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RESTRICTING IRREDUCIBLE CHARACTERS TO SYLOW p-SUBGROUPS

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Abstract. We restrict irreducible characters of finite groups of degree divisible by p to their Sylow p-subgroups and study the number of linear constituents.

1. Introduction

Suppose that P is a p-group. What do the characters ψ of P that are restrictions of irreducible characters of the groups that contain P as a Sylow p-subgroup look like? This is too vast a hypothesis to expect to draw any general conclusion, and yet it seems that something can be said in some special cases. Of course, $\psi(1)_p$ has to divide $|P|$ or, if we wish to mention G and the fusion in P, we have that $\psi(x) = \psi(y)$ whenever $x, y \in P$ are G-conjugate, to name two obvious conditions.

The relevance of the McKay conjecture has led us to study how irreducible characters of degree not divisible by p restrict to their Sylow p-subgroups and to count in these characters linear constituents. In some special cases, it can be proved that these restrictions have a unique linear constituent (see [G1], [N], or [NTV]), and this has served to prove strong forms of the McKay conjecture for particular classes of groups.

Also in this paper we are interested in counting linear constituents of the restricted characters χ_P, where $\chi \in \mathrm{Irr}(G)$ and $P \in \mathrm{Syl}_p(G)$, but in the opposite instance where p divides $\chi(1)$. Contrary to the case where χ has degree not divisible by p, in the new situation we might have that the number of linear constituents is zero. But not in symmetric groups, however. In the first main result of this paper, we prove the following.

Theorem A. *Suppose that n is a positive integer and let P be a Sylow p-subgroup of the symmetric group $\mathfrak{S}_n$. If $\chi \in \mathrm{Irr}(\mathfrak{S}_n)$ has degree divisible by p, then the restriction χ_P has at least p different linear constituents.*

The number of linear constituents of the characters χ_P in $\mathfrak{S}_n$ tends to be very large, but, as we shall point out, there are arbitrarily large integers n for which this number is exactly p. To classify these cases seems an interesting problem on its own, and we shall comment on this later. (See Remark 3.13.)

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Of course, the fact that χ_P has *any* linear constituent is a feature of the symmetric groups. There are many nonsolvable (and solvable) families of examples where χ_P has none. If we assume, however, that χ_P has at least one linear constituent, then we have evidence that χ_P should at least have p different ones. In the next theorem, we prove a strong form of this in a different class of groups.

Theorem B. *Suppose that G is a p-solvable group. Let $P \in \mathrm{Syl}_p(G)$, and let $\chi \in \mathrm{Irr}(G)$ of degree divisible by p. If χ_P contains a linear constituent λ, then there exists a subgroup $U < P$ of index p such that χ_P contains the character $(\lambda_U)^P$.*

We can only prove our next theorem by using the Kessar–Malle ([KM]) solution of one implication of Brauer's Height Zero conjecture on p-blocks, and a celebrated theorem of Knörr on vertices ([K]).

Theorem C. *Suppose that G has an abelian Sylow p-subgroup P. Let $\chi \in \mathrm{Irr}(G)$. If $D \leq P$ is a defect group of the p-block of χ, then there exists $\nu \in \mathrm{Irr}(P)$ such that χ_P contains $(\nu_D)^P$. In particular, if p divides $\chi(1)$, then χ_P contains at least p different linear constituents.*

If $|G|_p = p$, notice that Theorem C is a consequence of a well-known theorem of Brauer on p-defect zero vanishing on nontrivial p-elements.

We have gathered enough evidence to guess that the following might be true.

Conjecture D. *Suppose that $\chi \in \mathrm{Irr}(G)$ has degree divisible by p, and let $P \in \mathrm{Syl}_p(G)$. If χ_P has a linear constituent, then χ_P has at least p different linear constituents.*

Theorems C suggests that perhaps Theorem B might be true without any p-solvability hypothesis, showing a strong form of Conjecture D. If there is an argument to prove this or a counterexample, we are not aware of it at the time of this writing.

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2. Abelian Sylow p-subgroups

We start by proving Theorem C. Although its statement is elementary, our proof uses deep machinery. Our notation here is from [NT].

Theorem 2.1. *Suppose that G has an abelian Sylow p-subgroup P. Let $\chi \in \mathrm{Irr}(G)$. If $D \leq P$ is a defect group of the block of χ, then there exists $\nu \in \mathrm{Irr}(P)$ such that χ_P contains $(\nu_D)^P$. In particular, χ_P contains at least $|P : D|$ different linear constituents.*

Proof. Suppose that (K, R, F) is a p-modular system, and suppose that M is an RG-module affording χ. We can take F to be algebraically closed (using Theorem 1.13.27 of [NT]). By Knörr's Theorem (see [K]), we can choose M such that the vertex of M is D. By Theorem 3 of [B], let W be an indecomposable RP-module with vertex D and $W|M_P$. By the same theorem, let W_1 be an RD-module with vertex D such that $W|(W_1)^P$. Notice that by Lemma 4.7.1 of [NT], we have that indecomposable RG-modules are absolutely indecomposable. By Green's indecomposability theorem 4.7.2 of [NT], we have that $(W_1)^P$ is indecomposable. Thus $W = (W_1)^P$. Now, suppose that W_1 affords the character τ of D. Then $(W_1)^P$ affords the character τ^P, which is therefore contained in χ_P. If $\lambda \in \mathrm{Irr}(D)$ is an irreducible constituent of τ, then we have that χ_P contains λ^P. Now, since P is abelian, we have that $\lambda = \nu_D$ for some $\nu \in \mathrm{Irr}(P)$, and it follows that χ_P contains $\nu(1_D)^P$. Now, this character has $|P : D|$ distinct irreducible constituents. $\square$

Corollary 2.2. *Suppose that G has an abelian Sylow p-subgroup P. Let $\chi \in \mathrm{Irr}(G)$. If p divides $\chi(1)$, then χ_P contains at least p different constituents.*

Proof. If D is a defect group of the block of χ, by the main result of [KM], we have that $D < P$. $\square$

3. Symmetric groups

The aim of this section is to prove Theorem A. In order to do this, we will first show that the following fact holds. This was first conjectured in [G2].

Theorem 3.1. *Let $\chi \in \mathrm{Irr}(\mathfrak{S}_n)$ and let P_n be a Sylow p-subgroup of $\mathfrak{S}_n$. Then χ_{P_n} has a linear constituent.*

The first part of this section is devoted to the proof of Proposition 3.7 below. This is the fundamental combinatorial step that will allow us to prove Theorem 3.1 in the second part of the section.

3.1. **Combinatorics of partitions.** We start by recalling some very basic combinatorial definitions and notation in the framework of the representation theory of symmetric groups. We refer the reader to [J] or [O1] for a more detailed account.

A composition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$ is a finite sequence of positive integers. We say that λ_i is a part of λ and that λ is a composition of $|\lambda| = \sum \lambda_i$. We say that λ is a partition if its parts are nonincreasingly ordered. The Young diagram associated to λ is the set $[\lambda] := \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq s, 1 \leq j \leq \lambda_i\}$. For any natural number n we denote by $\mathcal{P}(n)$ (respectively, $\mathcal{C}(n)$) the set of partitions (respectively, compositions) of n. We will sometimes write $\lambda \vdash n$ if $\lambda \in \mathcal{P}(n)$. For $\mu \in \mathcal{C}(n)$ we denote by $\mu^\star$ the unique partition obtained by reordering the parts of μ. Moreover, if $\nu \in \mathcal{C}(m)$, we denote by $\mu \circ \nu$ the composition of $m+n$ obtained by the concatenation of μ and ν. Let μ and λ be partitions. We say that μ is a *subpartition* of λ, written $\mu \subseteq \lambda$, if $\mu_i \leq \lambda_i$, for all $i \geq 1$. When this occurs, we let the *skew Young diagram* $\lambda \smallsetminus \mu$ be the subdiagram of $[\lambda]$ defined by

$$\lambda \smallsetminus \mu = \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq s, \mu_i < j \leq \lambda_i\}.$$

For any $j \in \{1, \dots, s\}$ we let $(\lambda \smallsetminus \mu)_j$ be the size of the jth row of $\lambda \smallsetminus \mu$. More precisely, we have $(\lambda \smallsetminus \mu)_j = \lambda_j - \mu_j$.

Throughout this section we let n and q be fixed natural numbers, with $q \geq 2$. For any $\lambda \in \mathcal{P}(qn)$ we can uniquely write λ as $\lambda = (\mu \circ \nu)^\star$, where μ is the partition consisting of all the parts of λ that are not divisible by q and ν is the partition consisting of all the parts of λ that are multiples of q. In particular we have that

$$(*) \qquad \lambda = \big((k_1 q + x_1, \dots, k_t q + x_t) \circ (r_1 q, \dots, r_s q)\big)^\star,$$

where $k_1 \geq k_2 \geq \cdots \geq k_t \geq 0$, $r_1 \geq r_2 \geq \cdots \geq r_s > 0$, and where $x_j \in \{1, \dots, q-1\}$ for all $j \in \{1, \dots, t\}$. Since $\lambda \vdash qn$ there exists $\zeta_q(\lambda) \in \mathbb{N}_0$ such that

$x_1 + x_2 + \cdots + x_t = \zeta_q(\lambda)q$. Notice that $\zeta_q(\lambda) = \frac{1}{q}(x_1 + \cdots + x_t) \leq t \cdot \frac{q-1}{q} \leq t$. The notation introduced above will be kept for the rest of the section.

The following three definitions are central in the proof of Theorem 3.1. These are described in a specific situation in Example 3.5 below.

Definition 3.2. Given λ as in equation $(*)$ above, we let $\Delta^q(\lambda)$ be the partition of n defined by

$$\Delta^q(\lambda) = \big((k_1+1, k_2+1, \ldots, k_{\zeta_q(\lambda)}+1, k_{\zeta_q(\lambda)+1}, \ldots, k_t) \circ (r_1, r_2, \ldots, r_s)\big)^\star.$$

Definition 3.3. We denote by A_λ the multiset of q-residues defined by $A_\lambda = \{x_1, \ldots, x_t\}$. We define a total order $\gg$ on the indexing set $\{1, 2, \ldots, t\}$ as follows. Let i, j be distinct elements in $\{1, \ldots, t\}$. If $x_i > x_j$, then $i \gg j$. When $x_i = x_j$, then we let $i \gg j$ if and only if $i > j$.

We denote by $\lambda_\gg$ the composition defined by

$$\lambda_\gg = (k_{i_1}q + x_{i_1}, k_{i_2}q + x_{i_2}, \ldots, k_{i_t}q + x_{i_t}),$$

where $i_1, \ldots, i_t \in \{1, \ldots, t\}$ are such that $i_1 \gg i_2 \gg \cdots \gg i_t$.

Definition 3.4. Given λ as in equation $(*)$ above we let $\Omega(\lambda)_\gg$ be the composition defined by

$$\Omega(\lambda)_\gg = \big(\lambda_\gg - (k_{i_1}+1, \ldots, k_{i_{\zeta_q(\lambda)}}+1, k_{i_{\zeta_q(\lambda)}}, \ldots, k_{i_t})\big).$$

Moreover we denote by $\Omega(\lambda)$ the partition of $(q-1)n$ defined by

$$\Omega(\lambda) = \big[\Omega(\lambda)_\gg \circ (r_1(q-1), \ldots r_s(q-1))\big]^\star.$$

In the following example we explicitly describe in a concrete situation all the combinatorial objects introduced in Definitions 3.2, 3.3, and 3.4.

Example 3.5. Let $q = 5$, let $n = 10$, and let $\lambda = (12, 9, 8, 6, 6, 5, 4) \vdash 50$. We observe that λ has one part of size divisible by 5 and that $x_1 = 2, x_2 = 4, x_3 = 3, x_4 = 1, x_5 = 1, x_6 = 4$. In particular, we get $\zeta_5(\lambda) = 3$, and $\Delta^5(\lambda) = (3, 2, 2, 1, 1, 1, 0) \vdash 10$, by Definition 3.2. Following Definition 3.3 we observe that $6 \gg 2 \gg 3 \gg 1 \gg 5 \gg 4$. This shows that $\lambda_\gg = (4, 9, 8, 12, 6, 6)$ and that $\Omega(\lambda)_\gg = \lambda_\gg - (1, 2, 2, 2, 1, 1) = (3, 7, 6, 10, 5, 5)$. From Definition 3.4 we finally get that $\Omega(\lambda) = (\Omega(\lambda)_\gg \circ (4))^\star = (10, 7, 6, 5, 5, 4, 3) \vdash 40$. We conclude the example by observing that $\Delta^4(\Omega(\lambda)) = (3, 2, 2, 1, 1, 1, 0) = \Delta^5(\lambda)$. As we will show below, this is not a coincidence.

Our aim is to show that $\Delta^q(\lambda) = \Delta^{q-1}(\Omega(\lambda))$, for all $q, n \in \mathbb{N}$ and all $\lambda \in \mathcal{P}(qn)$. In order to do this, we first need to state the following technical lemma.

Lemma 3.6. *Let $q \geq 3$ and let $\lambda = (k_1q + x_1, \ldots, k_tq + x_t)$ be a partition of qn such that $x_1, \ldots, x_t \in \{1, 2, \ldots, q-1\}$. As usual we let $\zeta_q(\lambda) = (x_1 + \cdots + x_t)/q$. For $j \in \{1, \ldots, q-1\}$ let $z_j = |\{i \in \{1, \ldots, t\} \mid x_i = j\}|$. The following hold:*

(i) *If $z_{q-1} > \zeta_q(\lambda)$, then $2\zeta_q(\lambda) > t$.*

(ii) *If $z_1 \geq t - \zeta_q(\lambda)$, then $2\zeta_q(\lambda) \leq t$.*

Proof. We first prove (i). Let $I := \{j \in \{1, \ldots, t\} \mid x_j \neq q-1\}$. Clearly $t = |I| + z_{q-1}$ and

$$(\star) \qquad q\zeta_q(\lambda) = \sum_{j \in I} x_j + z_{q-1}(q-1) \geq |I| + z_{q-1}(q-1).$$

Using ($\star$) we see that the following chain of inequalities holds:

$$\begin{aligned} 2\zeta_q(\lambda)+\frac{q-2}{q}|I| &= \frac{2}{q}\Big(\sum_{j\in I}x_j+z_{q-1}(q-1)\Big)+\frac{q-2}{q}|I| \\ &\geq |I|+2\Big(\frac{q-1}{q}\Big)z_{q-1} \\ &= t+\Big(\frac{q-2}{q}\Big)z_{q-1}. \end{aligned}$$

In particular, we have that $2\zeta_q(\lambda)-t\geq \frac{q-2}{q}(z_{q-1}-|I|)$. Suppose for a contradiction that $z_{q-1}-|I|\leq 0$. Then from equation ($\star$) we would have that

$$q(\zeta_q(\lambda)-z_{q-1})\geq |I|-z_{q-1}\geq 0,$$

which contradicts the hypothesis of (i). We conclude that $2\zeta_q(\lambda)>t$, as required.

In order to prove (ii) we set $J:=\{j\in\{1,\dots,t\}\mid x_j=1\}$ and we recall that

$$q\zeta_q(\lambda)=x_1+\cdots+x_t=\sum_{j\in J}x_j+\sum_{j\notin J}x_j.$$

Since $|J|=z_1\geq t-\zeta_q(\lambda)$ and since every x_j is at most $q-1$ we have that

$$q\zeta_q(\lambda)\leq t-\zeta_q(\lambda)+(q-1)\zeta_q(\lambda).$$

Hence $2\zeta_q(\lambda)\leq t$. □

Proposition 3.7. *Let $q\geq 3$. Then $\Delta^{q-1}(\Omega(\lambda))=\Delta^q(\lambda)$ for all $\lambda\in\mathcal{P}(qn)$.*

Proof. Keeping the notation introduced at the beginning of the section we write

$$\lambda=\big((k_1q+x_1,\dots,k_tq+x_t)\circ(r_1q,\dots r_sq)\big)^{\star},$$

where $k_1\geq k_2\geq\cdots\geq k_t\geq 0$, $r_1\geq r_2\geq\cdots\geq r_s>0$, and where $x_j\in\{1,\dots,q-1\}$ for all $j\in\{1,\dots,t\}$. Again $\zeta_q(\lambda)=\frac{1}{q}\cdot(x_1+\cdots x_t)\in\mathbb{N}_0$. From Definitions 3.2 and 3.4, it is easy to see that

$$\Delta^{q-1}(\Omega(\lambda))=\big[\Delta^{q-1}\big((\Omega(\lambda)_{\gg})^{\star}\big)\circ(r_1,\dots r_s)\big]^{\star}.$$

On the other hand we have

$$\Delta^q(\lambda)=\big((k_1+1,k_2+2,\dots,k_{\zeta_q(\lambda)}+1,k_{\zeta_q(\lambda)+1},\dots,k_t)\circ(r_1,r_2,\dots,r_s)\big)^{\star}.$$

Hence it suffices to show that

$$\Delta^{q-1}\big((\Omega(\lambda)_{\gg})^{\star}\big)=(k_1+1,k_2+2,\dots,k_{\zeta_q(\lambda)}+1,k_{\zeta_q(\lambda)+1},\dots,k_t).$$

In particular we can assume that $\lambda=(\lambda_{\gg})^{\star}$ has no parts of size divisible by q (hence we are in the setting of Lemma 3.6).

For $j\in\{1,\dots,q-1\}$ we let $Z_j:=\{y\in\{1,\dots,t\}\mid x_y=j\}$ and $z_j:=|Z_j|$. To ease the notation we will let $z=z_{q-1}$. Let $j_1,\dots,j_t\in\{1,\dots,t\}$ be such that

$$j_1\gg j_2\gg\cdots\gg j_t.$$

We denote by $\mathcal{B}$ the subset of $\{1,\dots,t\}$ defined by $\mathcal{B}=\{j_1,\dots,j_{\zeta_q(\lambda)}\}$. We proceed by computing $\Delta^{q-1}(\Omega(\lambda))$ in two separate cases.

(i) First assume that $z \leq \zeta_q(\lambda)$. From the definition of the total order $\gg$, we see that if $y \in \{1, \ldots, t\}$ and $x_y = q-1$, then y is one of the greatest $\zeta_q(\lambda)$ elements in $\{1, \ldots, t\}$. Namely $y \in \mathcal{B}$. In particular, we have that

$$\Omega(\lambda)_y = \begin{cases} (q-1)k_y + x_y & \text{if } y \in \{1, \ldots, t\} \smallsetminus \mathcal{B}, \\ (q-1)k_y + (x_y - 1) & \text{if } y \in \mathcal{B} \smallsetminus V_0, \\ (q-1)k_y + 0 & \text{if } y \in V_0\,, \end{cases}$$

where $V_0 = Z_1 \cap \mathcal{B}$. Then we have that $A_{\Omega(\lambda)} = \{x_u, (x_v - 1) \mid u \notin \mathcal{B},\ v \in \mathcal{B} \smallsetminus V_0\}$. Hence we deduce that

$$(q-1)\zeta_{q-1}(\Omega(\lambda)) = \sum_{u \notin \mathcal{B}} x_u + \sum_{v \in \mathcal{B} \smallsetminus V_0} (x_v - 1) = \sum_{u \notin \mathcal{B}} x_u + \sum_{v \in \mathcal{B}} (x_v - 1) = (q-1)\zeta_q(\lambda).$$

Therefore we get that $\zeta_{q-1}(\Omega(\lambda)) = \zeta_q(\lambda)$.

Moreover, we claim that $V_0 \subseteq \{\zeta_q(\lambda)+1, \zeta_q(\lambda)+2, \ldots, t\}$. In order to prove this, we observe that from the definition of the total order $\gg$ on $\{1, \ldots, t\}$ it follows that if $y \in V_0 \neq \emptyset$, then for all $y' \notin \mathcal{B}$ we have that $x_{y'} = 1$ and $y' < y$. This shows that $z_1 = t - \zeta_q(\lambda) + |V_0| \geq t - \zeta_q(\lambda) + 1$ and that $y \geq t - \zeta_q(\lambda) + 1$. By Lemma 3.6(ii) we obtain that $2\zeta_q(\lambda) \leq t$. This implies that $y \geq \zeta_q(\lambda) + 1$.

Therefore we obtain that $\Delta^{q-1}(\Omega(\lambda))$ can be written as follows:

$$\Delta^{q-1}(\Omega(\lambda))_y = \begin{cases} k_y + 1 & \text{if } y \in \{1, \ldots, \zeta_q(\lambda)\}, \\ k_y & \text{if } y \in \{\zeta_q(\lambda)+1, \ldots, t\} \smallsetminus V_0, \\ k_y & \text{if } y \in V_0\,. \end{cases}$$

It follows that $\Delta^{q-1}(\Omega(\lambda)) = \Delta^q(\lambda)$, as required.

(ii) Suppose now that $z > \zeta_q(\lambda)$. Then by Lemma 3.6(i), we have that $t < 2\zeta_q(\lambda)$. Moreover we observe that in this case we have that $x_y = q-1$ for all $y \in \mathcal{B}$. Hence we have that $t \geq j_1 > j_2 > \cdots > j_{\zeta_q(\lambda)}$ and therefore that $j_{\zeta_q(\lambda)} \leq t - \zeta_q(\lambda) + 1$. Let V_{q-1} be the subset of $\{1, \ldots, t\}$ defined by

$$V_{q-1} = \big(\{1, \ldots, t\} \smallsetminus \mathcal{B}\big) \cap Z_{q-1}.$$

We claim that $V_{q-1} \subseteq \{1, \ldots, \zeta_q(\lambda)\}$. This can be seen by observing that for $y \in V_{q-1}$ we have that $y \ll u$ for all $u \in \mathcal{B}$. Equivalently this shows that $y < u$ for all $u \in \mathcal{B}$. Therefore we have that $y < j_{\zeta_q(\lambda)} \leq t - \zeta_q(\lambda) + 1 \leq \zeta_q(\lambda)$.

This shows that we can express $\Omega(\lambda)$ as follows:

$$\Omega(\lambda)_y = \begin{cases} (q-1)k_y + x_y & \text{if } y \in \big(\{1, \ldots, t\} \smallsetminus \mathcal{B}\big) \smallsetminus V_{q-1}, \\ (q-1)k_y + (q-1) & \text{if } y \in V_{q-1}, \\ (q-1)k_y + (x_y - 1) & \text{if } y \in \mathcal{B}\,. \end{cases}$$

Since $x_y - 1 = q - 2 > 0$ for all $y \in \mathcal{B}$, we obtain that

$$(q-1)\zeta_{q-1}(\Omega(\lambda)) = q\zeta_q(\lambda) - (q-1)|V_{q-1}| - |\mathcal{B}| = (q-1)\big(\zeta_q(\lambda) - |V_{q-1}|\big).$$

We deduce that $\zeta_{q-1}(\Omega(\lambda)) = \zeta_q(\lambda) - |V_{q-1}|$. This implies the following equality:

$$\Delta^{q-1}(\Omega(\lambda))_y = \begin{cases} k_y + 1 & \text{if } y \in \{1, \ldots, \zeta_q(\lambda)\} \smallsetminus V_{q-1}, \\ k_y + 1 & \text{if } y \in V_{q-1}, \\ k_y & \text{if } y \in \{\zeta_q(\lambda)+1, \ldots, t\}\,. \end{cases}$$

We conclude that $\Delta^{q-1}(\Omega(\lambda)) = \Delta^q(\lambda)$ also in this case. $\square$

3.2. **Characters of $\mathfrak{S}_n$.** In this second part of the section we apply the combinatorial ideas introduced above (especially Proposition 3.7) to prove Theorem 3.1. A key ingredient in our proof will be a rather sophisticated use of the Littlewood–Richardson rule (see [J, Chapter 16]). For the convenience of the reader we recall this here.

Definition 3.8. Let $\mathcal{A} = a_1, \ldots, a_k$ be a sequence of positive integers. The type of $\mathcal{A}$ is the sequence of nonnegative integers $m_1, m_2, \ldots$, where m_i is the number of occurrences of i in $a_1, \ldots, a_k$. We say that $\mathcal{A}$ is a *reverse lattice sequence* if the type of its prefix $a_1, \ldots, a_j$ is a partition, for all $j \geq 1$. Equivalently, for each $j = 1, \ldots, k$ such that $a_j = i \geq 2$, we require that

$$|\{u \mid 1 \leq u \leq j, a_u = i-1\}| \geq |\{v \mid 1 \leq v \leq j, a_v = i\}|.$$

In this case we say that the element a_j is *good* in $\mathcal{A}$.

Let $\omega \vdash n$ and $\delta \vdash m$. The outer tensor product $\chi^\omega \otimes \chi^\delta$ is an irreducible character of $\mathfrak{S}_n \times \mathfrak{S}_m$. Inducing this character to $\mathfrak{S}_{n+m}$ we may write

$$(\chi^\omega \otimes \chi^\delta)^{\mathfrak{S}_{n+m}} = \sum_{\lambda \vdash (n+m)} C^\lambda_{\omega,\delta} \chi^\lambda.$$

The *Littlewood–Richardson rule* asserts that $C^\lambda_{\omega,\delta}$ is zero if $\omega \not\subseteq \lambda$ and otherwise equals the number of ways to replace the nodes of the skew Young diagram $\lambda \smallsetminus \omega$ by natural numbers such that

(1) The numbers are weakly increasing along rows.
(2) The numbers are strictly increasing down the columns.
(3) The sequence obtained by reading the numbers from right to left and top to bottom is a reverse lattice sequence of type δ.

We call any such configuration a *Littlewood–Richardson tableau of type* δ for $\lambda \smallsetminus \omega$.

Let H be a finite group. From now on we denote by H^n the n-fold direct product $H \times H \times \cdots \times H$. Similarly, for $\theta \in \mathrm{Irr}(H)$ we denote by $\theta^{\otimes n}$ the irreducible character of H^n defined by $\theta^{\otimes n} = \underbrace{\theta \otimes \cdots \otimes \theta}_{n}$.

The following statement is a fundamental step to complete the proof of Theorem 3.1 and was first conjectured in [G2, Conjecture B].

Theorem 3.9. *Let n and q be natural numbers, with $q \geq 2$. Let $\lambda \in \mathcal{P}(qn)$ and let $\mu = \Delta^q(\lambda) \in \mathcal{P}(n)$. Then $(\chi^\mu)^{\otimes q}$ is an irreducible constituent of $(\chi^\lambda)_{\mathfrak{S}_n^q}$.*

Proof. We proceed by induction on $q \in \mathbb{N}_{\geq 2}$. Let $q = 2$ and let

$$\lambda = \big((2k_1+1, 2k_2+1, \ldots, 2k_t+1) \circ (2r_1, \ldots, 2r_s)\big)^\star \in \mathcal{P}(2n),$$

for some $k_1 \geq k_2 \geq \cdots \geq k_t \geq 0$, $r_1 \geq r_2 \geq \cdots \geq r_s > 0$. Clearly t must be even. We write $t = 2\zeta_2(\lambda)$ for some $\zeta_2(\lambda) \in \mathbb{N}_0$. It will be convenient to write $\{1, 2, \ldots, t+s\} = \mathcal{O} \cup \mathcal{E}$, where $\mathcal{O} = \{j \in \{1, 2, \ldots, t+s\} \mid \lambda_j \text{ is odd}\}$ and $\mathcal{E} = \{j \in \{1, 2, \ldots, t+s\} \mid \lambda_j \text{ is even}\}$. Clearly $|\mathcal{O}| = t$, and it is convenient to write it as

$$\mathcal{O} = \{h_1, h_2, \ldots, h_t\},$$

where $1 \leq h_1 < h_2 < \cdots < h_t \leq t+s$. In particular, we have that $\lambda_{h_i} = 2k_i + 1$, for all $i \in \{1, \ldots, t\}$.

By Definition 3.2 we obtain that

$$\mu := \Delta^2(\lambda) = \big((k_1+1,\dots,k_{\zeta_2(\lambda)}+1,k_{\zeta_2(\lambda)+1},\dots,k_t)\circ(r_1,\dots,r_s)\big)^\star \in \mathcal{P}(n).$$

Clearly μ is a subpartition of λ. Let $S(\lambda)$ be the skew Young diagram defined by $S(\lambda)=\lambda\smallsetminus\mu$. We observe that $S(\lambda)_{h_i}=k_i$ for $1\leq i\leq\zeta_2(\lambda)$, $S(\lambda)_{h_i}=k_i+1$ for $\zeta_2(\lambda)+1\leq i\leq t$, and $S(\lambda)_j=\frac{\lambda_j}{2}$ for all $j\in\mathcal{E}$.

To prove that $(\chi^\mu)^{\otimes 2}$ is a consitutent of $(\chi^\lambda)_{\mathfrak{S}_n^2}$, it is enough to show that there exists a way to replace the nodes of $S(\lambda)$ by integers in order to obtain a Littlewood–Richardson tableau of type μ. This is done as follows.

Step 1. Let $j\in\{1,\dots,t+s\}$. If $j\in\mathcal{E}$, then replace all the $\frac{\lambda_j}{2}$ nodes of row j of $S(\lambda)$ by j. If $j\in\mathcal{O}$, then replace the rightmost $\frac{\lambda_j-1}{2}$ nodes of row j of $S(\lambda)$ by j.

At this stage we still have $\zeta_2(\lambda)$ empty nodes $b_1,\dots,b_{\zeta_2(\lambda)}$ in $S(\lambda)$. Each of these nodes is the leftmost node in its row. More precisely, b_i is the leftmost node of row $h_{\zeta_2(\lambda)+i}$ of $S(\lambda)$. Moreover each b_i is either at the top of its column or lies below the empty node b_{i-1}. This observation follows easily from the construction of $S(\lambda)=\lambda\smallsetminus\mu$.

Step 2. Replace node b_i by h_i, for all $i\in\{1,\dots,\zeta_2(\lambda)\}$.

We now have that for all $j\notin\{h_{\zeta_2(\lambda)+1},h_{\zeta_2(\lambda)+2},\dots,h_t\}$ all nodes of row j of $S(\lambda)$ have been replaced by j. On the other hand, row $h_{\zeta_2(\lambda)+i}$ has h_i in the leftmost node and $h_{\zeta_2(\lambda)+i}$ in all other nodes. We conclude that numbers are weakly increasing along rows. Since each node b_i (empty node after Step 1) is either at the top of its column or lies below b_{i-1}, it follows that after Step 2 the numbers are strictly increasing down the columns. Finally let $\mathcal{A}$ be the sequence obtained by reading the numbers from right to left and top to bottom. It is clear by construction that $\mathcal{A}$ is a sequence of type μ. Moreover, for $j\geq 2$ we have that the number of nodes replaced by j in row j of $S(\lambda)$ is smaller than or equal to the number of nodes replaced by $j-1$ in row $j-1$ of $S(\lambda)$. Hence every j appearing in row j is good. If j replaces a node lying below row j, then $j=h_i$ replaces node b_i in row $h_{\zeta_2(\lambda)+i}$, for some $i\in\{1,\dots,\zeta_2(\lambda)\}$. By construction we know that the nodes of $S(\lambda)$ that are replaced by $j-1$ are all lying in higher rows than $h_{\zeta_2(\lambda)+i}$. In fact at most one $j-1$ lies in row $h_{\zeta_2(\lambda)+(i-1)}$ (if $j=h_{i-1}$), while all other $j-1$ replace nodes of row $j-1$. Hence also in this case j is good, and therefore $\mathcal{A}$ is a good sequence of type μ.

We conclude that Steps 1 and 2 provide us with a Littlewood–Richardson tableau of type μ. Therefore we deduce that $(\chi^\mu)^{\otimes 2}$ is a constituent of $(\chi^\lambda)_{\mathfrak{S}_n\times\mathfrak{S}_n}$.

Suppose now that $q\geq 3$. As usual let

$$\lambda=\big((k_1q+x_1,\dots,k_tq+x_t)\circ(r_1q,\dots,r_sq)\big)^\star,$$

where $k_1\geq k_2\geq\cdots\geq k_t\geq 0$, $r_1\geq r_2\geq\cdots\geq r_s>0$, and where $x_j\in\{1,\dots,q-1\}$ for all $j\in\{1,\dots,t\}$. Let $\zeta_q(\lambda)=\frac{1}{q}\cdot(x_1+\cdots+x_t)\in\mathbb{N}_0$. Again we write $\{1,2,\dots,t+s\}=\mathcal{O}\cup\mathcal{E}$, where $\mathcal{O}=\{j\in\{1,2,\dots,t+s\}\ :\ q\nmid\lambda_j\}$ and $\mathcal{E}=\{j\in\{1,2,\dots,t+s\}\ :\ q|\lambda_j\}$. Clearly $|\mathcal{O}|=t$, and it is convenient to write it in two different ways, as follows:

$$\mathcal{O}=\{h_1,h_2,\dots,h_t\}=\{g_1,g_2,\dots,g_t\},$$

where $1\leq g_1<g_2<\cdots<g_t\leq t+s$ and $h_1\ll h_2\ll\cdots\ll h_t$ (see Definition 3.3).

By Definition 3.2 we obtain that

$$\mu := \Delta^q(\lambda) = \big((k_1+1, k_2+1, \dots, k_{\zeta_q(\lambda)}+1, k_{\zeta_q(\lambda)+1}, \dots, k_t) \circ (r_1, r_2, \dots, r_s)\big)^\star.$$

In particular we have $\mu_{g_j} = k_j + 1$ for all $1 \leq j \leq \zeta_q(\lambda)$, $\mu_{g_j} = k_j$ for all $\zeta_q(\lambda)+1 \leq j \leq t$, and $\mu_i = \frac{\lambda_i}{q}$ for all $i \in \mathcal{E}$.

Let $\Omega := \Omega(\lambda)$ be the partition of $(q-1)n$ introduced in Definition 3.4. Using the inductive hypothesis together with Proposition 3.7 we deduce that $(\chi^\mu)^{\otimes q-1}$ is an irreducible constituent of $(\chi^\Omega)_{\mathfrak{S}_n^{q-1}}$.

Clearly Ω is a subpartition of λ, hence we let $S(\lambda) := \lambda \smallsetminus \Omega$. Observe in particular that row h_j of $S(\lambda)$ consists of $k_j + 1$ nodes for all $t - \zeta_q(\lambda) + 1 \leq j \leq t$. On the other hand, if $1 \leq j \leq t - \zeta_q(\lambda)$, then row h_j of $S(\lambda)$ consists of k_j nodes.

To conclude the proof it is enough to show that there exists a way to replace the nodes of $S(\lambda)$ by integers in order to obtain a Littlewood–Richardson tableau of type μ. This is done using the following algorithm (it might be convenient for the reader to read the following steps together with Example 3.10 below).

Step 1. For $j \in \mathcal{E}$ replace all the $\frac{\lambda_j}{q}$ nodes of row j of $S(\lambda)$ by j. For $j \in \mathcal{O}$ we have that $\lambda_j = qk_i + x_i$ for some $i \in \{1, \dots, t\}$. Then replace the k_i rightmost nodes of row j of $S(\lambda)$ by j.

After Step 1 is completed, we have precisely $\zeta_q(\lambda)$ nodes $b_1, \dots, b_{\zeta_q(\lambda)}$ of $S(\lambda)$ that have not yet been replaced by integers. Each b_i is the leftmost node of row f_i, where $f_i \in \{h_{t-\zeta_q(\lambda)+1}, h_{t-\zeta_q(\lambda)+2}, \dots, h_t\}$ is defined such that $f_1 < f_2 < \cdots < f_{\zeta_q(\lambda)}$. (This third change of notation for some of the elements of $\mathcal{O}$ is necessary to understand the relative ordering of the rows having an empty node at this stage).

Moreover, for all $i \in \{1, \dots, \zeta_q(\lambda)\}$ we have that b_i is either at the top of its column or b_{i-1} is the node above b_i. This is proved as follows. Let $\lambda_{f_i} = kq + z$ and $\lambda_{f_i - 1} = wq + y$ for some $k, z, w, y \in \mathbb{N}$ such that $k \leq w$, $1 \leq z \leq q-1$, and $0 \leq y \leq q-1$. Suppose that b_i is not at the top of its column in $S(\lambda)$. Then we necessarily have $k = w$ and hence that $1 \leq z \leq y \leq q-1$. If $y > z$, then from Definition 3.3 we deduce that $y \gg z$ and therefore that $i \geq 2$ and $f_i - 1 = f_{i-1}$ and hence that

$$\Omega_{f_i - 1} = \lambda_{f_i-1} - (k+1) = k(q-1) + (y-1) > k(q-1) + (z-1) = \Omega_{f_i}.$$

This would imply that b_i is a the top of its column. Hence we must discard this situation as well and assume that $\lambda_{f_i} = \lambda_{f_i - 1}$. If $f_i - 1 \notin \{h_{t-\zeta_q(\lambda)+1}, h_{t-\zeta_q(\lambda)+2}, \dots, h_t\}$, then again we would have

$$\Omega_{f_i-1} = \lambda_{f_i-1} - k = k(q-1) + y > k(q-1) + (y-1) = \Omega_{f_i},$$

which would imply that b_i is at the top of its column. Hence $i \geq 2$, $f_i - 1 = f_{i-1}$, and

$$\Omega_{f_{i-1}} = k(q-1) + (y-1) = \Omega_{f_i},$$

which implies that b_i lies below the mostleft node of row $f_i - 1 = f_{i-1}$ of $S(\lambda)$, that is, node b_{i-1}.

Step 2. For every $i \in \{1, \dots, \zeta_q(\lambda)\}$ replace node b_i by g_i.

The remarks after Step 1 guarantee that the configuration obtained has integers strictly increasing down the columns. Since $g_i \leq f_i$ for all $i \in \{1, \dots, \zeta_q(\lambda)\}$ we easily deduce that the integers are weakly increasing along the rows (from left to right). It is not difficult to see that the sequence obtained by reading the integers from right to left and top to bottom is a good sequence of type μ (a full proof of this fact is completely similar to the one given above for the base case $q = 2$).

We conclude that $\chi^{\Omega} \otimes \chi^{\mu}$ is an irreducible constituent of $(\chi^{\lambda})_{\mathfrak{S}_{(q-1)n} \times \mathfrak{S}_n}$. This completes the proof. $\square$

Before proceeding with the proof of Theorem 3.1, we illustrate in a concrete example Steps 1 and 2 described in the proof of Theorem 3.9.

Example 3.10. Let $q = 5$, $n = 10$, and let $\lambda = (12, 9, 8, 6, 6, 5, 4) \vdash 50$ be as in Example 3.5. Then the Littlewood–Richardson tableau of type $\Delta^5(\lambda)$ for $\lambda \smallsetminus \Omega(\lambda)$, described in the proof of Theorem 3.9, is depicted in Figure 1.

$$[\lambda] = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
\mathrm{X} & \mathrm{X} & \mathrm{X} & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & 1 & 1 \\
\hline
\mathrm{X} & \mathrm{X} & \bullet & \bullet & \bullet & \bullet & \bullet & \underline{1} & 2 \\
\cline{1-9}
\mathrm{X} & \mathrm{X} & \bullet & \bullet & \bullet & \bullet & \underline{2} & 3 \\
\cline{1-8}
\mathrm{X} & \bullet & \bullet & \bullet & \bullet & 4 \\
\cline{1-6}
\mathrm{X} & \bullet & \bullet & \bullet & \bullet & 5 \\
\cline{1-6}
\mathrm{X} & \bullet & \bullet & \bullet & 6 \\
\cline{1-5}
\bullet & \bullet & \bullet & \underline{3} \\
\cline{1-4}
\end{array}$$

FIGURE 1. The Young diagram $[\Delta^5(\lambda)]$ consists of those boxes containing an X. The Young diagram $[\Omega(\lambda)]$ is the union of the boxes of $[\Delta^5(\lambda)]$ and of those boxes containing a black dot. The skew Young diagram $S(\lambda)$ consists of the remaining boxes containing numbers. Nonunderlined numbers correspond to Step 1 of the algorithm described in the proof of Theorem 3.9. Underlined numbers are assigned to the boxes of $S(\lambda)$ according to Step 2 of the algorithm.

We can now prove Theorem 3.1, using Theorem 3.9.

Proof of Theorem 3.1. Let p be a prime number and let $k \in \mathbb{N}$. We let $P := P_{p^k}$ be a Sylow p-subgroup of $\mathfrak{S}_{p^k}$. Then $P = B \rtimes C_p$, where

$$B = P_{p^{k-1}} \times \cdots \times P_{p^{k-1}} \leq \mathfrak{S}_{p^{k-1}} \times \cdots \times \mathfrak{S}_{p^{k-1}} = \mathfrak{S}_{p^{k-1}}^{p},$$

and C_p acts on B permuting its p direct factors. We show by induction on k that χ_P has a linear constituent. If $k = 1$ this is obvious since P is abelian. Suppose that $k \geq 2$. By Theorem 3.9 there exists $\theta \in \mathrm{Irr}(\mathfrak{S}_{p^{k-1}})$ such that $\theta^{\otimes p}$ is an irreducible constituent of $\chi_{\mathfrak{S}_{p^{k-1}}^{p}}$. By inductive hypothesis there exists a linear character ϕ of P_{p^k-1} appearing as an irreducible constituent of $\theta_{P_{p^{k-1}}}$. Hence there exists an irreducible constituent ψ of χ_P lying above $\phi^{\otimes p} \in \mathrm{Irr}(B)$. Now, $\phi^{\otimes p}$ is C_p-invariant and it naturally extends to an irreducible character of P. This observation allows us to use Gallagher's Corollary (see [I, 6.17]) to deduce that ψ is linear, since $P/B \cong C_p$ is abelian.

Now let n be any natural number, with p-adic expansion $n = \sum_j a_j p^j$. If P_n is a Sylow p-subgroup of $\mathfrak{S}_n$, then we have that

$$P_n = \prod_{j=0}^{k} (P_{p^j})^{a_j} \leq \prod_{j=0}^{k} (\mathfrak{S}_{p^j})^{a_j} =: H.$$

Since $\chi_P = (\chi_H)_P$ the result follows. $\square$

In order to prove Theorem A for symmetric groups, we need the following observation.

Lemma 3.11. *Let k be a positive integer. There exists a p^k-cycle $g \in P_{p^k} \leq \mathfrak{S}_{p^k}$ such that the following hold:*

(i) *$\theta(g)$ is a pth root of unity for every linear character θ of P_{p^k}.*
(ii) *$\delta(g) = 0$ for all $\delta \in \mathrm{Irr}(P_{p^k})$ such that $p \mid \delta(1)$.*

Proof. Recall that $P_{p^k} = P_{p^{k-1}} \wr C_p = B \rtimes C_p$, where $B = P_{p^{k-1}} \times \cdots \times P_{p^{k-1}}$. Let $g \in P_{p^k} = P_{p^{k-1}} \wr C_p$ be a p^k-cycle of the form

$$g = (h, 1, \ldots, 1; k),$$

for some $h \in P_{p^{k-1}}$ such that h is a p^{k-1}-cycle in $\mathfrak{S}_{p^{k-1}}$ and $k \in C_p$.

To prove (i), we proceed by induction on k. If $k = 1$ the result follows trivially. Suppose that $k \geq 2$. For $\phi \in \mathrm{Irr}_{p'}(P_{p^{k-1}})$ we let $\hat{\phi}$ be the natural extension of $\phi^{\otimes p} \in \mathrm{Irr}(B)$ to $P_{p^{k-1}}$. By Gallagher's Corollary (see [I, 6.17]) we have that

$$\mathrm{Irr}(P_{p^k} \mid \phi^{\otimes p}) = \{\mathcal{X}(\phi; \psi) \mid \psi \in \mathrm{Irr}(C_p)\},$$

where $\mathcal{X}(\phi; \psi) = \hat{\phi} \cdot \inf_{C_p}^{P_{p^k}}(\psi)$. In particular, we observe that

$$\mathrm{Irr}_{p'}(P_{p^k}) = \{\mathcal{X}(\phi; \psi) \mid \phi \in \mathrm{Irr}_{p'}(P_{p^{k-1}}),\ \psi \in \mathrm{Irr}(C_p)\}.$$

If $\theta = \mathcal{X}(\phi; \psi)$ is a linear character of P_{p^k}, then we have that $\theta(x) = \mathcal{X}(\phi; \psi)(g) = \phi(h)\psi(k)$ (see for instance [JK, Lemma 4.3.9]). By inductive hypothesis we deduce that $\theta(x)$ is a pth root of unity.

Statement (ii) is proved similarly. $\square$

Remark 3.12. Let $k \in \mathbb{N}$. For $s \in \{0, 1, \ldots, k-1\}$ let γ_s be the element of $\mathfrak{S}_{p^k}$ defined by

$$\gamma_s = \prod_{j=1}^{p^s} (j, j + p^s, j + 2p^s, \ldots, j + (p-1)p^s).$$

Then $P := \langle \gamma_0, \gamma_1, \ldots, \gamma_{k-1} \rangle$ is a Sylow p-subgroup of $\mathfrak{S}_{p^k}$ and $g = \gamma_0 \gamma_1 \cdots \gamma_{k-1} \in P$ is a p^k-cycle satisfying conditions (i) and (ii) of Lemma 3.11.

We are now ready to prove Theorem A.

Proof of Theorem A. Let us first deal with the case where $n = p^k$, for some $k \in \mathbb{N}$. Since p divides $\chi(1)$ we know that $\chi = \chi^\lambda$ for some $\lambda \in \mathcal{P}(n)$ such that λ is not a hook partition. By Theorem 3.1 we know that there exists a linear constituent θ_1 of χ_P. Suppose for a contradiction that there exists $1 \leq t < p$ such that

$$\chi_P = \sum_{j=1}^{t} c_j \theta_j + \Delta,$$

where for all $1 \leq j < p$ we have that $\theta_j(1) = 1$, $c_j \in \mathbb{N}_{\geq 1}$ and where Δ is a sum of irreducible characters of P of degree divisible by p. Let $g \in P$ be a p^k-cycle satisfying conditions (i) and (ii) of Lemma 3.11. Since λ is not a hook partition we have that $\chi(g) = 0$. Moreover, from Lemma 3.11 we obtain that

$$0 = \chi(g) = c_1\theta_1(g) + c_2\theta_2(g) + \cdots + c_t\theta_t(g).$$

This is a contradiction since no $\mathbb{N}$-linear combination of t pth roots of unity can be equal to 0.

Let us now consider the case where n is not a power of p. Let $n = \sum_{j=0}^{k} a_j p^j$ be the p-adic expansion of n and let $g \in P_n$ be the product of a_j p^j-cycles for $j \in \{0, 1, \ldots, k\}$. By [MNO, Theorem 4.1] we have that $\chi(g) = 0$ for all $\chi \in \mathrm{Irr}(\mathfrak{S}_n)$ such that $p \mid \chi(1)$. Since

$$P_n = \prod_{j=0}^{k} (P_{p^j})^{a_j},$$

we deduce from Lemma 3.11 (and Remark 3.12) that g can be chosen such that:

- $\delta(g) = 0$ for all $\delta \in \mathrm{Irr}(P_n)$ such that $p \mid \delta(1)$;
- $\theta(g)$ is a pth root of unity for any linear character θ of P_n.

Arguing exactly as in the case where $n = p^k$ we obtain that χ_{P_n} must have at least p distinct linear constituents. □

Remark 3.13. As mentioned in the introduction, we can always find an arbitrarily large natural number n such that $\mathfrak{S}_n$ admits irreducible even-degree characters whose restriction to a Sylow 2-subgroup has exactly 2 constituents. An example follows. Let $p = 2$ and let $n = 2^k + 1$ for some $k \in \mathbb{N}$ such that $k \geq 2$. Let $\lambda \in \{(n-x, 1^x) \mid x \in \{1, \ldots, n-2\}\}$ and let P be a Sylow 2-subgroup of $\mathfrak{S}_n$. Then it is easy to see that $(\chi^\lambda)_P = \psi_x + \psi_{x-1} + \Delta$, where Δ is a sum (possibly empty) of irreducible characters of P of even degree and where $\psi_j \in \mathrm{Irr}(P)$ is the unique linear constituent of $(\chi^{(2^k-j,1^j)})_P$ for any $j \in \{0, 1, \ldots, 2^k - 1\}$ (as prescribed by [G1]).

A full classification of these instances will be the subject of a different article. This will also include a proof of Conjecture D for alternating groups. When p is odd, this is just an easy consequence of the result proved for symmetric groups. For $p = 2$ more attention and more combinatorics are required.

4. p-solvable groups

In this section, we prove Theorem B. First, we need a lemma.

Lemma 4.1. *Suppose that G/M is a p-group, and let $\chi \in \mathrm{Irr}(G)$ such that χ_M is not irreducible. Then there exists $M \subseteq U \lhd G$ of index p such that $\chi = \gamma^G$, for some $\gamma \in \mathrm{Irr}(U)$.*

Proof. Let $N/M \lhd G/M$ such that χ_N is irreducible such that $|N/M|$ is as small as possible. We know that $N > M$. Since G/M is a p-group, let $N > E \geq M$ be such that $E \lhd G$ and $|N : E| = p$. Write $\tau = \chi_N$. Since τ_E is not irreducible, it follows by Corollary 6.19 of [I] that τ_E is not homogeneous. So χ_E is not homogeneous. Thus, by the Clifford correspondence, χ is induced from a character of a proper subgroup containing M, which we may assume has index p, and therefore is normal in G. □

Next we prove a relative version of Theorem B, which is obtained in the case where $K = 1$.

Theorem 4.2. *Suppose that $K \lhd G$, and let P/K be a nontrivial Sylow p-subgroup of G/K. Let $\chi \in \mathrm{Irr}(G)$ be such that χ_P contains $\lambda \in \mathrm{Irr}(P)$ such that λ_K is irreducible and $\chi(1)/\lambda(1)$ is divisible by p. If G/K is p-solvable, then there exists $K \subseteq V \subseteq P$ of index p such that χ_P contains $(\lambda_V)^P$.*

Proof. We argue by induction on $|G : K|$. Suppose that there exists $P \subseteq H < G$ with a character $\psi \in \mathrm{Irr}(H)$ under χ and over λ such that p divides $\psi(1)/\lambda(1)$. Then χ_P contains ψ_P, and we are done by induction.

Let $\nu = \lambda_K \in \mathrm{Irr}(K)$, which lies under χ. Let T be the stabilizer of ν in G. Now $P \subseteq T \subseteq G$. Let $\psi \in \mathrm{Irr}(T)$ be under χ and over λ. Thus $\psi^G = \chi$ by the Clifford correspondence. Also, p divides $\psi(1)/\nu(1)$. Hence, by induction we may assume that $T = G$. Now we use the properties of character triple isomorphisms in Definition 11.23 of [I] and Theorem 11.28 of [I] to show that we may assume that K is central. Suppose now that (G^*, K^*, ν^*) is a triple isomorphic to (G, K, ν) with $K^* \subseteq \mathbf{Z}(G^*)$. Write

$$^* : G/K \to G^*/K^*$$

for the associated isomorphism, and for subgroups $K \leq H \leq G$, write $(H/K)^* = H^*/K^*$. Also, write $\psi^* \in \mathrm{Irr}(H^*|\nu^*)$ for the image of $\psi \in \mathrm{Irr}(H|\nu)$ under the isomorphism. By Definition 11.23(b) of [I], we have that λ^* is an irreducible constituent of χ^*. By Lemma 11.24 of [I], we have that $(\lambda^*)_{K^*}$ is irreducible because $\lambda_K = \nu$. By the same lemma, p divides $\chi^*(1)/\nu^*(1) = \chi^*(1)$. We show next that if the theorem is true in G^*, then it is true in K^*. Suppose that $K^* \leq V^* \leq P^*$ has index p and that $(\chi^*)_{P^*}$ contains $((\lambda^*)_{V^*})^{P^*}$. Notice that $((\lambda^*)_{V^*})^{P^*} = ((\lambda_V)^*)^{P^*}$ by Definition 11.23(b). Again by Definition 11.23(b) and using Frobenius reciprocity, we have that

$$((\lambda_V)^*)^{P^*} = ((\lambda_V)^P)^*.$$

Therefore $(\chi^*)_{P^*}$ contains $((\lambda_V)^P)^*$, and therefore χ_P contains $(\lambda_V)^P$, again by Definition 11.23(b) and Frobenius reciprocity.

Notice that G/K is not a p-group, because otherwise $P = K$, $\chi = \lambda$, and p does not divide $\chi(1)/\lambda(1)$. Let $M/K = \mathbf{O}^p(G/K)$, $N/K = \mathbf{O}^{p'}(M/K)$, and $H = PN$. Notice that $H < G$, because G/K is not a p-group. Now, let $\psi \in \mathrm{Irr}(H)$ be under χ and over λ. Then p does not divide $\psi(1)$ by the first paragraph. Thus, by Corollary 11.29 of [I], we have that $\rho = \psi_N \in \mathrm{Irr}(N)$ has p'-degree, lies under χ. If χ_M is irreducible, then using that M/N is a p'-group and that ρ has p'-degree, we would deduce that $\chi(1)$ has p'-degree, again by Corollary 11.29 of [I]. So we deduce that χ_M is not irreducible. By Lemma 4.1, there exists $M \subseteq U \triangleleft G$ of index p, and $\gamma \in \mathrm{Irr}(U)$ such that $\gamma^G = \chi$. Let $K \subseteq V = P \cap U$. Notice that $UP = G$ and that V has index p in P. Now, by Mackey,

$$\chi_P = (\gamma_V)^P$$

contains λ. So γ_V contains λ_V, and χ_P contains $(\lambda_V)^P$. □

References

[B] David W. Burry, *A strengthened theory of vertices and sources*, J. Algebra **59** (1979), no. 2, 330–344, DOI 10.1016/0021-8693(79)90131-5. MR543254

[G1] Eugenio Giannelli, *Characters of odd degree of symmetric groups*, J. Lond. Math. Soc. (2) **96** (2017), no. 1, 1–14. MR3687936

[G2] Eugenio Giannelli, *On the restriction of irreducible characters of symmetric groups to Sylow p-subgroups*, J. Algebra **483** (2017), 37–57, DOI 10.1016/j.jalgebra.2017.03.037. MR3649811

[I] I. Martin Isaacs, *Character theory of finite groups*, AMS Chelsea Publishing, Providence, RI, 2006. Corrected reprint of the 1976 original [Academic Press, New York; MR0460423]. MR2270898

[J] G. D. James, *The representation theory of the symmetric groups*, Lecture Notes in Mathematics, vol. 682, Springer, Berlin, 1978. MR513828

[JK] Gordon James and Adalbert Kerber, *The representation theory of the symmetric group*, with a foreword by P. M. Cohn; with an introduction by Gilbert de B. Robinson. Encyclopedia of Mathematics and its Applications, vol. 16, Addison-Wesley Publishing Co., Reading, Mass., 1981. MR644144

[KM] Radha Kessar and Gunter Malle, *Quasi-isolated blocks and Brauer's height zero conjecture*, Ann. of Math. (2) **178** (2013), no. 1, 321–384, DOI 10.4007/annals.2013.178.1.6. MR3043583

[K] Reinhard Knörr, *On the vertices of irreducible modules*, Ann. of Math. (2) **110** (1979), no. 3, 487–499, DOI 10.2307/1971234. MR554380

[MNO] Gunter Malle, Gabriel Navarro, and Jørn B. Olsson, *Zeros of characters of finite groups*, J. Group Theory **3** (2000), no. 4, 353–368, DOI 10.1515/jgth.2000.028. MR1790334

[NT] Hirosi Nagao and Yukio Tsushima, *Representations of finite groups*, Academic Press, Inc., Boston, MA, 1989. Translated from the Japanese. MR998775

[N] Gabriel Navarro, *Linear characters of Sylow subgroups*, J. Algebra **269** (2003), no. 2, 589–598, DOI 10.1016/S0021-8693(03)00391-0. MR2015855

[NTV] Gabriel Navarro, Pham Huu Tiep, and Carolina Vallejo, *McKay natural correspondences on characters*, Algebra Number Theory **8** (2014), no. 8, 1839–1856, DOI 10.2140/ant.2014.8.1839. MR3285617

[O1] Jørn B. Olsson, *Combinatorics and representations of finite groups*, Vorlesungen aus dem Fachbereich Mathematik der Universität GH Essen [Lecture Notes in Mathematics at the University of Essen], vol. 20, Universität Essen, Fachbereich Mathematik, Essen, 1993. MR1264418

[O2] Jørn B. Olsson, *McKay numbers and heights of characters*, Math. Scand. **38** (1976), no. 1, 25–42, DOI 10.7146/math.scand.a-11614. MR0409625

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