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EXPRESSION OF TIME ALMOST PERIODIC TRAVELING WAVE SOLUTIONS TO A CLASS OF COMPETITION DIFFUSION SYSTEMS

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ABSTRACT. In this paper we consider a class of competition diffusion systems with time almost periodic coefficients. We show that any *almost periodic traveling wave solution* to such a system is given by a decomposition formula, that is, each component of the solution equals the product of the corresponding diffusion coefficient and the classical traveling wave solution (with a different time scale).

1. Introduction

In this paper we present the expression of traveling wave solutions to the following time almost periodic system:

$$(u_i)_t = \mu_i d(t)(u_i)_{xx} + u_i\Big[r(t) - \sum_{j=1}^{n} a_{ij}u_j\Big], \quad i = 1, 2, \cdots, n, \tag{P}$$

where $d(t)$ and $r(t)$ are continuous almost periodic functions (see Definition 3.1) with positive lower bounds, $\mu_i,\ a_{ij}$ are positive real numbers, and $u_i(x,t)$ denotes the population density of the i-th competing species at position x and time t. This model is a special kind of competition diffusion system, which is used in ecology problems to describe n species moving by diffusion, with competition amongst the species. Time almost periodic coefficients allow one to take into account general seasonal variations.

We are interested in the traveling wave solutions to the system (P). In case $n = 1$, that is, the system is a scalar equation, the study of traveling wave solutions traces back to the pioneer works of R. Fisher [6] and A. Kolmogorov et al. [13]. They proved that the diffusive logistic equation,

$$u_t = u_{xx} + u(1-u),$$

has (classical) traveling wave solutions of the form $u(x,t) = \phi(x-ct)$ for each $c \geq 2$. In the 2000s the authors of [9, 10, 14–16] studied the equation,

$$u_t = u_{xx} + f(u,t), \tag{1.1}$$

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with $f(u,t)$ being monostable in u and periodic in t, like $f = u(r(t) - u)$ for some positive periodic r (they actually studied much more general equations involving shear flows or time-delays). Among others, they obtained the *time periodic traveling wave solutions*, which have the form $\phi(x - ct, t)$ for some bounded function $\phi(z,t)$ being monotone in z and periodic in t. In fact, this kind of solution was first studied in [1] in 1999, for the equation (1.1) with bistable nonlinearity. Also in 1999, in a series of papers [17, 18], W. Shen considered (1.1) with f being bistable in u and *almost periodic in* t. It turns out that, the traveling wave solution in this case is a time almost periodic one (see Definition 1.1 below).

In case $n \geq 2$, (P) is a competition diffusion system. There are also a lot of studies about the traveling wave solutions, especially for the homogeneous system:

$$(P0) \qquad (w_i)_t = \mu_i (w_i)_{xx} + w_i \Big[r_0 - \sum_{j=1}^{n} k_{ij} w_j \Big], \quad i = 1, 2, \cdots, n,$$

where μ_i, r_0, k_{ij} are all positive real numbers. A traveling wave solution to this system is a solution $(w_1, \cdots, w_n)$ with the form

$$w_i(x,t) = \phi_i(x - ct) \quad \text{for some } c \in \mathbb{R} \text{ and all } i = 1, \cdots, n,$$

$\phi_i(\pm\infty) = \alpha_i^{\pm}$, both $(\alpha_1^-, \cdots, \alpha_n^-)$ and $(\alpha_1^+, \cdots, \alpha_n^+)$ are constant stationary solutions to (P0) (cf. [5, 7, 11, 12, 19, 20], etc.). When $d(t)$ and $r(t)$ in (P) are T-periodic functions, the traveling wave solutions should be periodic ones with the form

$$u_i(x,t) = \phi_i(x - ct, t) \quad \text{for some } c \in \mathbb{R} \text{ and all } i = 1, \cdots, n,$$

with $\phi_i(z,t)$ being T-periodic in t, $\phi_i(\pm\infty, t) = \alpha_i^{\pm}(t)$, both $(\alpha_1^-(t), \cdots, \alpha_n^-(t))$ and $(\alpha_1^+(t), \cdots, \alpha_n^+(t))$ are T-periodic solutions to the related kinetic system

$$(1.2) \qquad (u_i)_t = u_i \Big[r(t) - \sum_{j=1}^{n} a_{ij} u_j \Big], \quad i = 1, 2, \cdots, n.$$

This kind of wave has been studied in [2, 21], etc. Furthermore, if d and r in (P) are almost periodic in t, the traveling wave solutions should be almost periodic ones, and can be defined as follows (cf. [17, 18]):

Definition 1.1. A solution $(u_1, \cdots, u_n)$ of (P) is called an *almost periodic traveling wave solution* connecting $(\alpha_1^-(t), \cdots, \alpha_n^-(t))$ with $(\alpha_1^+(t), \cdots, \alpha_n^+(t))$ if

(1) for each $i = 1, 2, \cdots, n$, $u_i(x,t) = \Phi_i(x - c(t), t)$ for some $\Phi_i(z,t) \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, which is almost periodic in t uniformly with respect to z in bounded sets;
(2) for each $i = 1, 2, \cdots, n$, $\Phi_i(\pm\infty, t) = \alpha_i^{\pm}(t)$, where $(\alpha_1^-(t), \cdots, \alpha_n^-(t))$ and $(\alpha_1^+(t), \cdots, \alpha_n^+(t))$ are almost periodic solutions to the kinetic system (1.2);
(3) $c'(t)$ is almost periodic in t.

Note that most studies on traveling wave solutions concern their existence and stability rather than their explicit expressions. Especially, in time-dependent cases, very little is known about the wave profiles and the propagating speeds. The aim of this paper is to give the expressions for almost periodic traveling wave solutions to (P) by using a decomposition formula. As can be seen in Section 3, one can specify the profiles, the instantaneous speeds and the limiting functions of the traveling wave solutions clearly by such expressions. The expressions are *explicit* in the sense that each component of the wave is expressed as the product of the diffusion coefficient $\mu_i d(t)$ and the classical traveling wave solution to the homogeneous system (P0) (with different time scale).

This paper is organized as follows. In Section 2 we present a decomposition formula by using the idea in [4]. In Section 3 we give the expression of almost periodic traveling wave solutions to (P). In Section 4 we present some examples, whose almost periodic traveling wave solutions are given by exact formulas.

2. A decomposition formula

In this section we present a decomposition formula for the solutions to (P).

Theorem 2.1. *Let* $(w_1, \cdots, w_n)$ *be a solution to* (P0). *Assume* $r(t)$ *is a positive continuous function (not necessarily to be almost periodic) and* $d(t)$ *is a positive solution to the following scalar logistic equation:*

$$d'(t) = d(t)\Big[r(t) - r_0 d(t)\Big]. \tag{2.1}$$

If a_{ij} *in* (P) *satisfies*

$$\mu_j a_{ij} = k_{ij}, \quad i, j = 1, 2, \cdots, n, \tag{2.2}$$

then $(u_1, u_2, \cdots, u_n)$ *given by*

$$u_i(x,t) = \mu_i d(t) \cdot w_i\Big(x, \int_0^t d(s)ds\Big), \qquad i = 1, 2, \cdots, n, \tag{2.3}$$

is a solution to (P).

Proof. Denote $\tau := \int_0^t d(s)ds$. For each $i = 1, \cdots, n$, by a direct calculation we have

$$(u_i)_t = \mu_i d'(t) w_i(x,\tau) + \mu_i d^2(t)(w_i)_\tau(x,\tau), \quad (u_i)_{xx} = \mu_i d(t)(w_i)_{xx}(x,\tau).$$

Using (2.1) and (2.2) one concludes that

$$\begin{aligned}(u_i)_t - \mu_i d(t)(u_i)_{xx} &= \mu_i d(t)[r(t) - r_0 d(t)]w_i + \mu_i d^2(t)\ w_i\Big[r_0 - \sum_{j=1}^n k_{ij} w_j\Big] \\ &= u_i\Big[r(t) - \sum_{j=1}^n k_{ij} d(t) w_j\Big] = u_i\Big[r(t) - \sum_{j=1}^n a_{ij} u_j\Big].\end{aligned}$$

This shows that $(u_1, \cdots u_n)$ is a solution to (P). □

Remark 2.2. The equality in (2.3) is a diffusive version of the decomposition formula for time-perturbed Lotka-Volterra systems of ordinary differential equations, which was first discovered by X. Chen, J. Jiang and L. Niu in [4], where the authors proved that every solution for time-perturbed Lotka-Volterra systems with identical intrinsic growth rate is expressed as the product of a solution for the corresponding deterministic Lotka-Volterra system without perturbation and the solution of the scalar logistic equation. Using this formula, they proved that the skew-product flow generated by the almost periodic Lotka-Volterra systems is totally determined by the long-term behavior for the corresponding deterministic systems and thoroughly classified the dynamics on three dimensional almost periodic Lotka-Volterra systems.

Remark 2.3. It is not difficult to verify that the decomposition formula (2.3) remains valid for some other systems. For example, it applies to

(1). reaction-diffusion-advection systems like

$$(u_i)_t = \mu_i d(t)(u_i)_{xx} + \beta_i d(t)(u_i)_x + u_i\Big[r(t) - \sum_{j=1}^{n} a_{ij}u_j\Big], \quad i = 1, \cdots, n;$$

(2). competition systems with nonlocal dispersal

$$\begin{aligned}(u_i)_t = \mu_i d(t) &\int_{\mathbb{R}} \kappa(y-x)u_i(y,t)dy \\ &- \mu_i d(t)u_i(x,t) + u_i\Big[r(t) - \sum_{j=1}^{n} a_{ij}u_j\Big], \quad i = 1, \cdots, n.\end{aligned}$$

Hence, the decomposition formula for the almost periodic traveling wave solutions to (P) (see (3.1) in the next section) remains valid for the traveling wave solutions to these two systems.

Remark 2.4. The system (P) we considered here is a special one in the sense that all the species have synchronic diffusion coefficients $\mu_i d(t)$, the same intrinsic growth rate $r(t)$, and time-independent interacting coefficients a_{ij}. These restrictions, however, cannot be omitted simply. For example, if we replace $\mu_i d(t)$ by $d_i(t)$ as the diffusion coefficients, then the new time scale τ should be replaced by $\tau_i = \int_0^t d_i(s)ds$ which are different from each other, and so, $w_i(x,\tau_i)$ cannot be unified in one formula since they have different time scales. For the same reason, neither $r(t)$ nor a_{ij} can be extended to more general cases simply.

3. Expression of almost periodic traveling wave solutions

In this section we show that each almost periodic traveling wave solution is given by a decomposition formula.

3.1. **Almost periodicity.** A set $A \subset \mathbb{R}$ is called *relatively dense* in $\mathbb{R}$ if there exists $M > 0$ such that any interval of the form $[b, b+M]$ contains a point in A.

Definition 3.1. A bounded continuous function $g : \mathbb{R} \to \mathbb{R}$ is called *almost periodic* in the sense of Bohr if, for any $\varepsilon > 0$, the following set is relatively dense in $\mathbb{R}$:

$$A_\varepsilon := \{a \in \mathbb{R} \mid \|g(a+\cdot) - g(\cdot)\|_{L^\infty(\mathbb{R})} < \varepsilon\}.$$

For any almost periodic function $g : \mathbb{R} \to \mathbb{R}$, it has a *mean value* which is defined by

$$\mathcal{M}\{g\} := \lim_{T\to+\infty} \frac{1}{T}\int_0^T g(s)ds.$$

For a condition to guarantee that an indefinite integral for an almost periodic function with zero mean value is also almost periodic, we refer to [4, Proposition 5.8]. Denote

$$\mathcal{A} := \{g(t) \mid g : \mathbb{R} \to \mathbb{R} \text{ is almost periodic, with } \mathcal{M}\{g\} > 0\}.$$

From [4, Lemma 5.6], we have the following result.

Lemma 3.2. *Assume $r(t) \in \mathcal{A}$. Then the equation* (2.1) *has a unique, stable almost periodic positive solution*

$$d(t) = \frac{1}{r_0}\Big(\int_{-\infty}^{t} \exp\{\int_t^s r(\tau)d\tau\}ds\Big)^{-1}.$$

Furthermore, $\mathcal{M}\{d\} = \mathcal{M}\{r\}$ and $\int_0^t d(s)ds = +\infty$.

3.2. **Expression of the traveling wave solutions.**

Theorem 3.3. *Let $(w_1, \cdots, w_n)$ with $w_i(x,t) = \phi_i(x - ct)$ $(i = 1, 2, \cdots, n)$ be a traveling wave solution of* (P0) *with $\phi_i(\pm\infty) = \alpha_i^{\pm}$, where $(\alpha_1^-, \cdots, \alpha_n^-)$ and $(\alpha_1^+, \cdots, \alpha_n^+)$ are constant stationary solutions to* (P0)*. Assume $r \in \mathcal{A}$, $d(t)$ is the unique positive almost periodic solution of* (2.1) *and $\mu_j a_{ij} = k_{ij}$ $(i, j = 1, \cdots, n)$. Then $(u_1, \cdots, u_n)$ given by*

$$u_i(x,t) = \mu_i d(t) \cdot \phi_i\Big(x - c\int_0^t d(s)ds\Big), \quad i = 1, \cdots, n, \tag{3.1}$$

is an almost periodic traveling wave solution *to* (P)*.*

Proof. By Theorem 2.1, $(u_1, \cdots, u_n)$ with u_i defined by (3.1) is a solution to (P). We only need to show that it is an almost periodic traveling wave solution as in Definition 1.1. Denote $c(t) := c\int_0^t d(s)ds$. Then for each $i = 1, \cdots, n$,

$$u_i(x,t) = \Phi_i(x - c(t), t) := \mu_i d(t) \cdot \phi_i(x - c(t)).$$

Clearly, $\Phi_i(z,t) = \mu_i d(t)\phi_i(z)$ is almost periodic in t, and $\Phi_i(\pm\infty, t) = \mu_i d(t)\alpha_i^{\pm}$.

Now we show that $(\mu_1\alpha_1^{\pm}d(t), \cdots, \mu_n\alpha_n^{\pm}d(t))$ are solutions to the kinetic system (1.2). In fact, $(\alpha_1^-, \cdots, \alpha_n^-)$ is a constant stationary solution to (P0), so we have

$$\alpha_i^-\Big(r_0 - \sum_{j=1}^n k_{ij}\alpha_j^-\Big) = 0, \quad i = 1, \cdots, n. \tag{3.2}$$

For any $i = 1, \cdots, n$, if $\alpha_i^- = 0$, then $\mu_i\alpha_i^- d(t) \equiv 0$, it clearly satisfies the i-th equation in (1.2). If $\alpha_i^- \neq 0$, then $r_0 = \sum_{j=1}^n k_{ij}\alpha_j^-$ by (3.2). Combining this equality with (2.1) and (2.2) we have

$$[\mu_i\alpha_i^- d(t)]' = [\mu_i\alpha_i^- d(t)]\Big(r(t) - r_0 d(t)\Big) = [\mu_i\alpha_i^- d(t)]\Big(r(t) - \sum_{j=1}^n a_{ij}[\mu_j\alpha_j^- d(t)]\Big),$$

that is, the i-th equation in (1.2) holds. This proves that $(\mu_1\alpha_1^- d, \cdots, \mu_n\alpha_n^- d)$ is a solution to (1.2). The proof for $(\mu_1\alpha_1^+ d, \cdots, \mu_n\alpha_n^+ d)$ is similar.

Finally, $c'(t) = cd(t)$ is almost periodic by Lemma 3.2. □

Remark 3.4. From this theorem we see that, if a traveling wave solution can be expressed by (3.1), then its profile, instantaneous speed and the limiting functions $\alpha_i^{\pm}(t)$ are given by

$$(\mu_1 d(t)\phi_1(z), \cdots, \mu_n d(t)\phi_n(z)), \qquad cd(t), \qquad (\mu_1\alpha_1^{\pm}d(t), \cdots, \mu_n\alpha_n^{\pm}d(t)),$$

explicitly. Hence, using the decomposition formula (3.1), one can capture the feature of the almost periodic traveling wave solution clearly.

In the above theorem we use the assumption that (P0) has traveling wave solutions, which is actually the truth in many cases. First, for scalar equation ($n = 1$), we have the following result.

Corollary 3.5. *Assume $n = 1$. Then for any $c \geq 2\sqrt{r_0}$, the (single) equation* (P0) *has a traveling wave solution $\phi_1(x - ct)$, which satisfies $\phi_1'(z) < 0$, $\phi_1(-\infty) = r_0/k_{11}$, $\phi_1(+\infty) = 0$. Moreover, when $r \in \mathcal{A}$, d satisfies* (2.1) *and $\mu_1 a_{11} = k_{11}$, and the function*

$$u_1(x,t) = \mu_1 d(t) \cdot \phi_1\Big(x - c\int_0^t d(s)ds\Big)$$

is an almost periodic traveling wave solution to (P).

For case $n = 2$, the related kinetic system of (P0) always has three equilibria $E_0 := (0,0)$, $E_1 := (\frac{r_0}{k_{11}}, 0)$, $E_2 := (0, \frac{r_0}{k_{22}})$, and when

$$g := \frac{k_{12}}{k_{11}}, \quad h := \frac{k_{21}}{k_{22}}, \tag{3.3}$$

satisfy $h, g < 1$ or $h, g > 1$, the kinetic system has a unique coexistence equilibrium $E^* := (w_1^*, w_2^*)$ given by

$$\begin{pmatrix} w_1^* \\ w_2^* \end{pmatrix} = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}^{-1} \cdot \begin{pmatrix} r_0 \\ r_0 \end{pmatrix}.$$

By a phase plane analysis, one knows that E_1 is stable and E_2 is unstable when $0 < g < 1 < h$, and E_2 is stable and E_1 is unstable when $0 < h < 1 < g$, both are called *monostable cases*; when $h, g < 1$, E^* is stable and this case is called an *coexistence case*; both E_1 and E_2 are stable when $h, g > 1$ and this is called a *bistable case* (cf. [8]). In all of these cases, traveling wave solutions to (P0) (with $n = 2$) have been studied extensively. Combining with the decomposition formula in Theorem 3.3 we have the following results.

Corollary 3.6. *Consider* (P) *and* (P0) *with $n = 2$. Let E_0, E_1, E_2, E^*, h and g be defined as above. Assume $r \in \mathcal{A}$, d satisfies* (2.1) *and $\mu_j a_{ij} = k_{ij}$ $(i, j = 1, 2)$.*

(1) *If $0 < g < 1 < h$ (resp. $0 < h < 1 < g$), then there exists $c^* < 0$ such that the system* (P0) *has a traveling wave solution $(\phi_1(x - ct), \phi_2(x - ct))$ for each $c \leq c^*$, with $(\phi_1(z), \phi_2(z))$ connecting E_2 at $-\infty$ and E_1 at $+\infty$ (resp. connecting E_1 at $-\infty$ and E_2 at $+\infty$; cf.* [11, 12]*). Moreover, (u_1, u_2) defined by* (3.1) *is an almost periodic traveling wave solution of* (P).

(2) *If $g, h > 1$, then the system* (P0) *has a traveling wave solution $(\phi_1(x - ct), \phi_2(x - ct))$ for some $c \in \mathbb{R}$, with $(\phi_1(z), \phi_2(z))$ connecting E_1 at $-\infty$ and E_2 at $+\infty$ (cf.* [5, 7]*). Moreover, (u_1, u_2) defined by* (3.1) *is an almost periodic traveling wave solution of* (P).

(3) *If $0 < g, h < 1$, then the system* (P0) *has a traveling wave solution $(\phi_1(x - ct), \phi_2(x - ct))$ for some $c > 0$, with $(\phi_1(z), \phi_2(z))$ connecting E^* at $-\infty$ and E_0 at $+\infty$ (cf.* [19, 20]*). Moreover, (u_1, u_2) defined by* (3.1) *is an almost periodic traveling wave solution of* (P).

4. Examples

Now we give some examples, whose almost periodic traveling wave solutions are given by exact formulas.

Example 4.1. Consider (P) and (P0) with $n = 1$. Assume $\mu_1 = r_0 = k_{11} = 1$. By a direct calculation one can see that (P0) has a traveling wave solution

$$\phi\Big(x + \frac{5}{\sqrt{6}}t\Big) = \left[\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\sqrt{6}}{12}\Big(x + \frac{5}{\sqrt{6}}t\Big)\right)\right]^2.$$

By Theorem 3.3,

$$u_1(x,t) = d(t)\cdot\left[\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\sqrt{6}}{12}\left(x + \frac{5}{\sqrt{6}}\int_0^t d(s)ds\right)\right)\right]^2$$

is an almost periodic traveling wave solution of (P) if d is the solution of (2.1).

Example 4.2. Consider (P) and (P0) with $n = 2$. From [3] we know that, for any $b \in (1, \frac{8}{3})$, let $\mu_1 = 1$, $\mu_2 = \frac{1}{3b}$, $r_0 = 1$, $k_{11} = k_{22} = 1$, $k_{12} = b$ and $k_{21} = \frac{11}{3} - b$, then the system (P0) has a traveling wave solution given by

$$\begin{cases} w_1(z) = \dfrac{1}{2} + \dfrac{1}{2}\tanh\dfrac{\sqrt{2b}}{4}z, \\ w_2(z) = \dfrac{1}{4}\left[1 - \tanh\dfrac{\sqrt{2b}}{4}z\right]^2, \end{cases}$$

with $z = x + \frac{2-b}{\sqrt{2b}}t$, and

$$(w_1(-\infty), w_2(-\infty)) = (0,1),\ (w_1(+\infty), w_2(+\infty)) = (1,0).$$

By Theorem 3.3, we know that

$$\begin{cases} u_1(x,t) = d(t)\left[\dfrac{1}{2} + \dfrac{1}{2}\tanh\left(\dfrac{\sqrt{2b}}{4}\left(x + \dfrac{2-b}{\sqrt{2b}}\displaystyle\int_0^t d(s)ds\right)\right)\right], \\ u_2(x,t) = \dfrac{d(t)}{12b}\left[1 - \tanh\left(\dfrac{\sqrt{2b}}{4}\left(x + \dfrac{2-b}{\sqrt{2b}}\displaystyle\int_0^t d(s)ds\right)\right)\right]^2, \end{cases}$$

is an almost periodic traveling wave solution of (P) if d is the solution of (2.1) as in Lemma 3.2, and it satisfies

$$\begin{cases} u_1(-\infty,t) = 0, \\ u_2(-\infty,t) = \frac{d(t)}{3b}, \end{cases} \qquad \begin{cases} u_1(+\infty,t) = d(t), \\ u_2(+\infty,t) = 0. \end{cases}$$

Example 4.3. Consider (P) and (P0) with $n = 3$. Assume

$$\mu_1 = \mu_2 = \mu_3 = 1,\ r_0 = 28,\ k_{11} = k_{22} = k_{33} = 1,$$
$$k_{12} = \frac{22}{21},\ k_{13} = 4,\ k_{21} = \frac{37}{21},\ k_{23} = \frac{3}{4},\ k_{31} = \frac{26}{21},\ k_{32} = \frac{22}{21};$$

then (P0) has a traveling wave solution

$$\begin{cases} w_1(z) = 14(1 + \tanh z), \\ w_2(z) = 7(1 - \tanh z)^2, \\ w_3(z) = \frac{4}{3}(1 - \tanh^2 z), \end{cases}$$

with $z = x + \frac{4}{3}t$ (cf. [3]), and

$$(w_1(-\infty), w_2(-\infty), w_3(-\infty)) = (0, 28, 0),$$
$$(w_1(+\infty), w_2(+\infty), w_3(+\infty)) = (28, 0, 0).$$

By Theorem 3.3 we know that

$$\begin{cases} u_1(x,t) = 14d(t)\left[1+\tanh\left(x+\frac{4}{3}\int_0^t d(s)ds\right)\right], \\ u_2(x,t) = 7d(t)\left[1-\tanh\left(x+\frac{4}{3}\int_0^t d(s)ds\right)\right]^2, \\ u_3(x,t) = \frac{4}{3}d(t)\left[1-\tanh^2\left(x+\frac{4}{3}\int_0^t d(s)ds\right)\right], \end{cases}$$

is an almost periodic traveling wave solution of (P) if d is the solution of (2.1) as in Lemma 3.2, and it satisfies

$$\begin{cases} u_1(-\infty,t) = 0, \\ u_2(-\infty,t) = 28d(t), \\ u_3(-\infty,t) = 0, \end{cases} \qquad \begin{cases} u_1(+\infty,t) = 28d(t), \\ u_2(+\infty,t) = 0, \\ u_3(+\infty,t) = 0. \end{cases}$$

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