A $p$-TH YAMABE EQUATION ON GRAPH

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Abstract. Assume $\alpha \geq p > 1$. Consider the following $p$-th Yamabe equation on a connected finite graph $G$:

$\Delta_p \varphi + h \varphi^{p-1} = \lambda f \varphi^{\alpha-1}$,

where $\Delta_p$ is the discrete $p$-Laplacian, $h$ and $f > 0$ are known real functions defined on all vertices. We show that the above equation always has a positive solution $\varphi$ for some constant $\lambda \in \mathbb{R}$.

1. Introduction

The well-known smooth Yamabe problem asks one to consider of the smooth Yamabe equation \cite{1,5,6}

$\Delta \varphi + h(x) \varphi = \lambda f(x) \varphi^{N-1}$

on a $C^\infty$ compact Riemannian manifold $M$ of dimension $n \geq 3$, where $h(x)$ and $f(x)$ are $C^\infty$ functions on $M$, with $f(x)$ everywhere strictly positive and $N = 2n/(n-2)$. The problem is to prove the existence of a real number $\lambda$ and of a $C^\infty$ function $\varphi$, everywhere strictly positive, satisfying the above Yamabe equation. In this short paper, we consider the corresponding discrete Yamabe equation

$\Delta \varphi + h \varphi = \lambda f \varphi^{\alpha-1}, \ \alpha \geq 2$,

on a finite graph. More generally, we shall establish the existence results of the following $p$-th discrete Yamabe equation

$\Delta_p \varphi + h \varphi^{p-1} = \lambda f \varphi^{\alpha-1}$

on a finite graph $G$ with $\alpha \geq p > 1$. This work is inspired by Grigor’yan, Lin, and Yang’s pioneer papers \cite{3,4}, where they studied similar equations on finite or locally finite graphs.

2. Settings and main results

Let $G = (V, E)$ be a finite graph, where $V$ denotes the vertex set and $E$ denotes the edge set. Fix a vertex measure $\mu : V \to (0, +\infty)$ and an edge measure $\omega : E \to (0, +\infty)$ on $G$. The edge measure $\omega$ is assumed to be symmetric, that is, $\omega_{ij} = \omega_{ji}$ for each edge $i \sim j$. The research is supported by National Natural Science Foundation of China under Grant No.11501027.
Denote $C(V)$ as the set of all real functions defined on $V$. Then $C(V)$ is a finite-dimensional linear space with the usual function additions and scalar multiplications. For any $p > 1$, the $p$-th discrete graph Laplacian $\Delta_p : C(V) \rightarrow C(V)$ is

$$\Delta_p f_i = \frac{1}{\mu_i} \sum_{j \sim i} \omega_{ij} |f_j - f_i|^{p-2}(f_j - f_i)$$

for any $f \in C(V)$ and $i \in V$. $\Delta_p$ is a nonlinear operator when $p \neq 2$ (see [2] for more properties about $\Delta_p$).

**Theorem 2.1.** Let $G = (V, E)$ be a finite connected graph. Given $h, f \in C(V)$ with $f > 0$. Assume $\alpha \geq p > 1$. Then the following $p$-th Yamabe equation

$$\Delta_p \varphi + h \varphi^{p-1} = \lambda f \varphi^{\alpha-1}$$

on $G$ always has a positive solution $\varphi$ for some constant $\lambda \in \mathbb{R}$.

Taking $p = 2$, we get the following.

**Corollary 2.2.** Let $G = (V, E)$ be a finite connected graph. Given $h, f \in C(V)$ with $f > 0$. Assume $\alpha \geq 2$. Then the following Yamabe equation

$$\Delta \varphi + h \varphi = \lambda f \varphi^{\alpha-1}$$

on $G$ always has a positive solution $\varphi$ for some constant $\lambda \in \mathbb{R}$.

**Remark 1.** Grigor’yan, Lin and Yang [4] established similar results for the following equation

$$-\Delta u + hu = |u|^{\alpha-2}u, \quad \alpha > 2,$$

on a finite graph under the assumption $h > 0$. They showed that the above equation (2.3) always has a positive solution. They also studied the equation

$$-\Delta_p u + h|u|^{p-2}u = f(x, u), \quad p > 1,$$

and established some existence results under certain assumptions about $f(x, u)$. However, it is remarkable that their $\Delta_p$ considered in equation (2.4) is different from ours when $p \neq 2$. It is also remarkable that our Theorem 2.1 doesn’t require $h > 0$.

3. Proofs of Theorem 2.1

3.1. **Sobolev embedding.** For any $f \in C(V)$, define the integral of $f$ over $V$ with respect to the vertex weight $\mu$ by

$$\int_V f \mu = \sum_{i \in V} \mu_i f_i.$$ 

Set $\text{Vol}(G) = \int_V d\mu$. Similarly, for any function $g$ defined on the edge set $E$, we define the integral of $g$ over $E$ with respect to the edge weight $\omega$ by

$$\int_E g d\omega = \sum_{i \sim j} \omega_{ij} g_{ij}.$$ 

Especially, for any $f \in C(V)$,

$$\int_E |\nabla f|^p d\omega = \sum_{i \sim j} \omega_{ij} |f_j - f_i|^p.$$
where $|\nabla f|$ is defined on the edge set $E$, and $|\nabla f|_{ij} = |f_j - f_i|$ for each edge $i \sim j$.

Next we consider the Sobolev space $W^{1,p}(G)$ on the graph $G$. Define

$$W^{1,p}(G) = \left\{ \varphi \in C(V) : \int_E |\nabla \varphi|^p d\omega + \int_V |\varphi|^p d\mu < +\infty \right\}$$

and

$$\|\varphi\|_{W^{1,p}(G)} = \left( \int_E |\nabla \varphi|^p d\omega + \int_V |\varphi|^p d\mu \right)^{\frac{1}{p}}.$$

Since $G$ is a finite graph, $W^{1,p}(G)$ is exactly $C(V)$, a finite-dimensional linear space. This implies the following Sobolev embedding.

**Lemma 3.1** (Sobolev embedding). Let $G = (V,E)$ be a finite graph. The Sobolev space $W^{1,p}(G)$ is precompact. Namely, if $\{\varphi_n\}$ is bounded in $W^{1,p}(G)$, then there exist some $\varphi \in W^{1,p}(G)$ such that up to a subsequence, $\varphi_n \rightarrow \varphi$ in $W^{1,p}(G)$.

**Remark 2.** The convergence in $W^{1,p}(G)$ is in fact pointwise convergence.

### 3.2. Proofs step by step.

We follow the original approach pioneered by Yamabe [6]. Denote an energy functional

$$I(\varphi) = \left( \int_E |\nabla \varphi|^p d\omega - \int_V h|\varphi|^p d\mu \right) \left( \int_V f|\varphi|^\alpha d\mu \right)^{-\frac{p}{\alpha}},$$

where $\varphi \in W^{1,p}(G)$, $\varphi \geq 0$, and $\varphi \not\equiv 0$. Denote

$$\beta = \inf \left\{ I(\varphi) : \varphi \geq 0, \varphi \not\equiv 0 \right\}.$$

We shall find a solution to (2.1) step by step as follows.

**Step 1.** $I(\varphi)$ is bounded below for all $\varphi \geq 0$, $\varphi \not\equiv 0$. Hence $\beta \neq -\infty$ and $\beta \in \mathbb{R}$. In fact, it’s easy to see that

$$0 < \left( \int_V f|\varphi|^\alpha d\mu \right)^{\frac{p}{\alpha}} \leq f_M^{\frac{p}{\alpha}} \left( \int_V |\varphi|^\alpha d\mu \right)^{\frac{p}{\alpha}} = f_M^{\frac{p}{\alpha}} \|\varphi\|_\alpha^p,$$

where $f_M = \max_{i \in V} f_i > 0$. Hence

$$0 < \left( \int_V f|\varphi|^\alpha d\mu \right)^{\frac{p}{\alpha}} \geq f_M^{-\frac{p}{\alpha}} \|\varphi\|_\alpha^{-p} > 0.$$

Similarly, we also have

$$- \int_V h|\varphi|^p d\mu \geq (-h)_m \int_V |\varphi|^p d\mu = (-h)_m \|\varphi\|_p^p,$$

where $(-h)_m = \min_{i \in V} (-h_i)$. Then it follows that

$$\int_E |\nabla \varphi|^p d\omega - \int_V h|\varphi|^p d\mu \geq (-h)_m \|\varphi\|_p^p.$$

By the estimates (3.3) and (3.4), we get

$$I(\varphi) \geq (-h)_m \|\varphi\|_p^p f_M^\frac{p}{\alpha} \|\varphi\|_\alpha^{-p},$$

and further

$$I(\varphi) \geq ((-h)_m \wedge 0) \|\varphi\|_p^p f_M^\frac{p}{\alpha} \|\varphi\|_\alpha^{-p},$$

where $\|\varphi\|_p = \left( \int_V |\varphi|^p d\mu \right)^{\frac{1}{p}}$. This completes the proof.
where \((-h)_m \land 0\) is the minimum of \((-h)_m\) and 0. Since \(\alpha \geq p\),

\[
0 < \|\varphi\|_p^p \leq \left( \int_V (\varphi^p)^{\frac{p}{p}} \, d\mu \right)^{\frac{p}{p}} \left( \int_V 1^{\frac{p}{\alpha-p}} \, d\mu \right)^{\frac{\alpha-p}{\alpha}} = \|\varphi\|_\alpha^p \text{Vol}(G)^{1-\frac{p}{\alpha}},
\]

which leads to

\[
0 < \|\varphi\|_\alpha^p \|\varphi\|_\alpha^{-p} \leq \text{Vol}(G)^{1-\frac{p}{\alpha}}.
\]

Thus by the estimates (3.6) and (3.7), we obtain

\[
I(\varphi) \geq ((-h)_m \land 0) f_M^{\frac{p}{\alpha}} \text{Vol}(G)^{1-\frac{p}{\alpha}} = C_{\alpha,p,h,f,G},
\]

where \(C_{\alpha,p,h,f,G} \leq 0\) is a constant depending only on the information of \(\alpha, p, h, f,\) and \(G\). Note that the information of \(G\) contains \(V, E, \mu,\) and \(\omega\). Hence \(I(\varphi)\) is bounded below by a universal constant.

**Step 2.** There exists a \(\hat{\varphi} \geq 0\) such that \(\beta = I(\hat{\varphi})\). To find such a \(\hat{\varphi}\), we choose \(\varphi_n \geq 0\), satisfying

\[
\int_V f \varphi_n^p \, d\mu = 1
\]

and

\[
I(\varphi_n) \to \beta
\]

as \(n \to \infty\). We may well suppose \(I(\varphi_n) \leq 1 + \beta\) for all \(n\). Note that

\[
1 = \int_V f \varphi_n^p \, d\mu \geq f_m \int_V \varphi_n^p \, d\mu = f_m \|\varphi_n\|_\alpha^p,
\]

where \(f_m = \min_{i \in V} f_i\). Hence

\[
\|\varphi_n\|_\alpha^p \leq f_m^{\frac{p}{\alpha}}.
\]

Denote \(|h|_M = \max_{i \in V} |h_i|\). Then by the estimates (3.6) and (3.9), we obtain

\[
\|\varphi_n\|_{W^{1,p}(G)}^p = \int_E |\nabla \varphi|^p \, d\omega + \int_V |\varphi|^p \, d\mu
\]

\[
= I(\varphi_n) + \int_V h \varphi_n^p \, d\mu + \|\varphi_n\|_p^p
\]

\[
\leq 1 + \beta + (1 + |h|_M) \|\varphi_n\|_p^p
\]

\[
\leq 1 + \beta + (1 + |h|_M) \text{Vol}(G)^{1-\frac{p}{\alpha}} \|\varphi_n\|_\alpha^p
\]

\[
\leq 1 + \beta + (1 + |h|_M) \text{Vol}(G)^{1-\frac{p}{\alpha}} f_m^{\frac{p}{\alpha}},
\]

which implies that \(\{\varphi_n\}\) is bounded in \(W^{1,p}(G)\). Therefore by the Sobolev embedding Lemma 3.1, there exists some \(\hat{\varphi} \in C(V)\) such that up to a subsequence, \(\varphi_n \to \hat{\varphi}\) in \(W^{1,p}(G)\). We may well denote this subsequence as \(\varphi_n\). Note that \(\varphi_n \geq 0\) and \(\int_V f \varphi_n^p \, d\mu = 1\). Let \(n \to +\infty\); we obtain \(\hat{\varphi} \geq 0\) and \(\int_V f \hat{\varphi}^p \, d\mu = 1\). This implies that \(\hat{\varphi} \neq 0\). Since the energy functional \(I(\varphi)\) is continuous, we have \(\beta = I(\hat{\varphi})\).

**Step 3.** \(\hat{\varphi} > 0\).

Calculating the Euler–Lagrange equation of \(I(\varphi)\), we get

\[
\frac{d}{dt} \bigg|_{t=0} I(\varphi + t\phi) = -p \left( \int_V f \varphi^p \, d\mu \right)^{-\frac{p}{\alpha}} \int_V (\Delta \phi + h \varphi^{p-1} - \lambda \phi \varphi^{\alpha-1}) \phi \, d\mu,
\]
where
\[
\lambda_\varphi = -\frac{\int_E |\nabla \varphi|^p d\omega - \int_V h\varphi^p d\mu}{\int_V f\varphi^\alpha d\mu}
\]
for any \( \varphi \geq 0, \varphi \not\equiv 0 \). Thus
\[
\frac{\partial I}{\partial \varphi_i} = -p\mu_i (\Delta_p \varphi_i + h\varphi_i^{p-1} - \lambda_\varphi f_i \varphi_i^{\alpha-1}) \left( \int_V f\varphi^\alpha d\mu \right)^{-\frac{p}{\alpha}}.
\]
Note that the graph \( G \) is connected. If \( \hat{\varphi} > 0 \) is not satisfied, since \( \hat{\varphi} \geq 0 \) and not identically zero, then there is an edge \( i \sim j \) such that \( \hat{\varphi}_i = 0 \) but \( \hat{\varphi}_j > 0 \). Now look at \( \Delta_p \hat{\varphi}_i \):
\[
\Delta_p \hat{\varphi}_i = \frac{1}{\mu_i} \sum_{k \sim i} \omega_{ik} |\hat{\varphi}_k - \hat{\varphi}_i|^{p-2}(\hat{\varphi}_k - \hat{\varphi}_i) > 0.
\]
Therefore by (3.12), we have
\[
\left. \frac{\partial I}{\partial \varphi_i} \right|_{\varphi = \hat{\varphi}} = -p\mu_i \Delta_p \hat{\varphi}_i \left( \int_V f\hat{\varphi}^\alpha d\mu \right)^{-\frac{p}{\alpha}} < 0.
\]
Recall we had proved that \( \hat{\varphi} \) is the minimum value of \( I(\varphi) \). Hence there should be
\[
\left. \frac{\partial I}{\partial \varphi_i} \right|_{\varphi = \hat{\varphi}} \geq 0,
\]
which is a contradiction. Thus \( \hat{\varphi} > 0 \).

**Step 4.** \( \hat{\varphi} \) satisfied equation (2.1), that is,
\[
\Delta_p \hat{\varphi} + h\hat{\varphi}^{p-1} = \lambda_{\hat{\varphi}} f \hat{\varphi}^\alpha - 1,
\]
where \( \lambda_{\hat{\varphi}} \) is defined according to (3.11). Because \( I(\varphi) \) attains its minimum value at \( \hat{\varphi} \), which lies in the interior of \( \{ \varphi \in C(V) : \varphi \geq 0 \} \),
\[
\left. \frac{d}{dt} \right|_{t=0} I(\hat{\varphi} + t\phi) = 0
\]
for all \( \phi \in C(V) \). This leads to (3.13).

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**References**

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