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LEFSCHETZ DECOMPOSITIONS FOR EIGENFORMS ON A KÄHLER MANIFOLD

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ABSTRACT. We show that the eigenspaces of the Laplacian Δ_k on k-forms on a compact Kähler manifold carry Hodge and Lefschetz decompositions. Among other consequences, we show that the positive part of the spectrum of Δ_k lies in the spectrum of Δ_{k+1} for $k < \dim X$.

Given a compact Riemannian manifold X without boundary, let Δ_k denote the Laplacian on k-forms and $\lambda_1^{(k)}$ its smallest positive eigenvalue. We can ask how these numbers vary with k. By differentiating eigenfunctions, we easily see that $\lambda_1^{(0)} \geq \lambda_1^{(1)}$. For $k > 1$, the situation is more complicated: Takahashi [T2] has shown that the sign of $\lambda_1^{(k)} - \lambda_1^{(0)}$ can be arbitrary for compact Riemannian manifolds. More generally, Guerini and Savo [G, GS] have constructed examples, where the sequence $\lambda_1^{(2)}, \lambda_1^{(3)}, \ldots, \lambda_1^{([\dim X/2])}$ can do just about anything. The goal of this note is to show that when X is compact Kähler, the eigenspaces carry extra structure, and that this imposes strong constraints on the eigenvalues and their multiplicities. For instance, we show that the eigenvalues of Δ_k occur with even multiplicities when k is odd. We also show that the positive part of the spectrum of Δ_k is contained in the spectrum of Δ_{k+1} for all $k < \dim_{\mathbb{C}} X$. Therefore the sequence $\lambda_1^{(0)}, \ldots, \lambda_1^{(\dim_{\mathbb{C}} X)}$ is weakly decreasing.

After this paper was submitted, it was brought to the author's attention that Jakobson, Strohmaier, and Zelditch [JSZ] have also studied the spectra of Kähler manifolds, although for rather different reasons.

1. MAIN THEOREMS

For the remainder of this paper, X will denote a compact Kähler manifold of complex dimension n, with Kähler form ω. Let $\Delta = d^*d + dd^*$ be the Laplacian on complex valued forms $\mathcal{E}^*$. Standard arguments in Hodge theory guarantee that the spectrum of Δ is discrete, and the eigenspaces

$$\mathcal{E}^*_\lambda = \{\alpha \in \mathcal{E}^* \mid \Delta\alpha = \lambda\alpha\}$$

are finite dimensional. Since Δ is positive and self adjoint, the eigenvalues are nonnegative real. We let $\mathcal{E}^k_\lambda$ and $\mathcal{E}^{(p,q)}_\lambda$ denote the intersection of $\mathcal{E}^*_\lambda$ with the space of k forms and (p,q)-forms respectively.

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The proofs of the following statements will naturally hinge on the Kähler identities [GH, W], which we recall below. We have

$$\Delta = 2(\partial\partial^* + \partial^*\partial) = 2(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)$$

which implies that it commutes with the projections $\pi^{pq}, \pi^k : \mathcal{E}^* \to \mathcal{E}^{(p,q)}, \mathcal{E}^k$. The Laplacian Δ also commutes with the Lefschetz operator $L(-) = \omega \wedge -$ and its adjoint Λ. An additional set of identities implies that L, Λ and $H = \sum(n-k)\pi^k$ together determine an action of the Lie algebra $sl_2(\mathbb{C})$ on $\mathcal{E}^*$.

Theorem 1.1. *For each λ, there is a Hodge decomposition*

$$\mathcal{E}^k_\lambda = \bigoplus_{p+q=k} \mathcal{E}^{(p,q)}_\lambda, \tag{1}$$

$$\overline{\mathcal{E}^{(p,q)}_\lambda} = \mathcal{E}^{(q,p)}_\lambda. \tag{2}$$

If $i > 0$, there is a hard Lefschetz isomorphism

$$\omega^i \wedge : \mathcal{E}^{n-i}_\lambda \xrightarrow{\sim} \mathcal{E}^{n+i}_\lambda. \tag{3}$$

Proof. (1) follows from the fact that Δ commutes with π^{pq}. Since Δ and λ are real, we obtain (2). The proof of (3) is identical to the usual proof of the hard Lefschetz theorem [GH, pp. 118-122]. The key point is that by representation theory, L^i maps $V \cap \mathcal{E}^{n-i}$ isomorphically to $V \cap \mathcal{E}^{n+i}$ for any $sl_2(\mathbb{C})$-submodule $V \subset \mathcal{E}^*$. Applying this to the subspace $V = \mathcal{E}_\lambda$, which is an $sl_2(\mathbb{C})$-submodule because L, Λ, H commute with Δ, proves (3). □

The first part of the theorem can be rephrased as saying that $\mathcal{E}^k_\lambda$ is a real Hodge structure of weight k. We define the multiplicities $h^{pq}_\lambda = \dim \mathcal{E}^{(p,q)}_\lambda$ and $b^i_\lambda = \dim \mathcal{E}^i_\lambda$. When $\lambda = 0$, these are the usual Hodge and Betti numbers. In general, they depend on the metric. These numbers share many properties of ordinary Hodge and Betti numbers:

Corollary 1.2. *For each λ,*

(a) $b^k_\lambda = \sum_{p+q=k} h^{pq}_\lambda$,
(b) $h^{pq}_\lambda = h^{qp}_\lambda$,
(c) *b^k_λ is even if k is odd,*
(d) $b^{2n-k}_\lambda = b^k_\lambda$,
(e) *if $k < n$, $b^k_\lambda \le b^{k+2}_\lambda$,*
(f) *if $p + q < n$, $h^{pq}_\lambda = h^{n-p,n-q}_\lambda$,*
(g) *if $p + q < n$, $h^{pq}_\lambda \le h^{p+1,q+1}_\lambda$.*

Proof. The first four statements are immediate. For (e) we use the fact that $\omega\wedge\colon \mathcal{E}^k_\lambda \to \mathcal{E}^{k+2}_\lambda$ is an injection by (3). For (f) and (g), we use (3), and observe that $\omega^i \wedge -$ shifts the bigrading by (i, i). □

The above results can be visualized in terms of the geometry of the "Hodge diamond". When $\lambda > 0$, there are some new patterns as well. We start with a warm up.

Lemma 1.3. *If λ is a positive eigenvalue of Δ_0, then $h^{0,1}_\lambda \ge h^{0,0}_\lambda = b^0_\lambda$ and $b^1_\lambda \ge 2b^0_\lambda$.*

Proof. The map $\bar\partial : \mathcal{E}^0_\lambda \to \mathcal{E}^{0,1}_\lambda$ is injective because the kernel consists of global holomorphic eigenfunctions which are necessarily constant and therefore 0. This implies the first inequality, which in turn implies the second. □

We will give an extension to higher degrees, but first we start with a lemma.

Lemma 1.4. *If* $\lambda > 0$,

$$\mathcal{E}^k_\lambda = d\mathcal{E}^{k-1}_\lambda \oplus d^*\mathcal{E}^{k+1}_\lambda$$

and

$$\mathcal{E}^{(p,q)}_\lambda = \bar\partial\mathcal{E}^{(p,q-1)}_\lambda \oplus \bar\partial^*\mathcal{E}^{(p,q+1)}_\lambda.$$

Proof. By standard Hodge theory [GH, W], we have the decompositions

$$\mathcal{E}^k = \mathcal{E}_0 \oplus d\mathcal{E}^{k-1} \oplus d^*\mathcal{E}^{k+1},$$

$$\mathcal{E}^{k\pm1} = \bigoplus_{i=1}^{\infty} \mathcal{E}^{k\pm1}_{\lambda_i^{(k\pm1)}}.$$

These can be combined to yield the decomposition

$$\mathcal{E}^k = \mathcal{E}^k_0 \oplus \bigoplus_{i=1}^{\infty} d\mathcal{E}^{k-1}_{\lambda_i^{(k-1)}} \oplus \bigoplus_{i=1}^{\infty} d^*\mathcal{E}^{k+1}_{\lambda_i^{(k+1)}}.$$

Since $d\mathcal{E}^{k-1}_\lambda, d^*\mathcal{E}^{k+1}_\lambda \subset \mathcal{E}^k_\lambda$, the first part of the lemma

$$\mathcal{E}^k_\lambda = d\mathcal{E}^{k-1}_\lambda \oplus d^*\mathcal{E}^{k+1}_\lambda$$

follows immediately. The proof of the second part is identical. □

Theorem 1.5. *Suppose that* $\lambda > 0$.

(a) *For all* k, $b^k_\lambda \le b^{k-1}_\lambda + b^{k+1}_\lambda$.
(b) *If* $p + q < n$, *then* $h^{pq}_\lambda \le h^{p+1,q}_\lambda + h^{p,q+1}_\lambda$.
(c) *If* $k < n$, *then* $b^k_\lambda \le b^{k+1}_\lambda$.

Proof. The first statement is an immediate consequence of Lemma 1.4.

By Lemma 1.4, we have a direct sum $\mathcal{E}^{(p,q)}_\lambda = \mathcal{E}^{(p,q)}_{\mathrm{im}\,\bar\partial,\lambda} \oplus \mathcal{E}^{(p,q)}_{\mathrm{im}\,\bar\partial^*,\lambda}$ of the $\bar\partial$-exact $\mathcal{E}^{(p,q)}_{\mathrm{im}\,\bar\partial,\lambda} := \bar\partial\mathcal{E}^{(p,q-1)}_\lambda$ and $\bar\partial$-coexact $\mathcal{E}^{(p,q)}_{\mathrm{im}\,\bar\partial^*,\lambda} := \bar\partial^*\mathcal{E}^{(p,q+1)}_\lambda$ parts. We denote the dimensions of these spaces by $h^{pq}_{\mathrm{im}\,\bar\partial,\lambda}$ and $h^{pq}_{\mathrm{im}\,\bar\partial^*,\lambda}$ respectively.

Suppose that $\alpha \in \mathcal{E}^{(p,q)}_{\mathrm{im}\,\bar\partial^*,\lambda}$; then we can write $\alpha = \bar\partial^*\beta$. We have $\bar\partial\alpha \in \mathcal{E}^{(p,q+1)}_\lambda$ because $\bar\partial$ and Δ commute. Suppose that $\bar\partial\alpha = 0$. Then

$$\alpha = \frac{1}{\lambda}\Delta\alpha = \frac{2}{\lambda}(\bar\partial^*\bar\partial + \bar\partial\bar\partial^*)\alpha = \frac{2}{\lambda}\bar\partial\bar\partial^*\alpha = \frac{2}{\lambda}\bar\partial(\bar\partial^*)^2\beta.$$

This is zero, because $\langle(\bar\partial^*)^2\beta, \xi\rangle = \langle\beta, \bar\partial^2\xi\rangle = 0$ for any ξ. Thus the map

$$\bar\partial : \mathcal{E}^{(p,q)}_{\mathrm{im}\,\bar\partial^*,\lambda} \hookrightarrow \mathcal{E}^{(p,q+1)}_{\mathrm{im}\,\bar\partial,\lambda}$$

is injective. Although, we will not need it, it is worth noting that it is also surjective because

$$\mathcal{E}^{(p,q+1)}_{\mathrm{im}\,\bar\partial,\lambda} = \bar\partial(\mathcal{E}^{(p,q)}_{\mathrm{im}\,\bar\partial,\lambda} \oplus \mathcal{E}^{(p,q)}_{\mathrm{im}\,\bar\partial^*,\lambda}) = \bar\partial\mathcal{E}^{(p,q)}_{\mathrm{im}\,\bar\partial^*,\lambda}.$$

Therefore

$$h^{p,q}_{\mathrm{im}\,\bar\partial^*,\lambda} = h^{p,q+1}_{\mathrm{im}\,\bar\partial,\lambda} \tag{4}$$

for all p, q. We now assume that $p+q<n$. We will also establish an inequality

$$h^{p,q}_{\mathrm{im}\,\bar\partial,\lambda} \le h^{p+1,q}_{\mathrm{im}\,\bar\partial^*\lambda}. \tag{5}$$

Let $\alpha = \bar\partial\beta \in \mathcal{E}^{(p,q)}_{\mathrm{im}\,\bar\partial,\lambda}$ be a nonzero element. The previous theorem shows that $\gamma = \omega\wedge\alpha$ is a nonzero element of $\mathcal{E}_\lambda$. The form $\bar\partial^*\gamma \neq 0$, since otherwise

$$\gamma = \frac{1}{\lambda}\Delta\gamma = \frac{2}{\lambda}\bar\partial^*\bar\partial^2(\omega\wedge\beta) = 0.$$

Thus we have proved that the map

$$\mathcal{E}^{(p,q)}_{\mathrm{im}\,\bar\partial,\lambda} \to \mathcal{E}^{(p+1,q)}_{\mathrm{im}\,\bar\partial^*,\lambda}$$

given by $\alpha \mapsto \bar\partial^*(\omega\wedge\alpha)$ is injective. Equation (5) is an immediate consequence. Adding (4) and (5) yields

$$h^{pq}_\lambda \le h^{p,q+1}_{\mathrm{im}\,\bar\partial,\lambda} + h^{p+1,q}_{\mathrm{im}\,\bar\partial^*\lambda} \tag{6}$$

which implies (b). Equation (6) also implies

$$\begin{aligned} b^k_\lambda &= h^{0,k}_\lambda + h^{1,k-1}_\lambda + \dots \\ &\le (h^{0,k+1}_{\mathrm{im}\,\bar\partial,\lambda} + h^{1,k}_{\mathrm{im}\,\bar\partial^*,\lambda}) + (h^{1,k}_{\mathrm{im}\,\bar\partial,\lambda} + h^{2,k-1}_{\mathrm{im}\,\bar\partial^*,\lambda}) + \dots \\ &= h^{0,k+1}_{\mathrm{im}\,\bar\partial,\lambda} + (h^{1,k}_{\mathrm{im}\,\bar\partial^*,\lambda} + h^{1,k}_{\mathrm{im}\,\bar\partial,\lambda}) + \dots \\ &\le h^{0,k+1}_\lambda + h^{1,k}_\lambda + \dots \\ &= b^{k+1}_\lambda. \end{aligned}$$

□

By the positive spectrum of an operator, we mean the set of its positive eigenvalues (considered without multiplicity).

Corollary 1.6. *The positive spectrum of Δ_k is contained in the union of the spectra of Δ_{k-1} and Δ_{k+1}.*

Let $\lambda_1^{(k)}$ denote the first strictly positive eigenvalue of $\Delta_k = \Delta|_{\mathcal{E}^k}$.

Corollary 1.7. *If $k<n$, the positive spectrum of Δ_k is contained in the positive spectrum of Δ_{k+1}. Consequently, $\lambda_1^{(0)} \ge \lambda_1^{(1)} \ge \ldots \ge \lambda_1^{(n)}$.*

We can show that the positive spectra of Δ_k coincide for certain values of k.

Corollary 1.8. *The positive spectra of Δ_{n-1}, Δ_n, and Δ_{n+1} coincide. In particular, when $n=1$, the positive spectra of all the Laplacians coincide.*

Proof. If $\lambda>0$, then the inequalities

$$b^{n-1}_\lambda = b^{n+1}_\lambda,$$
$$b^{n-1}_\lambda \le b^n_\lambda \le b^{n-1}_\lambda + b^{n+1}_\lambda = 2b^{n-1}_\lambda,$$

follow from Theorems 1.1 and 1.5. These imply the corollary. □

The spectra are difficult to calculate in general, although there is at least one case where it is straightforward.

Example 1.9. Let $L \subset \mathbb{C}^n$ be a lattice with dual lattice L^* with respect to the Euclidean inner product. The spectrum of each Δ_k on the flat torus $\mathbb{C}^n/L$ is easily calculated to be the same set $\{4\pi^2||v||^2 \mid v\in L^*\}$; cf. [BGM, pp. 146-148].

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