Let $k$ be an algebraically closed field, and $Q$ a finite acyclic quiver. The modules which we consider are the (finite-dimensional) $kQ$-modules, where $kQ$ is the path algebra of $Q$, thus the (finite-dimensional) representations of $Q$ (with coefficients in $k$). We denote by $\text{mod}kQ$ the corresponding module category.

Let $M$ be a representation of $Q$ and let $d$ be a dimension vector for $Q$. The quiver Grassmannian $G_d(M)$ is the set of submodules of $M$ with dimension vector $\text{dim} M = d$; this is a projective variety. A famous result of Zimmermann-Huisgen, Hille and Reineke asserts that any projective variety occurs as the quiver Grassmannian for some wild acyclic quiver $Q$; see for example [3]. We are going to show:

**Theorem.** Let $Q$ be any wild acyclic quiver. Any projective variety occurs as a quiver Grassmannian $G_d(M)$ for some representation $M$ of $Q$ and some dimension vector $d$.

Typical wild acyclic quivers are the Kronecker quivers $Q = K(n)$ with $n \geq 3$ (the Kronecker quiver $K(n)$ has two vertices 1 and 2 and $n$ arrows pointing from 2 to 1). A representation of $K(n)$ will be said to be reduced provided $N$ has no simple injective direct summand. In [3] we have shown that for any projective variety $V$ there is a natural number $n$ (depending on $V$) such that $V$ can be realized as the quiver Grassmannian $G_{(1,1)}(N)$ of a reduced representation $N$ of $K(n)$ (see also [1]). Our present investigation relies on this special case.

Note that the elements of $G_{(1,1)}(N)$ are certain submodules of $N$ of length 2, and all the indecomposable submodules of length 2 belong to $G_{(1,1)}(N)$. We call indecomposable modules of length 2 bristles. For any representation $N$ of $K(n)$, the set $\beta(N)$ of bristle submodules of $N$ is an open subset of $G_{(1,1)}(N)$ which we call the bristle variety of $N$. In general, $\beta(N)$ is a proper subset of $G_{(1,1)}(N)$, but for a reduced representation $N$, we have $\beta(N) = G_{(1,1)}(N)$.

The procedure of the present paper is as follows: Given any wild acyclic quiver $Q$, and a natural number $m$, we will construct for some $n \geq m$ an orthogonal pair $X, Y$ of bricks with $\text{dim Ext}^1(Y, X) = n$ (a brick is a module with endomorphism ring $k$ and $X, Y$ are said to be orthogonal provided $\text{Hom}(X, Y) = 0 = \text{Hom}(Y, X)$).
Always, \( x \) and \( y \) will denote the dimension vectors of \( X \) and \( Y \), respectively. Let \( \mathcal{E} = \mathcal{E}(Y, X) \) be the full subcategory of all \( kQ \)-modules \( M \) with an exact sequence of the form

\[
0 \rightarrow X^a \rightarrow M \rightarrow Y^b \rightarrow 0,
\]

where \( a, b \) are natural numbers. Note that \( \mathcal{E} \) is equivalent to \( \text{mod} \, kK(n) \) with an equivalence being given by an exact fully faithful functor

\[
\eta: \text{mod} \, kK(n) \rightarrow \text{mod} \, kQ
\]

with image \( \mathcal{E} \). We say that a module \( M \) in \( \mathcal{E} \) is \( \mathcal{E} \)-reduced provided it has no direct summand isomorphic to \( Y \), thus provided it is the image of a reduced \( kK(n) \)-module under \( \eta \).

An indecomposable \( kQ \)-module \( U \) will be called an \( \mathcal{E} \)-bristle provided there is an exact sequence of the form \( 0 \rightarrow X \rightarrow U \rightarrow Y \rightarrow 0 \), thus provided \( U \) is the image of a bristle in \( \text{mod} \, kK(n) \) under \( \eta \). For any \( kK(n) \)-module \( N \) with \( M = \eta N \), the functor \( \eta \) identifies the bristle variety \( \beta(N) \) of \( N \) with the set \( \beta_\mathcal{E}(M) \) of submodules of \( M \) which are \( \mathcal{E} \)-bristles. Since \( \mathcal{E} \)-bristles have dimension vector \( x + y \), we have \( \beta_\mathcal{E}(M) \subseteq \mathbb{G}_{x+y}(M) \). It remains to find conditions such that any submodule \( U \) of \( M \) with dimension vector \( x + y \) is indeed an \( \mathcal{E} \)-bristle.

To be precise, we are looking for \( kQ \)-modules \( X, Y \) so that the following closure condition \((C)\) is satisfied:

\((C)\) If \( M \) is an \( \mathcal{E} \)-reduced module in \( \mathcal{E}(Y, X) \) and \( U \) is a submodule of \( M \) with \( \dim U = x + y \), then \( U \) is an \( \mathcal{E} \)-bristle.

If the condition \((C)\) is satisfied, then for any reduced representation \( N \) of \( K(n) \), there is a canonical bijection between \( \mathbb{G}_{(1,1)}(N) \) and \( \mathbb{G}_{x+y}(M) \), where \( M = \eta N \). Namely, if \( B \) is a submodule of the \( kK(n) \)-module \( N \) with \( \dim B = (1, 1) \), then \( \eta B \) is a submodule of \( M \) with dimension vector \( x + y \). Conversely, if \( U \) is a submodule of \( M \) with \( \dim U = x + y \), then, by condition \((C)\), \( U \) belongs to \( \mathcal{E}(Y, X) \), say \( U = \eta B \) for some \( k(n) \)-submodule \( B \) and the dimension vector of \( B \) is \( (1, 1) \).

**The minimal wild acyclic quivers.** As we have mentioned, our aim is to exhibit for any wild acyclic quiver \( Q \) and any natural number \( m \) an orthogonal pair \( X, Y \) of \( kQ \)-modules which are bricks such that \( \dim_k \text{Ext}^1(Y, X) = n \geq m \) and such that the condition \((C)\) is satisfied. Of course, it is sufficient to deal with minimal wild acyclic quivers. (We recall that a quiver \( Q \) is wild provided it is not the disjoint union of Dynkin and Euclidean quivers, and \( Q \) is said to be minimal wild provided it is wild, and no quiver obtained from \( Q \) by deleting a vertex or an arrow is wild.)

The following well-known proposition suggests to deal with two different cases.

**Proposition.** A minimal wild acyclic quiver \( Q \) different from \( K(3) \) is obtained from a Euclidean quiver \( Q' \) by adding a vertex \( \omega \) and a single arrow which connects \( \omega \) with some vertex of \( Q' \) (in particular, \( \omega \) is a sink or a source).

**Sketch of proof.** If \( Q \) has cycles, then there is a subquiver \( Q' \) of type \( \widetilde{A}_n \) for some \( n \) such that \( Q' \) is obtained from \( Q \) by deleting one vertex and one arrow.

Now assume that \( Q \) is a tree. If there is a vertex with at least four neighbors, then \( Q' \) is obtained from a quiver of type \( \widetilde{D}_4 \) by deleting one vertex and one arrow. If \( Q \) has two vertices which have three neighbors each, then \( Q' \) is obtained from a quiver of type \( \widetilde{D}_n \) with \( n \geq 5 \) by deleting one vertex and one arrow. If \( Q \) has a star
with three arms, then $Q'$ is obtained from a quiver of type $\widetilde{E}_m$ with $m = 6, 7, 8$ by deleting one vertex and one arrow.

**Case 1** (One-point extensions of representation-infinite quivers). We assume now that $Q$ is a connected quiver with a vertex $\omega$ which is a sink or a source such that the quiver $Q'$ obtained from $Q$ by deleting $\omega$ and the arrows which start or end in $\omega$ is connected and representation-infinite. Up to duality, we can assume that $\omega$ is a source, thus there is an arrow $\omega \to p$ with $p \in Q'_0$.

Let $Y = S(\omega)$, the simple $kQ'$-module corresponding to the vertex $\omega$. Since $Q'$ is connected and representation-infinite, there is an exceptional $kQ'$-module $X$ with $\dim_k X_p = m$. The arrow $\omega \to p$ shows that $\dim_k \operatorname{Ext}^1(Y, X) \geq \dim_k X_p$. This pair $X,Y$ is the orthogonal pair of bricks which we use in order to look at $\mathcal{E}(Y, X)$.

**Lemma 1.** Let $a$ be a natural number. Any submodule $W$ of $X^a$ with $\dim W = x$ is isomorphic to $X$.

**Proof.** We denote by $\langle - , - \rangle$ the bilinear form on the Grothendieck group $K_0(kQ)$ with $\langle \dim M, \dim M' \rangle = \dim_k \operatorname{Hom}(M, M') - \dim_k \operatorname{Ext}^1(M, M')$. Since $X$ is exceptional, we have $\langle X, W \rangle = \langle X, X \rangle > 0$. Therefore, there is a non-zero homomorphism $f : X \to W$. Let $\iota : W \to X^a$ be the inclusion map. The composition $\iota f : X \to X^a$ is non-zero. Since $X$ is a brick, we see that $f : X \to W$ is a split monomorphism, in particular injective. Now $\dim X = \dim W$ implies that $f$ is an isomorphism. □

**Proof of condition (C).** Let $M$ be an $\mathcal{E}$-reduced $kQ$-module in $\mathcal{E}(Y, X)$, say with an exact sequence

$$
0 \longrightarrow X^a \xrightarrow{\mu} M \xrightarrow{\pi} Y^b \longrightarrow 0.
$$

Let $U$ be a submodule of $M$ with dimension vector $x + y$ and inclusion map $\iota : U \to M$. The composition $\pi \iota$ is non-zero, since otherwise $U$ would be a submodule of $X^a$, but $\dim_k U_\omega = 1$ whereas $X_\omega = 0$. If follows that the image of $\pi \iota$ is isomorphic to $Y$. If we denote the kernel of $\pi \iota$ by $W$, we obtain the following commutative diagram with exact rows and vertical monomorphisms:

$$
\begin{array}{ccc}
0 & \longrightarrow & W \\
\text{ } & \downarrow & \text{ } \\
0 & \longrightarrow & X^a \\
\end{array}
\quad \begin{array}{ccc}
M & \xrightarrow{\pi} & Y^b \\
\mu & \text{ } & \text{ } \\
\iota & \text{ } & \text{ } \\
0 & \longrightarrow & 0
\end{array}
$$

Of course, $\dim W = x$, thus Lemma □ shows that $W$ is isomorphic to $X$. In particular, $U$ belongs to $\mathcal{E}$.

It remains to show that $U$ is indecomposable. Otherwise, $U$ would be isomorphic to $W \oplus Y$. Thus $M$ would have a submodule isomorphic to $Y$. But $Y$ is relative injective inside $\mathcal{E}$, thus $M$ would have a direct summand isomorphic to $Y$, in contrast to our assumption that $M$ is $\mathcal{E}$-reduced. This shows that $U$ is indecomposable, thus an $\mathcal{E}$-bristle. □

**Case 2** (The Kronecker quiver $K(3)$). Here we consider the Kronecker quiver $Q = K(3)$, with the three arrows $\alpha, \beta, \gamma : 2 \to 1$. Let $\lambda_1, \ldots, \lambda_n$ be pairwise different non-zero elements of $k$ with $n \geq 2$. Let $X = X(\lambda_1, \ldots, \lambda_n) = (k^n, k^n; \alpha, \beta, \gamma)$ be defined by

$$
\alpha(e(i)) = e(i), \quad \beta(e(i)) = \lambda_i e(i), \quad \gamma(e(i)) = e(i+1),
$$

It remains to show that $X$ is injective. Otherwise, $X$ would be isomorphic to $k^n \oplus X$ and $X$ would have a submodule isomorphic to $k^n$. But $k^n$ is relative injective in $\mathcal{E}$, thus $X$ would have a direct summand isomorphic to $k^n$, in contrast to our assumption that $X$ is indecomposable. This shows that $X$ is injective. □
for $1 \leq i \leq n$, where $e(1), \ldots, e(n)$ is the canonical basis of $k^n$ and $e(n + 1) = e(1)$.

Let $Y = (k, k; 1, 0, 0)$. We denote by $Q'$ the subquiver of $Q$ with arrows $\alpha, \beta$, this is the 2-Kronecker quiver $K(2)$. For the structure of the category mod $K(2)$, see for example [2]. The restriction of $X, Y$ to $Q'$ shows that $\text{Hom}(X, Y) = \text{Hom}(Y, X) = 0$. The endomorphism ring of $X_{|Q'}$ is $k \times \cdots \times k$; and the only endomorphisms of $X_{|Q'}$ which commute with $\gamma$ are the scalar multiplications. This shows that $X$ is a brick. Also, it is easy to see that $\dim_k \text{Ext}^1(Y, X) = n$.

**Lemma 2.** Let $a$ be a natural number. Any submodule $W$ of $X^n$ with $\dim W$ of the form $(w, w)$ is isomorphic to $X^s$ for some $s$.

**Proof.** Let $M = X^n$ and decompose $M_{|Q'} = \bigoplus_{i=1}^n M(i)$, where $\beta(x) = \lambda_ix$ for $x \in M(i)_1$. Here, we use $\alpha$ in order to identify $M_1$ and $M_2$. Now we consider the submodule $W$ of $M$. Note that $W_{|Q'}$ has to be regular, since it cannot have any non-zero preinjective direct summand. As a regular submodule of a semisimple regular Kronecker module it has to be a direct summand of $M_{|Q'}$, thus we have a similar direct decomposition $W = \bigoplus W(i)$, where $W(i) = W \cap M(i)$.

The linear map $\gamma$ restricted to $W(i)_1$ is a monomorphism $W(i)_1 \rightarrow W(i_1+1)$ for $1 \leq i \leq n$; we obtain in this way a monomorphism $W(1)_1 \rightarrow W(1)$, which shows that all the monomorphisms $W(i)_1 \rightarrow W(i_1+1)$ are actually bijections. Let $\dim_k W(1)_1 = s$. It follows that $W$ is isomorphic to $X^s$. □

**Proof of condition (C).** Let $M$ be an $E$-reduced $kQ$-module in $E$ and let $U$ be a submodule of $M$ with dimension vector $x + y = (n + 1, n + 1)$ and with inclusion map $\iota : U \rightarrow M$.

Starting with the exact sequence $0 \rightarrow X^n \xrightarrow{\mu} M \xrightarrow{\pi} Y^b \rightarrow 0$ and the inclusion map $\iota : U \rightarrow M$, let $W$ be the kernel and let $\overline{U}$ be the image of $\pi \iota : U \rightarrow Y^b$. We obtain the following commutative diagram with exact rows and injective vertical maps:

$$
\begin{array}{cccccc}
0 & \rightarrow & W & \rightarrow & U & \rightarrow & \overline{U} & \rightarrow & 0 \\
0 & \rightarrow & X^n & \xrightarrow{\mu} & M & \xrightarrow{\pi} & Y^b & \rightarrow & 0.
\end{array}
$$

Let us consider the restriction of these modules to $Q'$. Since $M_{|Q'}$ is regular, it has no non-zero preinjective direct summand. Thus any submodule of $M_{|Q'}$ with dimension vector $(n + 1, n + 1)$ has to be regular. This shows that $U_{|Q'}$ is regular. Actually, $M_{|Q'}$ is semisimple regular, thus also its regular submodule $U_{|Q'}$ is semisimple regular (and a direct summand of $M_{|Q'}$). Next, $\pi \iota$ is a map between regular $kQ'$-modules. It follows that the kernel $W_{|Q'}$ and the image $\overline{U}_{|Q'}$ are regular $kQ'$-modules. In particular, the dimension vector of $W$ is of the form $\dim W = (w, w)$ for some $0 \leq w \leq n + 1$.

Now $\overline{U}_{|Q'}$ is a regular submodule of the semisimple regular $kQ'$-module $Y^b_{|Q'}$, thus $\overline{U}_{|Q'}$ is a direct sum of copies of $Y_{|Q'}$. By construction, $Y$ is annihilated by $\gamma$. Since $\overline{U}$ is a submodule of $Y^b$, it follows that $\overline{U}$ is annihilated by $\gamma$. Altogether, we see that $\overline{U}$ is the direct sum of copies of $Y$.

We claim that $W \neq 0$. Otherwise $U = \overline{U} = Y^{n+1}$, thus $Y$ is a direct summand of $M$. However, by assumption, $M$ is $E$-reduced. This contradiction shows that $W \neq 0$. 

Now $W$ is a submodule of $X^a$ with dimension vector $(w, w)$, thus, according to Lemma 2, $W$ is a direct summand of say $s$ copies of $X$ and $s \geq 1$. The equality $(w, w) = (sn, sn)$ implies that $s = 1$, since $w \leq n + 1$ and $n \geq 2$. In this way, we see that $W$ is isomorphic to $X$. It follows that $\dim \underline{U} = (1, 1)$ and therefore $\underline{U} = Y$.

Finally, as in Case 1, we see that $U$ is indecomposable, using again the assumption that $M$ is $E$-reduced. This shows that $U$ is an $E$-bristle. □

Remark. We should stress that given orthogonal bricks $X, Y$ in mod $kQ$, the condition (C) is usually not satisfied. Here is a typical example for $Q = K(3)$. As above, let $Y = (k, k; 1, 0, 0)$, but for $X$ we now take $X = X'(\lambda_1, \lambda_2) = (k^2, k^2; \alpha, \beta, \gamma)$, defined by

\[
\alpha(e(i)) = e(i), \quad \beta(e(i)) = \lambda_i e(i), \quad \gamma(e(1)) = e(2), \quad \gamma(e(2)) = 0
\]

for $1 \leq i \leq 2$. Again, $e(1), e(2)$ is the canonical basis of $k^2$ and $\lambda_1 \neq \lambda_2$ are assumed to be non-zero elements of $k$. Since $\dim_k \text{Ext}^1(Y, X) = 2$, there is an equivalence $\eta : \text{mod } kK(2) \to \mathcal{E}(Y, X)$. Let $N$ be an indecomposable $kK(2)$-module with dimension vector $(2, b)$ (note that $b$ has to be equal to 1, 2 or 3) and $M = \eta N$. Thus there is an exact sequence

\[
0 \longrightarrow X^2 \longrightarrow M \longrightarrow Y^b \longrightarrow 0.
\]

Since we assume that $N$ is indecomposable, it is reduced, thus $M$ is $E$-reduced. Note that $X$ has a (unique) $kQ$-submodule $V$ with dimension vector $(1, 1)$: the vector spaces $V_1$ and $V_2$ both are generated by $e(2)$. The submodule $U = X \oplus V$ of $X^2$ is a submodule of $M$ with dimension vector $(3, 3) = x + y$, and it is not an $E$-bristle. Thus, condition (C) is not satisfied. Here, $\eta$ defines a proper embedding of $\beta(N) = \mathbb{G}_{(1, 1)}(N)$ into $\mathbb{G}_{x+y}(M)$.

References


Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany
Email address: ringel@math.uni-bielefeld.de