ON PROJECTIVIZED VECTOR BUNDLES AND POSITIVE HOLOMORPHIC SECTIONAL CURVATURE

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ABSTRACT. We generalize a construction of Hitchin to prove that, given any compact Kähler manifold M with positive holomorphic sectional curvature and any holomorphic vector bundle E over M, the projectivized vector bundle $\mathbb{P}(E)$ admits a Kähler metric with positive holomorphic sectional curvature.

1. INTRODUCTION

Let us denote by $\mathcal{H}_n$ (resp. $\mathcal{H}'_n$) the set of all n-dimensional projective manifolds (resp. compact complex manifolds) which admit Kähler metrics with positive holomorphic sectional curvature H. It is a longstanding goal in complex geometry to understand these sets, which are conjectured to agree. As far as we know, this agreement is still open for $n \geq 3$.

First, let us recall some known facts. For any M in $\mathcal{H}'_n$, a theorem of Tsukamoto [Tsu57] states that M is simply-connected. Furthermore, by a theorem of Berger [Ber66], the scalar curvature of any Kähler metric on M is the average of the holomorphic sectional curvature. Therefore, $H > 0$ implies that the scalar curvature is positive. It was proven by Kobayashi and Wu [KW70, Corollary 2] that consequently the Kodaira dimension $\mathrm{kod}(M) = -\infty$. In recent work by the second-named author and Wong [HW15] it has been proven that any M in $\mathcal{H}_n$ is rationally connected, i.e., given any two points in M, there exist a connected chain of rational curves containing both points.

On the existence side, it is well known that the Fubini-Study metric on $\mathbb{P}^2$ has constant positive H, so it is in $\mathcal{H}_2$. In [Hit75], Hitchin showed that any Hirzebruch surface $\mathbb{F}_a = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1})$, $a \in \{0, 1, 2, \ldots\}$, is also in $\mathcal{H}_2$. For all other rational surfaces, including the del Pezzo surfaces, it is not known to the best of our knowledge whether or not they admit any Kähler metric with positive H. In general, there are only few known elements of $\mathcal{H}'_n$. Products of manifolds with positive H constitute a further obvious class of elements. Also, on Kähler C-spaces (i.e., simply-connected, compact, homogeneous Kähler manifolds), the standard Kähler-Einstein metric has positive holomorphic sectional curvature, so they all belong to $\mathcal{H}_n$.

The main purpose of this note is to generalize Hitchin's construction on Hirzebruch surfaces to arbitrary projectivized vector bundles over a compact Kähler

manifold with positive holomorphic sectional curvature and thus provide a new class of examples of manifolds in $\mathcal{H}'_n$ as stated in the following theorem.

Theorem 1.1. *Let M be a compact Kähler manifold with positive holomorphic sectional curvature. Let E be a holomorphic vector bundle over M and $P = \mathbb{P}(E)$ the projectivization of E. Then P admits a Kähler metric with positive holomorphic sectional curvature.*

If we replace $\mathbb{P}(E)$ by the Grassmannian bundle $G_k(E)$ of all k-dimensional subspaces of the fibers of E (here k is any positive integer less than the rank of E), then it is clear from the proof of the theorem that $G_k(E)$ also has positive holomorphic sectional curvature, and it can be holomorphically and isometrically embedded into $\mathbb{P}(\Lambda^k E)$.

Recall that in [Hit75], the proof of the existence of a metric of positive holomorphic sectional curvature on Hirzebruch surfaces is a stepping stone in the proof of the existence of metrics of positive scalar curvature on generic rational surfaces. As part of the argument, the following fundamental fact is established there as [Hit75, Corollary (5.18)].

Fact 1.2. Let X be a compact Kähler manifold of dimension > 2 with everywhere positive scalar curvature. Suppose we blow up any point $p \in X$ to obtain $\hat{X}$; then $\hat{X}$ admits a Kähler metric of positive scalar curvature.

Based on this fact and Berger's theorem, we observe the following immediate corollary to Theorem 1.1.

Corollary 1.3. *Let P be as in Theorem* 1.1. *Assume that the dimension of P is greater than* 2. *Let Y be obtained from P by a finite sequence of blow-ups of points. Then Y carries a Kähler metric of positive scalar curvature.*

In general, it is reasonable to conjecture that for any holomorphic fiber bundle with total space X, if both the fiber and the base admit Kähler metrics with positive holomorphic sectional curvature, then so does X. However, at this point we do not know how to construct a natural metric on X from the given metrics on the fiber and the base in a way that enables us to compute the holomorphic sectional curvature. In the case of negative holomorphic sectional curvature, an even more general result of this nature was obtained by Cheung [Che89, Theorem 1].

We conclude this introduction by remarking that in the paper [ACH15] an explicit analysis of the pinching constants for Hitchin's metrics on Hirzebruch surfaces was conducted. We leave such an explicit analysis for the present higher-dimensional case to a future occasion.

2. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. Let (M, g) be a compact n-dimensional Kähler manifold with positive holomorphic sectional curvature. Let (E, h) be a holomorphic vector of rank $r + 1$ equipped with a Hermitian metric. Denote by $P = \mathbb{P}(E)$ the projectivization of E, namely, for any $x \in M$, the fiber P_x is simply the projective space $\mathbb{P}(E_x)$ of the fiber E_x. In other words, P_x consists of the equivalence classes $[w]$ where w is any non-zero vector in E_x. Note that this is the differential-geometric notation – the algebro-geometric notation of $\mathbb{P}(E)$ would be the $\mathbb{P}(E^*)$ here.

As essentially observed in [Hit75, (4.1)], the metrics g and h naturally induce a closed $(1,1)$-form on P:

$$\omega_G = \lambda\, \pi^*(\omega_g) + \sqrt{-1}\, \partial\overline{\partial} \log h(v, \overline{v}),$$

where ω_g is the Kähler form of g, $\pi : P \to M$ the projection map, and $(x, [v])$ is a moving point in P. Since the restriction of the second term of the right hand side on a fiber of π is just the Fubini-Study metric, we know that for λ sufficiently large, ω_G is positive definite everywhere so $G = G_\lambda$ becomes a Kähler metric on P.

We claim that there is some constant $\lambda_0 > 0$ which depends on g and h, such that for any $\lambda \geq \lambda_0$, the metric $G = G_\lambda$ has positive holomorphic sectional curvature. This will complete the proof of our theorem.

To prove this, we need to compute the curvature of the metric G. Fix any $p = (x_0, [w])$ in P. Without loss of generality, we may assume that $|w| = 1$. Let $(z_1, \ldots, z_n)$ be a local holomorphic coordinate centered at x_0 which is normal with respect to g. That is, $x_0 = (0, \ldots, 0)$, and under the coordinates,

$$g_{i\overline{j}}(0) = \delta_{i\overline{j}}, \quad dg_{i\overline{j}}(0) = 0,$$

for any $1 \leq i, j \leq n$. Write Θ^h for the curvature of (E, h). Then by a constant unitary change of z if necessary, we may assume that the $(1,1)$-form $\Theta^h_{w\overline{w}}$ at x_0 is diagonal, namely,

$$\Theta^h_{w\overline{w}} = \sum_{i=1}^{n} \xi_i\, dz_i \wedge d\overline{z}_i.$$

Next let us choose a local holomorphic frame $\{e_0, e_1, \ldots, e_r\}$ of E near x_0. Write $h_{\alpha\overline{\beta}}$ for $h(e_\alpha, \overline{e_\beta})$. We may assume that at the origin, $e_0(0) = w$, and

$$h_{\alpha\overline{\beta}}(0) = \delta_{\alpha\beta}, \quad dh_{\alpha\overline{\beta}}(0) = 0.$$

It is also easy to see that we may further assume that $\partial_i\partial_k h_{\alpha\overline{\beta}}(0) = 0$. Let us write

$$v = e_0(z) + \sum_{\alpha=1}^{r} t_\alpha e_\alpha(z).$$

Then (z, t) becomes a local holomorphic coordinate in P centered at $p = (0, 0)$. We will use the index convention of z_i, t_α, and with index after comma denoting the partial derivatives. Because of the sesquilinearity of h, we may naturally write $h_{\alpha\overline{v}}$ instead of $h_{v\overline{v},\alpha}$ and $h_{v\overline{\beta}}$ instead of $h_{v\overline{v},\overline{\beta}}$. We have

$$
\begin{aligned}
G_{i\overline{j}} &= \lambda\, g_{i\overline{j}} + \frac{1}{h_{v\overline{v}}} h_{v\overline{v},i\overline{j}} - \frac{1}{(h_{v\overline{v}})^2} h_{v\overline{v},i}\, h_{v\overline{v},\overline{j}}, \\
G_{i\overline{\beta}} &= \frac{1}{h_{v\overline{v}}} h_{v\overline{\beta},i} - \frac{1}{(h_{v\overline{v}})^2} h_{v\overline{v},i}\, h_{v\overline{\beta}}, \\
G_{\alpha\overline{j}} &= \frac{1}{h_{v\overline{v}}} h_{\alpha\overline{v},\overline{j}} - \frac{1}{(h_{v\overline{v}})^2} h_{v\overline{v},\overline{j}}\, h_{\alpha\overline{v}}, \\
G_{\alpha\overline{\beta}} &= \frac{1}{h_{v\overline{v}}} h_{\alpha\overline{\beta},i} - \frac{1}{(h_{v\overline{v}})^2} h_{\alpha\overline{v}}\, h_{v\overline{\beta}}.
\end{aligned}
$$

So at the origin p, we have $G_{i\overline{j}}(0) = (\lambda - \xi_i)\, \delta_{ij}$, $G_{i\overline{\beta}}(0) = G_{\alpha\overline{j}}(0) = 0$, and $G_{\alpha\overline{\beta}}(0) = \delta_{\alpha\beta}$, for any $1 \leq i, j \leq n$ and any $1 \leq \alpha, \beta \leq r$. Next, we compute the

first derivatives of G:

$$
\begin{aligned}
G_{i\bar{j},k} &= \lambda\, g_{i\bar{j},k} + \frac{1}{h_{v\bar{v}}}h_{v\bar{v},i\bar{j}k} - \frac{1}{(h_{v\bar{v}})^2}h_{v\bar{v},k}\,h_{v\bar{v},i\bar{j}} - \frac{1}{(h_{v\bar{v}})^2}h_{v\bar{v},ik}\,h_{v\bar{v},\bar{j}} \\
&\quad - \frac{1}{(h_{v\bar{v}})^2}h_{v\bar{v},i}\,h_{v\bar{v},\bar{j}k} + \frac{2}{(h_{v\bar{v}})^3}h_{v\bar{v},k}\,h_{v\bar{v},i}\,h_{v\bar{v},\bar{j}}, \\
G_{i\bar{\beta},k} &= \frac{1}{h_{v\bar{v}}}h_{v\bar{\beta},ik} - \frac{1}{(h_{v\bar{v}})^2}h_{v\bar{v},k}\,h_{v\bar{\beta},i} - \frac{1}{(h_{v\bar{v}})^2}h_{v\bar{v},i}\,h_{v\bar{\beta},k} \\
&\quad - \frac{1}{(h_{v\bar{v}})^2}h_{v\bar{v},ik}\,h_{v\bar{\beta}} + \frac{2}{(h_{v\bar{v}})^3}h_{v\bar{v},i}\,h_{v\bar{v},k}\,h_{v\bar{\beta}}, \\
G_{\alpha\bar{j},k} &= \frac{1}{h_{v\bar{v}}}h_{\alpha\bar{v},\bar{j}k} - \frac{1}{(h_{v\bar{v}})^2}h_{v\bar{v},k}\,h_{\alpha\bar{v},\bar{j}} - \frac{1}{(h_{v\bar{v}})^2}h_{v\bar{v},\bar{j}}\,h_{\alpha\bar{v},k} \\
&\quad - \frac{1}{(h_{v\bar{v}})^2}h_{v\bar{v},\bar{j}k}\,h_{\alpha\bar{v}} + \frac{2}{(h_{v\bar{v}})^3}h_{\alpha\bar{v}}h_{v\bar{v},\bar{j}}\,h_{v\bar{v},k}, \\
G_{\alpha\bar{\beta},k} &= \frac{1}{h_{v\bar{v}}}h_{\alpha\bar{\beta},k} - \frac{1}{(h_{v\bar{v}})^2}h_{\alpha\bar{\beta}}\,h_{v\bar{v},k} - \frac{1}{(h_{v\bar{v}})^2}h_{\alpha\bar{v},k}\,h_{v\bar{\beta}} \\
&\quad - \frac{1}{(h_{v\bar{v}})^2}h_{\alpha\bar{v}}\,h_{v\bar{\beta},k} + \frac{2}{(h_{v\bar{v}})^3}h_{\alpha\bar{v}}\,h_{v\bar{\beta}}\,h_{v\bar{v},k}, \\
G_{\alpha\bar{j},\gamma} &= -\frac{1}{(h_{v\bar{v}})^2}h_{\gamma\bar{v}}\,h_{\alpha\bar{v},\bar{j}} - \frac{1}{(h_{v\bar{v}})^2}h_{\gamma\bar{v},\bar{j}}\,h_{\alpha\bar{v}} + \frac{2}{(h_{v\bar{v}})^3}h_{\alpha\bar{v}}\,h_{\gamma\bar{v}}\,h_{v\bar{v},\bar{j}}, \\
G_{\alpha\bar{\beta},\gamma} &= -\frac{1}{(h_{v\bar{v}})^2}h_{\alpha\bar{\beta}}\,h_{\gamma\bar{v}} - \frac{1}{(h_{v\bar{v}})^2}h_{\alpha\bar{v}}\,h_{\gamma\bar{\beta}} + \frac{2}{(h_{v\bar{v}})^3}h_{\alpha\bar{v}}\,h_{\gamma\bar{v}}\,h_{v\bar{\beta}}.
\end{aligned}
$$

We also have $G_{i\bar{j},\alpha} = G_{\alpha\bar{j},i}$ and $G_{i\bar{\beta},\alpha} = G_{\alpha\bar{\beta},i}$ by the Kählerness of G. At the origin p, we have $h_{v\bar{v}}(0) = |w|^2 = 1$, $h_{v\bar{\beta}}(0) = h_{\alpha\bar{v}}(0) = 0$, and all first order derivatives of h are zero, so by taking another derivative and evaluate at 0, we get the following at the point p:

$$
\begin{aligned}
G_{i\bar{j},k\bar{l}} &= \lambda g_{i\bar{j},k\bar{l}} + h_{v\bar{v},i\bar{j}k\bar{l}} - h_{v\bar{v},i\bar{j}}h_{v\bar{v},k\bar{l}} - h_{v\bar{v},i\bar{l}}h_{v\bar{v},k\bar{j}}, \\
G_{i\bar{j},k\bar{\beta}} &= h_{v\bar{\beta},i\bar{j}k}, \\
G_{i\bar{j},\alpha\bar{\beta}} &= h_{\alpha\bar{\beta},i\bar{j}} - h_{\alpha\bar{\beta}}h_{v\bar{v},i\bar{j}}, \\
G_{\alpha\bar{j},\gamma\bar{l}} &= 0, \\
G_{\alpha\bar{\beta},\gamma\bar{j}} &= 0, \\
G_{\alpha\bar{\beta},\gamma\bar{\delta}} &= -h_{\alpha\bar{\beta}}h_{\gamma\bar{\delta}} - h_{\alpha\bar{\delta}}h_{\gamma\bar{\beta}}.
\end{aligned}
$$

Now we are ready to compute the holomorphic sectional curvature of G at p. Let $0 \neq V = X + U$ be a type $(1,0)$ tangent vector at $p \in P$, where $X = \sum_{i=1}^{n} x_i \frac{\partial}{\partial z_i}$ and $U = \sum_{\alpha=1}^{r} u_\alpha \frac{\partial}{\partial t_\alpha}$. Denote by R, R^g, and R^h the curvature tensors of G, g, and h, respectively. We have

$$
\begin{aligned}
R_{V\bar{V}V\bar{V}} &= R_{X\bar{X}X\bar{X}} + 4R_{X\bar{X}U\bar{U}} + R_{U\bar{U}U\bar{U}} \\
&\quad + 2Re\{R_{X\bar{U}X\bar{U}} + 2R_{X\bar{X}X\bar{U}} + 2R_{U\bar{U}U\bar{X}}\}
\end{aligned}
$$

by the symmetry of the curvature tensor. At p the matrix of G is diagonal, so we have

$$
\begin{aligned}
R_{V\overline{V}V\overline{V}} &= -G_{V\overline{V},V\overline{V}} + \sum_{a=1}^{n+r} \frac{1}{G_{a\overline{a}}} |G_{V\overline{a},V}|^2 \\
&\geq -G_{V\overline{V},V\overline{V}} \\
&= -G_{X\overline{X},X\overline{X}} - 4G_{X\overline{X},U\overline{U}} - G_{U\overline{U},U\overline{U}} \\
&\quad - 2Re\{G_{X\overline{U},X\overline{U}} + 2G_{X\overline{X},X\overline{U}} + 2G_{U\overline{U},U\overline{X}}\} \\
&= -G_{X\overline{X},X\overline{X}} - 4G_{X\overline{X},U\overline{U}} - G_{U\overline{U},U\overline{U}} - 4Re\{G_{X\overline{X},X\overline{U}}\} \\
&= (\lambda R^g_{X\overline{X}X\overline{X}} - h_{v\overline{v},X\overline{X}X\overline{X}} + 2(R^h_{v\overline{v}X\overline{X}})^2) + 4(R^h_{U\overline{U}X\overline{X}} \\
&\quad - |U|^2 R^h_{v\overline{v}X\overline{X}}) + 2|U|^4 - 4Re\{h_{v\overline{U},X\overline{X}X}\}.
\end{aligned}
$$

Let $H_0 > 0$ be the minimum of holomorphic sectional curvature of g over M, and choose a constant $C > 0$ such that $|R^h|$ and $|\nabla R^h|$ are both bounded by C; then the above computation leads to

$$
R_{V\overline{V}V\overline{V}} \geq (\lambda H_0 - C)|X|^4 - 8C|X|^2|U|^2 + 2|U|^4 - 4C|X|^3|U|.
$$

Clearly, if λ is sufficiently large, this quantity will be positive when X and U are not both zero. This completes the proof of Theorem 1.1.

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