alpha_m^{-1}\}\to 0$, we may invoke Theorem 2.1 in order to find an integer $m_1>m_0$ so that

$$|C_n^{\alpha_m,\beta_m}(\theta)|<\epsilon,\quad m\geq m_1,\quad n=0,1,\ldots,n_0.$$

It is now clear that the assertion of the theorem is reached. □

3. The main result

In this section, we describe the main contribution of the paper, that is, a characterization for the semigroup class $\bigcap_m \mathcal{F}^{\alpha_m,\beta_m}$, for sequences $\{\alpha_m\}$ and $\{\beta_m\}$ satisfying the assumptions of Theorem 2.4.

Taking into account Theorem 2.1, the first result in this section takes the following form.

Theorem 3.1. *Let $\{\alpha_m\}$ and $\{\beta_m\}$ be sequences in $(-1/2,\infty)$ with $\{\alpha_m\}\to\infty$ and $\{\beta_m\alpha_m^{-1}\}\to 0$. If $f:[-1,1]\to\mathbb{R}$ is a continuous function belonging to $\bigcap_{m=1}^{\infty}\mathcal{F}^{\alpha_m,\beta_m}$, then f is representable in the form*

$$f(t)=\sum_{n=0}^{\infty}a_n\left(\frac{1+t}{2}\right)^n,\quad t\in[-1,1],$$

in which all the a_n are nonnegative and $\sum_{n=0}^{\infty}a_n<\infty$.

Proof. If f is a function in the intersection class quoted in the statement of the theorem, then for each m, we can write

$$f(t)=\sum_{n=0}^{\infty}a_n^mP_n^{(\alpha_m,\beta_m)}(t),\quad t\in[-1,1],$$

in which the coefficients a_n^m are all nonnegative and with convergence of the series for $t=1$. From the inequality

$$\left|a_n^mP_n^{(\alpha_m,\beta_m)}(1)\right|\le\sum_{k=0}^{\infty}a_k^mP_k^{(\alpha_m,\beta_m)}(1)=f(1),\quad m,n=0,1,\ldots,$$

it is promptly seen that the double indexed sequence $\{a_n^mP_n^{(\alpha_m,\beta_m)}(1)\}$ is uniformly bounded. Without loss of generality, we can assume that all the limits

$$\lim_{m\to\infty}a_n^mP_n^{(\alpha_m,\beta_m)}(1),\quad n=0,1,\ldots,$$

exist. Indeed, otherwise, we may use Cantor's diagonal process to replace the sequence with a convenient subsequence satisfying the above condition. If we put

$$a_n:=\lim_{m\to\infty}a_n^mP_n^{(\alpha_m,\beta_m)}(1),\quad n=0,1,\ldots,$$

then we obviously have that $a_n\ge 0$, $n=0,1,\ldots$. On the other hand, for $t\in[-1,1)$, the series $\sum_{n=0}^{\infty}a_n^mP_n^{(\alpha_m,\beta_m)}(1)2^{-n}(1+t)^n$ converges uniformly on m, due to the inequality

$$\left|\sum_{n=0}^{\infty}a_n^mP_n^{(\alpha_m,\beta_m)}(1)\left(\frac{1+t}{2}\right)^n\right|\le\sum_{n=0}^{\infty}f(1)\left|\frac{1+t}{2}\right|^n$$

and the convergence of the series in the right hand side of the inequality. In particular, the formula

$$\sum_{n=0}^{\infty}a_n\left(\frac{1+t}{2}\right)^n:=\lim_{m\to\infty}\sum_{n=0}^{\infty}a_n^mP_n^{(\alpha_m,\beta_m)}(1)\left(\frac{1+t}{2}\right)^n,\quad t\in[-1,1),$$

is meaningful. Next, we show that

$$\lim_{m\to\infty}\sum_{n=0}^{\infty}a_n^mP_n^{(\alpha_m,\beta_m)}(1)\left(\frac{1+t}{2}\right)^n=f(t),\quad t\in[-1,1].$$

First, we write ($m \geq 1$)

$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty} a_n^m P_n^{(\alpha_m,\beta_m)}(1) \left(\frac{1+t}{2}\right)^n \\ &+ \sum_{n=0}^{\infty} a_n^m P_n^{(\alpha_m,\beta_m)}(1) \left[R_n^{(\alpha_m,\beta_m)}(t) - \left(\frac{1+t}{2}\right)^n\right], \quad t \in [-1,1]. \end{aligned}$$

For $\epsilon > 0$ fixed, we may apply Theorem 2.4 to select $m_0 = m_0(\epsilon, t) > 0$ so that

$$\left| R_n^{(\alpha_m,\beta_m)}(t) - \left(\frac{1+t}{2}\right)^n \right| < \frac{\epsilon}{f(1)+1}, \quad m \geq m_0.$$

In particular,

$$\left| \sum_{n=0}^{\infty} a_n^m P_n^{(\alpha_m,\beta_m)}(1) \left[R_n^{(\alpha_m,\beta_m)}(t) - \left(\frac{1+t}{2}\right)^n\right] \right| < \epsilon, \quad m \geq m_0,$$

and, consequently,

$$f(t) = \sum_{n=0}^{\infty} a_n \left(\frac{1+t}{2}\right)^n, \quad t \in [-1,1).$$

In order to conclude the proof, we now handle the case in which $t = 1$, that is, we show that $\sum_{n=0}^{\infty} a_n$ converges. If this were not the case, we could select $n_0 \geq 0$ so that $\sum_{n=0}^{n_0} a_n \geq 2f(1)$. But then, choosing $\bar{t} \in [0,1)$ so that $(1+\bar{t})^{n_0} \geq 2^{n_0-1}$, we would reach

$$f(\bar{t}) \geq \sum_{n=0}^{n_0} a_n \left(\frac{1+\bar{t}}{2}\right)^n \geq f(1),$$

a contradiction. □

To proceed, we now invoke Bateman's formula quoted in [1, P. 11]. For $\alpha, \beta > -1$, it holds

$$\left(\frac{1+t}{2}\right)^n = \sum_{k=0}^{n} b_{k,n}^{\alpha,\beta} P_k^{(\alpha,\beta)}(t), \quad n = 0, 1, \ldots,$$

in which all coefficients $b_{k,n}^{\alpha,\beta}$ are nonnegative. With the help of this formula, we can prove the following result.

Theorem 3.2. *Let $f : [-1,1] \to \mathbb{R}$ be a function representable in the form*

$$f(t) = \sum_{n=0}^{\infty} a_n \left(\frac{1+t}{2}\right)^n, \quad t \in [-1,1],$$

in which all the a_n are nonnegative and $\sum_{n=0}^{\infty} a_n < \infty$. Then, f belongs to $\mathcal{F}^{\alpha,\beta}$, $\alpha \geq \beta > -1/2$.

Proof. Using Bateman's formula, we have

$$f(1) = \sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} b_{k,n}^{\alpha,\beta} P_k^{(\alpha,\beta)}(1) = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} a_n b_{k,n}^{\alpha,\beta} P_k^{(\alpha,\beta)}(1).$$

Since $|P_k^{(\alpha,\beta)}(t)| \leq P_k^{(\alpha,\beta)}(1)$, $n \geq 0$, we may apply Fubini's theorem for series to deduce that

$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty} a_n \left(\frac{1+t}{2}\right)^n \\ &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} b_{k,n}^{\alpha,\beta} P_k^{\alpha,\beta}(t) = \sum_{k=0}^{\infty} a_k^{\alpha,\beta} P_k^{\alpha,\beta}(t), \quad t \in [-1,1], \end{aligned}$$

where

$$a_k^{\alpha,\beta} = \sum_{n=k}^{\infty} a_n b_{k,n}^{\alpha,\beta} \geq 0, \quad k = 0, 1, \ldots.$$

The proof is complete. $\square$

This is the main result of the paper, which is now evident.

Theorem 3.3. *Let $\{\alpha_m\}$ and $\{\beta_m\}$ be sequences in $(-1/2,\infty)$ with $\{\alpha_m\} \to \infty$ and $\{\beta_m \alpha_m^{-1}\} \to 0$. A continuous function $f : [-1,1] \to \mathbb{R}$ belongs to $\bigcap_{m=1}^{\infty} \mathcal{F}^{\alpha_m,\beta_m}$ if, and only if, f has the representation*

$$f(t) = \sum_{n=0}^{\infty} a_n \left(\frac{1+t}{2}\right)^n, \quad t \in [-1,1],$$

in which all the a_n are nonnegative and $\sum_{n=0}^{\infty} a_n < \infty$.

For $\alpha, \beta \in (-1/2, \infty)$, an easy calculation reveals that the function

$$t \in [-1,1] \to \frac{1}{2}\left[c(t-1) + t + 1\right]$$

belongs to $\mathcal{F}^{\alpha,\beta}$ if, and only if,

$$\frac{\alpha-\beta}{\alpha+\beta+2} \leq \frac{1-c}{1+c}.$$

Thus, under the assumptions of the previous theorem, if we replace the condition $\{\beta_m \alpha_m^{-1}\} \to 0$ with $\{\beta_m \alpha_m^{-1}\} \to c > 0$, it may not be true that a continuous function $f : [-1,1] \to \mathbb{R}$ will belong to $\bigcap_{m=1}^{\infty} \mathcal{F}^{\alpha_m,\beta_m}$ if, and only if,

$$f(t) = \sum_{n=0}^{\infty} a_n \left[\frac{c(t-1)}{2} + \frac{t+1}{2}\right]^n, \quad t \in [-1,1],$$

with all the a_n nonnegative and the series being convergent at $t = 1$. In the case $c = 1$, the situation may be even worse because if $\beta_m < \alpha_m$, $m = 1, 2, \ldots$, then the monomials t^n, $n \geq 1$, do not belong to any of the classes $\mathcal{F}^{\alpha_m,\beta_m}$.

4. Application to positive defineteness on manifolds

In this section, we will translate the main result of the paper into the setting of positive definiteness on certain manifolds. Let us write $\mathbb{H}$ to denote a compact two-point homogeneous space and assume that the geodesic distance on $\mathbb{H}$ fulfills the following requirement: all geodesics have the same length 2π. We will consider isotropic kernels on $\mathbb{H}$, that is, kernels having the form

$$K(x,y) = K_i^d(\cos|xy|/2), \quad x, y \in \mathbb{H},$$

for some continuous function $K_i^d : [-1,1] \to \mathbb{R}$, where $|xy|$ denotes the geodesic distance from x to y on $\mathbb{H}$. The function K_i^d is usually called the *isotropic part* of K. The upper index d will refer to the dimension of the spaces in accordance with the well-known classification for them provided by Wang in [20] a long time ago: the unit spheres S^d, $d = 1, 2, \ldots$, the real projective spaces $\mathbb{P}^d(\mathbb{R})$, $d = 2, 3, \ldots$, the complex projective spaces $\mathbb{P}^d(\mathbb{C})$, $d = 4, 6, \ldots$, the quaternionic projective spaces $\mathbb{P}^d(\mathbb{Q})$, $d = 8, 12, \ldots$, and the Cayley projective plane $\mathbb{P}^d(Cay)$, $d = 16$.

According to R. Gangolli ([8]), a kernel K as above is positive definite if, and only if, its isotropic part has a representation in the form

$$K_i^d(t) = \sum_{k=0}^{\infty} a_k^{(d-2)/2,\beta} P_k^{((d-2)/2,\beta)}(t), \quad t \in [-1,1],$$

in which $a_k^{(d-2)/2,\beta} \in [0,\infty)$, $k \in \mathbb{Z}_+$ and $\sum_{k=0}^{\infty} a_k^{(d-2)/2,\beta} P_k^{((d-2)/2,\beta)}(1) < \infty$, with the number β assuming the values $\beta = (d-2)/2, -1/2, 0, 1, 3$, depending on the respective category $\mathbb{H}$ belongs to, among those mentioned above. In particular, this result encompasses Schoenberg's characterization for positive definite functions on spheres mentioned at the introduction and proved in [16]. In the case of real projective spaces ($\beta = -1/2$), which can be thought of as spheres with antipodal points identified, the quadratic transformation

$$\frac{P_{2k}^{(\alpha,\alpha)}(t)}{P_{2k}^{(\alpha,\alpha)}(1)} = \frac{P_k^{(\alpha,-1/2)}(2t^2-1)}{P_k^{(\alpha,-1/2)}(1)}, \quad t \in [-1,1], \quad k = 0, 1, \ldots,$$

provides an alternative series representation for K_i^d which agrees with the setting in [16].

Just for the record, a large class of concrete examples of kernels fitting Gangolli's representation theorem can be constructed via Theorem 3.4 in [13]. Indeed, let $F : [0,\infty) \to [0,\infty)$ be nonconstant and continuous in its domain and completely monotone in $(0,\infty)$. If $g : [-1,1] \to \mathbb{R}$ is of negative type, in the sense that, $g(0) \geq 0$ and the kernel $(x,y) \in \mathbb{H}^2 \to g(\cos(|xy|/2))$ is negative definite according to [4, P. 67], then the kernel

$$(x,y) \in \mathbb{H}^2 \to F(g(\cos(|xy|/2)))$$

is positive definite. In particular, the use of

$$g(t) = a + b\left[1 - R_1^{((d-2)/2,\beta)}(t)\right], \quad t \in [-1,1],$$

with $a, b \geq 0$, provides an easy way to construct parameterized families of positive definite kernels on $\mathbb{H}$ accordingly.

Since the semigroup class $\bigcap_{d\geq 2} \mathcal{F}^{(d-2)/2,(d-2)/2}$ was characterized by Schoenberg, below we will describe the corresponding results for the semigroup classes $\bigcap_{d=1}^{\infty} \mathcal{F}^{(d-2)/2,-1/2}$, $\bigcap_{d=2}^{\infty} \mathcal{F}^{(d-2)/2,0}$ and $\bigcap_{d=4}^{\infty} \mathcal{F}^{(d-2)/2,1}$.

Theorem 4.1. *Let $f : [-1,1] \to \mathbb{R}$ be continuous. The following assertions are equivalent:*
(i) f has a representation in the form

$$f(t) = \sum_{k=0}^{\infty} a_n \left(\frac{1+t}{2}\right)^n, \quad t \in [-1,1],$$

in which all the a_n *are nonnegative and* $\sum_{n=0}^{\infty} a_n < \infty$;
(ii) f *belongs to* $\bigcap_{d=2}^{\infty} \mathcal{F}^{(d-2)/2,-1/2}$, *that is, the kernel* $(x,y) \in \mathbb{P}^d(\mathbb{R}) \times \mathbb{P}^d(\mathbb{R}) \to f(\cos|xy|/2)$ *is positive definite for* $d = 2, 3, \ldots$;
(iii) f *belongs to* $\bigcap_{d=2}^{\infty} \mathcal{F}^{(d-2)/2,0}$, *that is, the kernel* $(x,y) \in \mathbb{P}^d(\mathbb{C}) \times \mathbb{P}^d(\mathbb{C}) \to f(\cos|xy|/2)$ *is positive definite for* $d = 4, 6, \ldots$;
(iv) f *belongs to* $\bigcap_{d=4}^{\infty} \mathcal{F}^{(d-2)/2,1}$, *that is, the kernel* $(x,y) \in \mathbb{P}^d(\mathbb{Q}) \times \mathbb{P}^d(\mathbb{Q}) \to f(\cos|xy|/2)$ *is positive definite for* $d = 8, 12, \ldots$.

Proof. In view of Theorems 3.2 and 3.3, we only have to prove that (ii) implies (i). Let f belong to $\bigcap_{d=2}^{\infty} \mathcal{F}^{(d-2)/2,-1/2}$, for some $d \geq 2$. Clearly, the function $g(t) := f(2t^2-1)$ belongs to $\mathcal{F}^{(d-2)/2,(d-2)/2}$. Since g is an even function, the odd indexed coefficients of g in its Fourier-Jacobi expansion as an element of $\mathcal{F}^{(d-2)/2,(d-2)/2}$ are zero. Thus,

$$g(s) = \sum_{n=0}^{\infty} a_n s^{2n}, \quad s \in [-1,1],$$

in which all the a_n are nonnegative and $\sum_{n=0}^{\infty} a_n < \infty$. But this corresponds to

$$f(t) = \sum_{n=0}^{\infty} a_n \left(\frac{1+t}{2}\right)^n, \quad t \in [-1,1],$$

and the proof is complete. □

By expanding the binomial $(1+t)^n$, it is easily seen that a function f satisfying any of the four conditions in the previous theorem also belongs to $\bigcap_{d\geq 2} \mathcal{F}^{(d-2)/2,(d-2)/2}$, for $d = 1, 2, \ldots$. The same is true for a function f as in the statement of Theorem 3.3.

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