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A p-TH YAMABE EQUATION ON GRAPH

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ABSTRACT. Assume $\alpha \geq p > 1$. Consider the following p-th Yamabe equation on a connected finite graph G:

$$\Delta_p \varphi + h\varphi^{p-1} = \lambda f \varphi^{\alpha-1},$$

where Δ_p is the discrete p-Laplacian, h and $f > 0$ are known real functions defined on all vertices. We show that the above equation always has a positive solution φ for some constant $\lambda \in \mathbb{R}$.

1. Introduction

The well-known smooth Yamabe problem asks one to consider of the smooth Yamabe equation [1, 5, 6]

$$\Delta\varphi + h(x)\varphi = \lambda f(x)\varphi^{N-1}$$

on a C^∞ compact Riemannian manifold M of dimension $n \geq 3$, where $h(x)$ and $f(x)$ are C^∞ functions on M, with $f(x)$ everywhere strictly positive and $N = 2n/(n-2)$. The problem is to prove the existence of a real number λ and of a C^∞ function φ, everywhere strictly positive, satisfying the above Yamabe equation. In this short paper, we consider the corresponding discrete Yamabe equation

$$\Delta\varphi + h\varphi = \lambda f\varphi^{\alpha-1}, \;\; \alpha \geq 2,$$

on a finite graph. More generally, we shall establish the existence results of the following p-th discrete Yamabe equation

$$\Delta_p\varphi + h\varphi^{p-1} = \lambda f\varphi^{\alpha-1}$$

on a finite graph G with $\alpha \geq p > 1$. This work is inspired by Grigor'yan, Lin, and Yang's pioneer papers [3, 4], where they studied similar equations on finite or locally finite graphs.

2. Settings and main results

Let $G = (V, E)$ be a finite graph, where V denotes the vertex set and E denotes the edge set. Fix a vertex measure $\mu : V \to (0, +\infty)$ and an edge measure $\omega : E \to (0, +\infty)$ on G. The edge measure ω is assumed to be symmetric, that is, $\omega_{ij} = \omega_{ji}$ for each edge $i \sim j$.

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Denote $C(V)$ as the set of all real functions defined on V. Then $C(V)$ is a finite-dimensional linear space with the usual function additions and scalar multiplications. For any $p > 1$, the p-th discrete graph Laplacian $\Delta_p : C(V) \to C(V)$ is

$$\Delta_p f_i = \frac{1}{\mu_i} \sum_{j \sim i} \omega_{ij} |f_j - f_i|^{p-2}(f_j - f_i)$$

for any $f \in C(V)$ and $i \in V$. Δ_p is a nonlinear operator when $p \neq 2$ (see [2] for more properties about Δ_p).

Theorem 2.1. *Let $G = (V, E)$ be a finite connected graph. Given $h, f \in C(V)$ with $f > 0$. Assume $\alpha \geq p > 1$. Then the following p-th Yamabe equation*

$$\Delta_p \varphi + h\varphi^{p-1} = \lambda f \varphi^{\alpha-1} \tag{2.1}$$

on G always has a positive solution φ for some constant $\lambda \in \mathbb{R}$.

Taking $p = 2$, we get the following.

Corollary 2.2. *Let $G = (V, E)$ be a finite connected graph. Given $h, f \in C(V)$ with $f > 0$. Assume $\alpha \geq 2$. Then the following Yamabe equation*

$$\Delta\varphi + h\varphi = \lambda f \varphi^{\alpha-1} \tag{2.2}$$

on G always has a positive solution φ for some constant $\lambda \in \mathbb{R}$.

Remark 1. Grigor'yan, Lin and Yang [4] established similar results for the following equation

$$-\Delta u + hu = |u|^{\alpha-2}u, \quad \alpha > 2, \tag{2.3}$$

on a finite graph under the assumption $h > 0$. They showed that the above equation (2.3) always has a positive solution. They also studied the equation

$$-\Delta_p u + h|u|^{p-2}u = f(x, u), \quad p > 1, \tag{2.4}$$

and established some existence results under certain assumptions about $f(x, u)$. However, it is remarkable that their Δ_p considered in equation (2.4) is different from ours when $p \neq 2$. It is also remarkable that our Theorem 2.1 doesn't require $h > 0$.

3. Proofs of Theorem 2.1

3.1. **Sobolev embedding.** For any $f \in C(V)$, define the integral of f over V with respect to the vertex weight μ by

$$\int_V f d\mu = \sum_{i \in V} \mu_i f_i.$$

Set $\mathrm{Vol}(G) = \int_V d\mu$. Similarly, for any function g defined on the edge set E, we define the integral of g over E with respect to the edge weight ω by

$$\int_E g d\omega = \sum_{i \sim j} \omega_{ij} g_{ij}.$$

Especially, for any $f \in C(V)$,

$$\int_E |\nabla f|^p d\omega = \sum_{i \sim j} \omega_{ij} |f_j - f_i|^p,$$

where $|\nabla f|$ is defined on the edge set E, and $|\nabla f|_{ij} = |f_j - f_i|$ for each edge $i \sim j$. Next we consider the Sobolev space $W^{1,\,p}$ on the graph G. Define

$$W^{1,\,p}(G) = \left\{ \varphi \in C(V) : \int_E |\nabla \varphi|^p d\omega + \int_V |\varphi|^p d\mu < +\infty \right\}$$

and

$$\|\varphi\|_{W^{1,\,p}(G)} = \left(\int_E |\nabla \varphi|^p d\omega + \int_V |\varphi|^p d\mu \right)^{\frac{1}{p}}.$$

Since G is a finite graph, $W^{1,\,p}(G)$ is exactly $C(V)$, a finite-dimensional linear space. This implies the following Sobolev embedding.

Lemma 3.1 (Sobolev embedding)**.** *Let $G = (V, E)$ be a finite graph. The Sobolev space $W^{1,\,p}(G)$ is precompact. Namely, if $\{\varphi_n\}$ is bounded in $W^{1,\,p}(G)$, then there exist some $\varphi \in W^{1,\,p}(G)$ such that up to a subsequence, $\varphi_n \to \varphi$ in $W^{1,\,p}(G)$.*

Remark 2. The convergence in $W^{1,\,p}(G)$ is in fact pointwise convergence.

3.2. **Proofs step by step.** We follow the original approach pioneered by Yamabe [6]. Denote an energy functional

$$I(\varphi) = \left(\int_E |\nabla \varphi|^p d\omega - \int_V h\varphi^p d\mu \right) \left(\int_V f\varphi^\alpha d\mu \right)^{-\frac{p}{\alpha}}, \tag{3.1}$$

where $\varphi \in W^{1,\,p}(G)$, $\varphi \geq 0$, and $\varphi \not\equiv 0$. Denote

$$\beta = \inf \big\{ I(\varphi) : \varphi \geq 0,\ \varphi \not\equiv 0 \big\}. \tag{3.2}$$

We shall find a solution to (2.1) step by step as follows.

Step 1. $I(\varphi)$ is bounded below for all $\varphi \geq 0$, $\varphi \not\equiv 0$. Hence $\beta \neq -\infty$ and $\beta \in \mathbb{R}$. In fact, it's easy to see that

$$0 < \left(\int_V f\varphi^\alpha d\mu \right)^{\frac{p}{\alpha}} \leq f_M^{\frac{p}{\alpha}} \left(\int_V \varphi^\alpha d\mu \right)^{\frac{p}{\alpha}} = f_M^{\frac{p}{\alpha}} \|\varphi\|_\alpha^p,$$

where $f_M = \max\limits_{i \in V} f_i > 0$. Hence

$$\left(\int_V f\varphi^\alpha d\mu \right)^{-\frac{p}{\alpha}} \geq f_M^{-\frac{p}{\alpha}} \|\varphi\|_\alpha^{-p} > 0. \tag{3.3}$$

Similarly, we also have

$$-\int_V h\varphi^p d\mu \geq (-h)_m \int_V \varphi^p d\mu = (-h)_m \|\varphi\|_p^p,$$

where $(-h)_m = \min\limits_{i \in V} (-h_i)$. Then it follows that

$$\int_E |\nabla \varphi|^p d\omega - \int_V h\varphi^p d\mu \geq (-h)_m \|\varphi\|_p^p. \tag{3.4}$$

By the estimates (3.3) and (3.4), we get

$$I(\varphi) \geq (-h)_m \|\varphi\|_p^p f_M^{-\frac{p}{\alpha}} \|\varphi\|_\alpha^{-p},$$

and further

$$I(\varphi) \geq \big((-h)_m \wedge 0 \big) \|\varphi\|_p^p f_M^{-\frac{p}{\alpha}} \|\varphi\|_\alpha^{-p}, \tag{3.5}$$

where $(-h)_m \wedge 0$ is the minimum of $(-h)_m$ and 0. Since $\alpha \geq p$,

$$0 < \|\varphi\|_p^p \leq \left(\int_V (\varphi^p)^{\frac{\alpha}{p}}\, d\mu\right)^{\frac{p}{\alpha}} \left(\int_V 1^{\frac{\alpha}{\alpha-p}} d\mu\right)^{\frac{\alpha-p}{\alpha}} = \|\varphi\|_\alpha^p \mathrm{Vol}(G)^{1-\frac{p}{\alpha}}, \tag{3.6}$$

which leads to

$$0 < \|\varphi\|_p^p \|\varphi\|_\alpha^{-p} \leq \mathrm{Vol}(G)^{1-\frac{p}{\alpha}}. \tag{3.7}$$

Thus by the estimates (3.5) and (3.7), we obtain

$$I(\varphi) \geq \big((-h)_m \wedge 0\big) f_M^{-\frac{p}{\alpha}} \mathrm{Vol}(G)^{1-\frac{p}{\alpha}} = C_{\alpha,p,h,f,G}, \tag{3.8}$$

where $C_{\alpha,p,h,f,G} \leq 0$ is a constant depending only on the information of α, p, h, f, and G. Note that the information of G contains V, E, μ, and ω. Hence $I(\varphi)$ is bounded below by a universal constant.

Step 2. There exists a $\hat{\varphi} \geq 0$ such that $\beta = I(\hat{\varphi})$. To find such a $\hat{\varphi}$, we choose $\varphi_n \geq 0$, satisfying

$$\int_V f\varphi_n^\alpha d\mu = 1$$

and

$$I(\varphi_n) \to \beta$$

as $n \to \infty$. We may well suppose $I(\varphi_n) \leq 1 + \beta$ for all n. Note that

$$1 = \int_V f\varphi_n^\alpha d\mu \geq f_m \int_V \varphi_n^\alpha d\mu = f_m \|\varphi_n\|_\alpha^\alpha,$$

where $f_m = \min_{i\in V} f_i$. Hence

$$\|\varphi_n\|_\alpha^p \leq f_m^{-\frac{p}{\alpha}}. \tag{3.9}$$

Denote $|h|_M = \max_{i\in V} |h_i|$. Then by the estimates (3.6) and (3.9), we obtain

$$\begin{aligned}
\|\varphi_n\|_{W^{1,p}(G)}^p &= \int_E |\nabla\varphi|^p d\omega + \int_V |\varphi|^p d\mu \\
&= I(\varphi_n) + \int_V h\varphi_n^p d\mu + \|\varphi_n\|_p^p \\
&\leq 1 + \beta + (1 + |h|_M)\|\varphi_n\|_p^p \\
&\leq 1 + \beta + (1 + |h|_M)\mathrm{Vol}(G)^{1-\frac{p}{\alpha}} \|\varphi_n\|_\alpha^p \\
&\leq 1 + \beta + (1 + |h|_M)\mathrm{Vol}(G)^{1-\frac{p}{\alpha}} f_m^{-\frac{p}{\alpha}},
\end{aligned}$$

which implies that $\{\varphi_n\}$ is bounded in $W^{1,p}(G)$. Therefore by the Sobolev embedding Lemma 3.1, there exists some $\hat{\varphi} \in C(V)$ such that up to a subsequence, $\varphi_n \to \hat{\varphi}$ in $W^{1,p}(G)$. We may well denote this subsequence as φ_n. Note that $\varphi_n \geq 0$ and $\int_V f\varphi_n^\alpha d\mu = 1$. Let $n \to +\infty$; we obtain $\hat{\varphi} \geq 0$ and $\int_V f\hat{\varphi}^\alpha d\mu = 1$. This implies that $\hat{\varphi} \not\equiv 0$. Since the energy functional $I(\varphi)$ is continuous, we have $\beta = I(\hat{\varphi})$.

Step 3. $\hat{\varphi} > 0$.

Calculating the Euler–Lagrange equation of $I(\varphi)$, we get

$$\frac{d}{dt}\Big|_{t=0} I(\varphi + t\phi) = -p\left(\int_V f\varphi^\alpha d\mu\right)^{-\frac{p}{\alpha}} \int_V \left(\Delta_p\varphi + h\varphi^{p-1} - \lambda_\varphi f\varphi^{\alpha-1}\right)\phi d\mu, \tag{3.10}$$

where

$$\lambda_\varphi = -\frac{\int_E |\nabla\varphi|^p d\omega - \int_V h\varphi^p d\mu}{\int_V f\varphi^\alpha d\mu} \tag{3.11}$$

for any $\varphi \geq 0$, $\varphi \not\equiv 0$. Thus

$$\frac{\partial I}{\partial \varphi_i} = -p\mu_i(\Delta_p\varphi_i + h\varphi_i^{p-1} - \lambda_\varphi f_i\varphi_i^{\alpha-1})\left(\int_V f\varphi^\alpha d\mu\right)^{-\frac{p}{\alpha}}. \tag{3.12}$$

Note that the graph G is connected. If $\hat{\varphi} > 0$ is not satisfied, since $\hat{\varphi} \geq 0$ and not identically zero, then there is an edge $i \sim j$ such that $\hat{\varphi}_i = 0$ but $\hat{\varphi}_j > 0$. Now look at $\Delta_p\hat{\varphi}_i$:

$$\Delta_p\hat{\varphi}_i = \frac{1}{\mu_i}\sum_{k\sim i}\omega_{ik}|\hat{\varphi}_k - \hat{\varphi}_i|^{p-2}(\hat{\varphi}_k - \hat{\varphi}_i) > 0.$$

Therefore by (3.12), we have

$$\frac{\partial I}{\partial \varphi_i}\Big|_{\varphi=\hat{\varphi}} = -p\mu_i\Delta_p\hat{\varphi}_i\left(\int_V f\hat{\varphi}^\alpha d\mu\right)^{-\frac{p}{\alpha}} < 0.$$

Recall we had proved that $\hat{\varphi}$ is the minimum value of $I(\varphi)$. Hence there should be

$$\frac{\partial I}{\partial \varphi_i}\Big|_{\varphi=\hat{\varphi}} \geq 0,$$

which is a contradiction. Thus $\hat{\varphi} > 0$.

Step 4. $\hat{\varphi}$ satisfied equation (2.1), that is,

$$\Delta_p\hat{\varphi} + h\hat{\varphi}^{p-1} = \lambda_{\hat{\varphi}} f\hat{\varphi}^{\alpha-1}, \tag{3.13}$$

where $\lambda_{\hat{\varphi}}$ is defined according to (3.11). Because $I(\varphi)$ attains its minimum value at $\hat{\varphi}$, which lies in the interior of $\{\varphi \in C(V) : \varphi \geq 0\}$,

$$\frac{d}{dt}\Big|_{t=0} I(\hat{\varphi} + t\phi) = 0$$

for all $\phi \in C(V)$. This leads to (3.13).

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