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ON THE $C^{1,\alpha}$ REGULARITY OF p-HARMONIC FUNCTIONS IN THE HEISENBERG GROUP

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ABSTRACT. We present a proof of the local Hölder regularity of the horizontal derivatives of weak solutions to the p-Laplace equation in the Heisenberg group $\mathbb{H}^1$ for $p > 4$.

1. INTRODUCTION

We present a proof of the local Hölder regularity of derivatives of weak solutions to the p-Laplace equation in the Heisenberg group $\mathbb{H}^1$ for the range $p > 4$. Our notation for the first Heisenberg group is $\mathbb{H} = \mathbb{H}^1 = (\mathbb{R}^3, *)$. Here, indicating points $x, y \in \mathbb{H}$ by $x = (x_1, x_2, z)$ and $y = (y_1, y_2, s)$, the group operation is

$$x * y = (x_1, x_2, z) * (y_1, y_2, s) = \left(x_1 + y_1, x_2 + y_2, z + s + \frac{1}{2}(x_1y_2 - x_2y_1)\right),$$

and a basis of left-invariant vector fields for the associated Lie algebra $\mathfrak{h}$ is given by

$$X_1 = \partial_{x_1} - \frac{x_2}{2}\partial_z, \quad X_2 = \partial_{x_2} + \frac{x_1}{2}\partial_z, \text{ and } T = \partial_z .$$

If $u : \Omega \longrightarrow \mathbb{R}$ is a function from an open subset of $\mathbb{H}$ we indicate by $\nabla_{\mathbb{H}} u = (X_1u, X_2u)$ the horizontal gradient of u. We denote by $HW^{1,p}(\Omega)$ the Sobolev space of functions u such that both u and $\nabla_{\mathbb{H}} u \in L^p(\Omega)$.

We study the regularity of solutions to the p-Laplace equation:

$$\sum_{i=1}^{2} X_i\Big(|\nabla_{\mathbb{H}} u|^{p-2} X_i u\Big) = 0 \quad \text{in } \Omega. \tag{1.1}$$

The main result is the following:

Theorem 1.1. *Let $u \in HW^{1,p}(\Omega)$ be a weak solution of the p-Laplace equation (1.1) for $p > 4$ and let $B_{R_0} \Subset \Omega$. Then there exists $\beta = \beta(p) \in (0,1)$ such that for every $l \in \{1,2\}$ we have*

$$\operatorname{osc}_{B_R}(X_l u) \le C_p \, \|\nabla_{\mathbb{H}} u\|_{L^\infty(B_{R_0})} \left(\frac{R}{R_0}\right)^{\beta} \quad \textit{for all } R \le \frac{R_0}{2},$$

where C_p is a constant depending only on p.

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In this work $B_r(x_0)$ denotes a Carnot-Carathéodory ball of radius r and center x_0 (we omit the center when it is not essential). We recall that the Carnot-Carathéodory distance between x and $y \in \mathbb{H}$ is defined as

$$d_{cc}(x,y) = \inf\{l(\Gamma) \mid \Gamma \in S(x,y)\} \, .$$

Here $S(x,y)$ denotes the set of all horizontal subunitary curves joining x and y, i.e., absolutely continuous curves $\Gamma : [0,T] \longrightarrow \Omega$ such that $\Gamma'(t) = \sum_{j=1}^{2} \alpha_j(t) X_j(\Gamma(t))$ for some real valued functions α_j with $\sum_{j=1}^{2} \alpha_j(t)^2 \le 1$. The length of such a curve is defined to be $l(\Gamma) = T$.

Moreover we recall that the Lebesgue measure is the Haar measure of the group and the homogeneous dimension is $Q = 4$.

To prove regularity results in general one considers a family of approximated nondegenerate problems and tries to produce estimates independent of the nondegeneracy parameter in such a way that they can be applied to the degenerate equation by passing to the limit. More precisely, here we consider the nondegenerate equations

$$\sum_{i=1}^{2} X_i\left(\left(\delta^2 + |\nabla_{\mathbb{H}} u|^2\right)^{\frac{p-2}{2}} X_i u \right) = 0 \quad \text{in } \Omega \tag{1.2}$$

for a parameter $\delta > 0$. Equation (1.1) corresponds to the degenerate case $\delta = 0$.

The Heisenberg group presents new challenges with respect to its Euclidean counterpart , since we only assume that u is in the horizontal Sobolev space $HW^{1,p}$, and differentiating the equation produces terms involving the vertical derivative Tu, due to the noncommutativity of the horizontal vector fields. This constitutes the main difficulty.

For $p = 2$ it is now classical that the solutions of equation (1.1) are C^∞ [14]. For $p \neq 2$ the Hölder regularity of solutions of equations modeled on (1.2) was established by Capogna and Garofalo [3] and Lu [18]. Later Manfredi and Mingione [20] were able to prove $C^{1,\alpha}$ regularity in the nondegenerate case for $2 \le p < c(n) < 4$, and by adapting an argument used by Capogna they achieve C^∞ regularity for this range of values of p. The starting point is the integrability result for the vertical derivative $Tu \in L^p$ established by Domokos for $1 < p < 4$ in [7], where he extends integrability results considered by Marchi for $1 + \frac{1}{\sqrt{5}} < p < 1 + \sqrt{5}$ in [21], [22].

Mingione, Zatorska-Goldstein, and Zhong proved in [23] that the Euclidean gradient of solutions to the nondegenerate equation are $C^{1,\alpha}$ for $2 \le p < 4$ and also that solutions to the degenerate equation are locally Lipschitz continuous for $2 \le p < 4$.

Zhong in [26] extended the Hilbert-Haar theory to the Heisenberg group setting and proved that solutions to the degenerate equation (1.1) are locally Lipschitz for the full range $1 < p < \infty$. For an account of this theory, further historical details and additional references see [24].

As for the Hölder continuity of the horizontal derivatives for the degenerate equation (1.1) the only published result for $p \neq 2$ has been obtained by Manfredi and Domokos in [9], [8] via the Cordes perturbation technique for p near 2.

The proof of the Hölder continuity of the horizontal derivatives contained in this work uses the particular form of the equation in $\mathbb{H}^1$ and new integration by parts for the second derivatives that produce weights of the form $(\delta^2 + |\nabla_{\mathbb{H}} u|)^{\frac{p-4}{2}}$. This is the reason why our proof is only valid in the first Heisenberg group $\mathbb{H}^1$ and for the range $p > 4$.

2. Preliminaries

2.1. **The p-Laplace equation.** We will consider the nondegenerate p-Laplace equation (1.2). Denoting $z = (z_1, z_2) \in \mathbb{R}^2$ and calling

$$a_i(z) = (\delta^2 + |z|^2)^{\frac{p-2}{2}} z_i \quad \text{and} \quad w = \delta^2 + |\nabla_{\mathbb{H}} u|^2 ,$$

equation (1.2) rewrites as

$$\sum_{i=1}^{2} X_i a_i(\nabla_{\mathbb{H}} u) = 0 \quad \text{in } \Omega \tag{2.1}$$

and satisfies the following ellipticity and growth conditions for all $p > 1$:

$$\begin{aligned} \sum_{i,j=1}^{2} \partial_{z_j} a_i(\nabla_{\mathbb{H}} u)\xi_i \xi_j &\geq c_p w^{\frac{p-2}{2}} |\xi|^2, \\ |a_i(\nabla_{\mathbb{H}} u)| &\leq w^{\frac{p-1}{2}}, \\ |\partial_{z_j} a_i(\nabla_{\mathbb{H}} u)| &\leq C_p w^{\frac{p-2}{2}}, \\ |\partial_{z_s} \partial_{z_j} a_i(\nabla_{\mathbb{H}} u)| &\leq C_p w^{\frac{p-3}{2}}, \end{aligned} \tag{2.2}$$

and

$$\left| \frac{\partial_{z_j} a_i(z)}{\partial_{z_l} a_l(z)} \right| \leq C_p \quad \text{for all } i, j, l \in \{1, 2\}. \tag{2.3}$$

We remark that the proofs presented in this work depend only on these properties; therefore they extend to more general equations of p-Laplacean type as in (2.1) for a_i of class C^2 satisfying (2.2) and (2.3).

We say that a function $u \in HW^{1,p}(\Omega)$ is a weak solution of (1.2) if

$$\int_{\Omega} w^{\frac{p-2}{2}} \langle \nabla_{\mathbb{H}} u, \nabla_{\mathbb{H}} \varphi \rangle \, \mathrm{d}x = 0 \quad \text{for all } \varphi \in HW_0^{1,p}(\Omega) , \tag{2.4}$$

where $HW_0^{1,p}(\Omega)$ is the closure of the space of C^∞ compactly supported functions with respect to the horizontal Sobolev norm.

2.2. **Previous results.** We now collect some known results about the nondegenerate equation (1.2) that will be used in the following sections. We refer to [24] for a detailed presentation and complete proofs.

First we have that solutions to the nondegenerate p-Laplace equation (1.2) are smooth. This was proved by Capogna in [2] for $p \geq 2$ and extended to the full range $1 < p < \infty$ in [24, Chapter 4] by adapting techniques of Domokos in [7].

As a consequence we have

$$\sum_{i,j=1}^{2} \partial_{z_j} a_i(\nabla_{\mathbb{H}} u) X_i X_j u = 0 \quad \text{a.e. in} \quad \Omega. \tag{2.5}$$

Hence we can express $X_1 X_1 u$ (respectively $X_2 X_2 u$) in terms of $X_i X_j u$ where at least one index is a 2 (respectively a 1). This will be a crucial point later.

We now collect the equations satisfied by the horizontal and vertical derivatives (see [24], Lemma 4.1):

Lemma 2.1. *The functions X_1u, X_2u, and Tu are weak solutions respectively of the following equations (in Ω):*

(2.6)
$$\sum_{i=1}^{2} X_i \left(\sum_{j=1}^{2} \partial_{z_j} a_i(\nabla_{\mathbb{H}} u) X_j X_1 u \right) + \sum_{i=1}^{2} X_i \left(\partial_{z_2} a_i(\nabla_{\mathbb{H}} u) Tu \right) + T\left(a_2(\nabla_{\mathbb{H}} u) \right) = 0,$$

(2.7)
$$\sum_{i=1}^{2} X_i \left(\sum_{j=1}^{2} \partial_{z_j} a_i(\nabla_{\mathbb{H}} u) X_j X_2 u \right) - \sum_{i=1}^{2} X_i \left(\partial_{z_1} a_i(\nabla_{\mathbb{H}} u) Tu \right) - T\left(a_1(\nabla_{\mathbb{H}} u) \right) = 0,$$

(2.8)
$$\sum_{i=1}^{2} X_i \left(\sum_{j=1}^{2} \partial_{z_j} a_i(\nabla_{\mathbb{H}} u) X_j Tu \right) = 0.$$

In [26] Zhong established the weighted higher order integrability of Tu as follows:

Lemma 2.2. *For all $q > 4$ and $\xi \in C_0^\infty(\Omega)$ we have*

$$\int_\Omega \xi^q \, w^{\frac{p-2}{2}} \, |Tu|^q \, dx \le C(q) \left(\|\nabla_{\mathbb{H}} \xi\|_{L^\infty}^2 + \|\xi T\xi\|_{L^\infty} \right)^{\frac{q}{2}} \int_{\mathrm{supp}(\xi)} w^{\frac{p-2+q}{2}} \, dx, \tag{2.9}$$

where $C(q) = C_p^{\frac{q-2}{2}} q^{q+8}$ and C_p depends only on p.

For the sake of completeness we give a proof in the Appendix.

3. De Giorgi classes in the Heisenberg group

We now describe a type of De Giorgi class in the Heisenberg group. These kinds of spaces were introduced and studied by De Giorgi in the Euclidean case (see [5]). We will use the standard notation for super- (sub)level sets of a measurable function

$$A_{k,r}^+ = A_{k,r}^+(f) = B_r \cap \{f > k\},$$
$$A_{k,r}^- = A_{k,r}^-(f) = B_r \cap \{f < k\}.$$

Definition 3.1 (De Giorgi class in the Heisenberg group)**.** Let $\Omega \subset \mathbb{H}$ be open, let γ, χ be positive real constants, and let $q > 4$. A function $f \in HW^{1,2}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ belongs to the De Giorgi class $DG^+(\Omega, \gamma, \chi, q)$ if

$$\int_{A_{h,r'}^+} |\nabla_{\mathbb{H}} f|^2 \, \mathrm{d}x \le \frac{\gamma}{(r - r')^2} \sup_{B_r} |(f-h)^+|^2 |A_{k,r}^+| + \chi |A_{k,r}^+|^{1-\frac{2}{q}} \tag{3.1}$$

for some concentric balls $B_{r'} \subset B_r \Subset \Omega$ and levels $h \in \mathbb{R}$.

In this section we consider an arbitrary ball $B_R \Subset \Omega$ and denote $M = M(R) = \sup_{B_R} f$ and $m = m(R) = \inf_{B_R} f$.

We take the following lemma from [15], Lemma 2.3, where it is proved in a more general setting.

Lemma 3.2. *Let $l > k$, $f \in HW^{1,1}_{\mathrm{loc}}(\Omega)$, $B_r \Subset \Omega$. Then if $|B_r \setminus A_{k,r}^+| > 0$ we have*

$$(l-k)|A_{l,r}^+|^{1-\frac{1}{4}} \le \frac{Cr^4}{|B_r \setminus A_{k,r}^+|} \int_{A_{k,r}^+ \setminus A_{l,r}^+} |\nabla_{\mathbb{H}} f| \, dx, \tag{3.2}$$

where C is a purely numeric constant.

The next lemma is adapted from Lemma 2.3 in [19] and Lemma 6.1 in [16].

Lemma 3.3. *Let $0 < \lambda_0, \lambda_1 < 1$, and let $k < M$. Suppose that $f \in DG^+(\Omega, \gamma, \chi, q)$ for $h \in [k, \lambda_0 k + (1-\lambda_0)M]$ and for $r' < r \in [\lambda_1 R, R]$. Then there exists $\theta = \theta(\gamma, \lambda_0, \lambda_1) \in (0,1)$ such that if*

$$M - k \geq \chi^{\frac{1}{2}} R^{1-\frac{4}{q}},$$

then

$$|A^+_{k,R}| \leq \theta |B_R| \quad \textit{implies} \quad f \leq \lambda_0 k + (1-\lambda_0)M \ \textit{ a.e. in } B_{\lambda_1 R}.$$

The following lemma is adapted from Lemma 2.4 in [19] and Lemma 6.2 in [16].

Lemma 3.4. *Let $0 < \lambda_1 < 1$ and $k < M$. Suppose $f \in DG^+(\Omega, \gamma, \chi, q)$ for $h \in [k, M]$ and for $r' = \lambda_1 R$, $r = R$. If there exists a constant $0 < C_0 < 1$ such that $|A^+_{k,\lambda_1 R}| \leq C_0 |B_{\lambda_1 R}|$, then given $0 < \theta < 1$ there exists $s = s(\gamma, \lambda_1, C_0, \theta) \in \mathbb{N}$ such that*

$$\textit{if} \quad M - k \geq 2^s \chi^{\frac{1}{2}} R^{1-\frac{4}{q}}, \quad \textit{then} \quad |A^+_{k_s,\lambda_1 R}| \leq \theta |B_{\lambda_1 R}|,$$

where $k_s = k + (1 - 2^{-s})(M - k)$ is a level set between k and M.

Combining the previous lemmas we get an estimate for the decay of the oscillation of functions in the De Giorgi class. We are adapting it from Lemma 2.5 in [19] and from [16].

Lemma 3.5 (Oscillation estimate)**.** *Let $0 < \lambda_1 < 1$, and suppose that for radii $r' < r \in [\lambda_1 R, R]$ we have $f \in DG^+(\Omega, \gamma, \chi, q)$ for $h \in [\frac{m+M}{2}, M]$ and $-f \in DG^+(\Omega, \gamma, \chi, q)$ for $h \in [-M, -\frac{m+M}{2}]$. Then there exists $A = A(\gamma, \lambda_1) \in (0,1)$ such that*

$$\operatorname{osc}_{B_{\lambda_1 R}} f \leq A \operatorname{osc}_{B_R} f + B R^{1-\frac{4}{q}}, \tag{3.3}$$

where

$$B = \frac{\chi^{\frac{1}{2}}}{4(1-A)}.$$

4. Main estimate

From now on we will fix a ball $B_{R_0} \Subset \Omega$, and for a concentric ball $B_R \subset B_{R_0}$ we introduce the notation

$$\mu(R) = \max_{1 \leq l \leq 2} \|X_l u\|_{L^\infty(B_R)} \quad \text{and} \quad \lambda(R) = \frac{1}{2}\mu(R).$$

In this section we prove the following proposition, which contains the main estimates and constitutes the novel contribution of this work.

Proposition 4.1. *Let $B_{R_0} \Subset \Omega$ and let $u \in HW^{1,p}(\Omega)$ be a weak solution of the nondegenerate equation (1.2) for $p > 4$. For every $0 < r' < r < \frac{R_0}{2}$, $l = 1, 2$, and* *for every* $q > \max\{4, 2 + \frac{4}{p-4}\}$ *we have*

$$\int_{B_{r'}} w^{\frac{p-2}{2}} \left|\nabla_{\mathbb{H}}(X_l u - k)^+\right|^2 \, dx \le \frac{C_p}{(r-r')^2} \int_{B_r} w^{\frac{p-2}{2}} |(X_l u - k)^+|^2 \, dx + \chi \left|A^+_{k,r}(X_l u)\right|^{1-\frac{2}{q}}, \tag{4.1}$$

$$\int_{B_{r'}} w^{\frac{p-2}{2}} \left|\nabla_{\mathbb{H}}(X_l u - k)^-\right|^2 \, dx \le \frac{C_p}{(r-r')^2} \int_{B_r} w^{\frac{p-2}{2}} |(X_l u - k)^-|^2 \, dx + \chi \left|A^-_{k,r}(X_l u)\right|^{1-\frac{2}{q}}. \tag{4.2}$$

The inequalities (4.1) *hold for levels* $k \ge -\mu(R_0)$*, while* (4.2) *hold for levels* $k \le \mu(R_0)$*. The constant* C_p *depends only on* p*, and the parameter* χ *is given by*

$$\chi = \frac{C_p q^6}{R_0^2} \left(\delta^2 + \mu(R_0)^2\right)^{\frac{p}{2}} |B_{R_0}|^{\frac{2}{q}}. \tag{4.3}$$

Proof. We will prove (4.1) for $l = 1$; the other estimates follow in a similar fashion. We use the notation $v_l = (X_l u - k)^+ = \max\{X_l u - k, 0\}$. Fix $0 < r' < r < \frac{R_0}{2}$ and let $\phi = \xi^2 v_1$, where ξ is a cut-off function between $B_{r'}$ and B_r with $|\nabla_{\mathbb{H}}\xi| \le \frac{C}{(r-r')}$. We denote $A^+_{k,r}(X_1 u)$ for simplicity by $A^+_{k,r}$ and we use the usual convention of sum on repeated indices. Test equation (2.6) with ϕ to get

$$\begin{aligned} J_1 := \int_{B_r} \xi^2 \, \partial_{z_j} a_i(\nabla_{\mathbb{H}} u) \, X_j X_1 u \, X_i v_1 \, \mathrm{d}x = & -2 \int_{B_r} \xi \, \partial_{z_j} a_i(\nabla_{\mathbb{H}} u) \, X_j X_1 u \, X_i \xi \, v_1 \, \mathrm{d}x \\ & - \int_{B_r} \xi^2 \, \partial_{z_2} a_i(\nabla_{\mathbb{H}} u) \, X_i v_1 \, Tu \, \mathrm{d}x \\ & - 2 \int_{B_r} \xi \, \partial_{z_2} a_i(\nabla_{\mathbb{H}} u) \, X_i \xi \, Tu \, v_1 \, \mathrm{d}x \\ & - \int_{B_r} a_2(\nabla_{\mathbb{H}} u) \, T(\xi^2 v_1) \, \mathrm{d}x \\ & =: J_2 + J_3 + J_4 + J_5. \end{aligned}$$

Routine calculations using Young's inequality and (2.2) allow us to estimate J_i for $1 \le i \le 4$ as follows:

$$\begin{aligned} \int_{B_r} \xi^2 \, w^{\frac{p-2}{2}} \, |\nabla_{\mathbb{H}} v_1|^2 \, \mathrm{d}x \le C \int_{B_r} |\nabla_{\mathbb{H}}\xi|^2 \, w^{\frac{p-2}{2}} \, v_1^2 \, \mathrm{d}x + C \int_{A^+_{k,r}} \xi^2 \, w^{\frac{p-2}{2}} \, |Tu|^2 \, \mathrm{d}x \\ + \left| \int_{B_r} a_2(\nabla_{\mathbb{H}} u) \, T(\xi^2 v_1) \, \mathrm{d}x \right|. \end{aligned} \tag{4.4}$$

The new idea is to estimate the last integral in the previous inequality by integrating by parts twice. First, integrating by parts with respect to the field T we get

$$\begin{aligned} \int_{B_r} a_2(\nabla_{\mathbb{H}} u) \, T(\xi^2 v_1) \, \mathrm{d}x &= - \int_{B_r} T(a_2(\nabla_{\mathbb{H}} u)) \, \xi^2 v_1 \, \mathrm{d}x \\ &= - \int_{B_r} \partial_{z_j} a_2(\nabla_{\mathbb{H}} u) \, X_j T u \, \xi^2 \, v_1 \, \mathrm{d}x, \end{aligned}$$

and then integrating with respect to the fields X_j for $j = 1, 2$ we obtain

$$\begin{aligned}
-\int_{B_r} \partial_{z_j} a_2(\nabla_{\mathbb{H}} u)\, X_j T u\, \xi^2\, v_1\, \mathrm{d}x &= \int_{B_r} Tu\, X_j \left(\partial_{z_j} a_2(\nabla_{\mathbb{H}} u) \xi^2 v_1\right)\, \mathrm{d}x \\
&= \int_{B_r} Tu\, \partial_{z_s} \partial_{z_j} a_2(\nabla_{\mathbb{H}} u)\, X_j X_s u\, \xi^2\, v_1\, \mathrm{d}x \\
&\quad + 2 \int_{B_r} Tu\, \partial_{z_j} a_2(\nabla_{\mathbb{H}} u)\, \xi\, X_j \xi\, v_1\, \mathrm{d}x \\
&\quad + \int_{B_r} Tu\, \partial_{z_j} a_2(\nabla_{\mathbb{H}} u)\, \xi^2\, X_j v_1\, \mathrm{d}x \\
&\quad =: J_{5,1} + J_{5,2} + J_{5,3}.
\end{aligned}$$

Note that $J_{5,2}$ and $J_{5,3}$ can be estimated respectively as J_4 and J_3.

Denoting $J_{5,1} := \sum_{s,j} J_{5,1}^{s,j}$ we have

(4.5)
$$\begin{aligned}
\left|\sum_j J_{5,1}^{1,j}\right| \le C_p \int_{B_r} \xi^2\, w^{\frac{p-3}{2}}\, |\nabla_{\mathbb{H}} v_1|\, v_1\, |Tu|\, \mathrm{d}x &\le C_p \varepsilon \int_{B_r} \xi^2\, w^{\frac{p-2}{2}}\, |\nabla_{\mathbb{H}} v_1|^2\, \mathrm{d}x \\
&\quad + \frac{C_p}{\varepsilon} \int_{B_r} \xi^2\, w^{\frac{p-4}{2}}\, |Tu|^2\, v_1^2\, \mathrm{d}x
\end{aligned}$$

and

$$\begin{aligned}
\left|J_{5,1}^{2,1}\right| &\le \int_{B_r} |\partial_{z_2} \partial_{z_1} a_2(\nabla_{\mathbb{H}} u)\, X_1 X_2 u\, Tu\, \xi^2\, v_1|\, \mathrm{d}x \\
&\le \int_{B_r} |\partial_{z_2} \partial_{z_1} a_2(\nabla_{\mathbb{H}} u)\, X_2 X_1 u\, Tu|\, \xi^2\, v_1\, \mathrm{d}x \\
&\quad + \int_{B_r} |\partial_{z_2} \partial_{z_1} a_2(\nabla_{\mathbb{H}} u)|\, |Tu|^2\, \xi^2\, v_1\, \mathrm{d}x.
\end{aligned}$$

The first term of the last inequality can be estimated as in (4.5). For the other term we have

$$\begin{aligned}
\int_{B_r} |\partial_{z_2} \partial_{z_1} a_2(\nabla_{\mathbb{H}} u)|\, |Tu|^2\, \xi^2\, v_1\, \mathrm{d}x &\le C_p \int_{B_r} \xi^2\, w^{\frac{p-3}{2}}\, |Tu|^2\, v_1\, \mathrm{d}x \\
&\le C_p \int_{A_{k,r}^+} \xi^2\, w^{\frac{p-2}{2}}\, |Tu|^2\, \mathrm{d}x \\
&\quad + C_p \int_{B_r} \xi^2\, w^{\frac{p-4}{2}}\, |Tu|^2\, v_1^2\, \mathrm{d}x.
\end{aligned}$$

Now another key step is to use the equation in (2.5) and (2.3) to get

$$
\begin{aligned}
|J_{5,1}^{2,2}| &= \left| \int_{B_r} \partial_{z_2}\partial_{z_2} a_2(\nabla_{\mathbb{H}} u)\, X_2 X_2 u\, \xi^2\, v_1\, Tu \,\mathrm{d}x \right| \\
&\le C_p \int_{B_r} |\partial_{z_2}\partial_{z_2} a_2(\nabla_{\mathbb{H}} u)\, X_1 X_1 u\, Tu|\, \xi^2\, v_1 \,\mathrm{d}x \\
&\quad + C_p \int_{B_r} |\partial_{z_2}\partial_{z_2} a_2(\nabla_{\mathbb{H}} u)\, X_2 X_1 u\, Tu|\, \xi^2\, v_1 \,\mathrm{d}x \\
&\quad + C_p \int_{B_r} |\partial_{z_2}\partial_{z_2} a_2(\nabla_{\mathbb{H}} u)\, X_1 X_2 u\, Tu|\, \xi^2\, v_1 \,\mathrm{d}x \\
&=: F_1 + F_2 + F_3.
\end{aligned}
$$

Note that F_1 and F_2 can be estimated as $J_{5,1}^{1,j}$, while F_3 can be estimated as $J_{5,1}^{2,1}$.

Choosing ε small enough (4.4) becomes

(4.6)

$$
\begin{aligned}
\int_{B_r} \xi^2\, w^{\frac{p-2}{2}}\, |\nabla_{\mathbb{H}} v_1|^2 \,\mathrm{d}x &\le C \int_{B_r} |\nabla_{\mathbb{H}} \xi|^2\, w^{\frac{p-2}{2}}\, v_1^2 \,\mathrm{d}x + C \int_{A_{k,r}^+} \xi^2\, w^{\frac{p-2}{2}}\, |Tu|^2 \,\mathrm{d}x \\
&\quad + C \int_{B_r} \xi^2\, w^{\frac{p-4}{2}}\, |Tu|^2\, v_1^2 \,\mathrm{d}x \\
&=: I_1 + I_2 + I_3.
\end{aligned}
$$

We only need to estimate I_2 and I_3. We use Hölder's inequality with exponent $q/2$ and Lemma 2.2:

$$
\begin{aligned}
I_2 &\le \left(\int_{A_{k,r}^+} \xi^q\, w^{\frac{p-2}{2}}\, |Tu|^q \,\mathrm{d}x \right)^{\frac{2}{q}} \left(\int_{A_{k,r}^+} w^{\frac{p-2}{2}} \,\mathrm{d}x \right)^{1-\frac{2}{q}} \\
&\le \left(\int_{B_{R_0}} \eta^q\, w^{\frac{p-2}{2}}\, |Tu|^q \,\mathrm{d}x \right)^{\frac{2}{q}} \left(\int_{A_{k,r}^+} w^{\frac{p-2}{2}} \,\mathrm{d}x \right)^{1-\frac{2}{q}} \\
&\le \left(\left(\|\nabla_{\mathbb{H}} \eta\|_{L^\infty}^2 + \|\eta T\eta\|_{L^\infty} \right)^{\frac{q}{2}} \int_{B_{R_0}} w^{\frac{p-2+q}{2}} \,\mathrm{d}x \right)^{\frac{2}{q}} \left(\delta^2 + \mu(r)^2\right)^{\frac{p-2}{2}(1-\frac{2}{q})} \left| A_{k,r}^+ \right|^{1-\frac{2}{q}} \\
&\le \frac{C_p\, q^6}{R_0^2} \left(\delta^2 + \mu(R_0)^2\right)^{\frac{p}{2}} |B_{R_0}|^{\frac{2}{q}} \left| A_{k,r}^+ \right|^{1-\frac{2}{q}},
\end{aligned}
$$

where η is a cut-off function between $B_{\frac{R_0}{2}}$ and B_{R_0} with $|\nabla_{\mathbb{H}} \eta| \le \frac{C}{R_0}$. In a similar way and noting that $v_1^2 \le 2(\delta^2 + \mu(R_0)^2)$ for $k \ge -\mu(R_0)$ we get

$$
\begin{aligned}
I_3 &\le \left(\delta^2 + \mu(R_0)^2\right) \left(\int_{A_{k,r}^+} \xi^q\, w^{\frac{p-2}{2}}\, |Tu|^q \,\mathrm{d}x \right)^{\frac{2}{q}} \left(\int_{A_{k,r}^+} w^{\frac{p-4}{2} - \frac{2}{q-2}} \,\mathrm{d}x \right)^{1-\frac{2}{q}} \\
&\le \left(\delta^2 + \mu(R_0)^2\right) \left(\left(\|\nabla_{\mathbb{H}} \eta\|_{L^\infty}^2 + \|\eta T\eta\|_{L^\infty} \right)^{\frac{q}{2}} \int_{B_{R_0}} w^{\frac{p-2+q}{2}} \,\mathrm{d}x \right)^{\frac{2}{q}} \\
&\quad \times \left(\delta^2 + \mu(r)^2\right)^{\left(\frac{p-4}{2} - \frac{2}{q-2}\right)(1-\frac{2}{q})} \left| A_{k,r}^+ \right|^{1-\frac{2}{q}} \\
&\le \frac{C_p\, q^6}{R_0^2} \left(\delta^2 + \mu(R_0)^2\right)^{\frac{p}{2}} |B_{R_0}|^{\frac{2}{q}} \left| A_{k,r}^+ \right|^{1-\frac{2}{q}}. \qquad \square
\end{aligned}
$$

Remark 4.2. Note that the main difficulty in the proof is estimating the terms containing Tu. In particular in J_5 we integrate by parts twice to avoid dealing with terms involving $\nabla_{\mathbb{H}} Tu$. Then we use the equation in order to estimate terms with X_2X_2u appropriately with quantities independent of δ or that can be absorbed in the right hand side.

5. Oscillation estimate

In this section we prove our main result Theorem 1.1. Recall that we fixed a ball $B_{R_0} \Subset \Omega$ and we now consider an arbitrary concentric ball $B_R \subset B_{\frac{R_0}{2}}$.

Remark 5.1. Let $u \in HW^{1,p}(\Omega)$ be a weak solution of the nondegenerate equation (1.2) for $p > 4$. For $\delta \geq \lambda(R)$ we easily get that for every $\lambda_1 \in (0,1)$ there exists $A = A(p, \lambda_1)$ such that

$$\operatorname{osc}_{B_{\lambda_1 R}}(X_l u) \leq A \operatorname{osc}_{B_R}(X_l u) + BR^{\alpha} \quad \text{for every } \ l \in \{1, 2\},$$

where

$$B = \frac{C_p q^{\frac{6}{p}} (\delta^2 + \mu(R_0)^2)^{\frac{1}{2}}}{4(1-A) R_0^{\alpha}} \quad \text{and} \quad \alpha = \left(1 - \frac{4}{q}\right) \frac{2}{p}.$$

Proof. Since $\delta \geq \lambda(R)$ we can get rid of the weight and obtain that $X_l u$ is in a De Giorgi class. Indeed from (4.1) we get

$$\int_{B_{r'}} |\nabla_{\mathbb{H}} v_l|^2 \, \mathrm{d}x \leq \frac{C_p}{(r - r')^2} \int_{B_r} v_l^2 \, \mathrm{d}x + \frac{2^{p-2}\chi}{\mu(R)^{p-2}} \left| A^+_{k,r}(X_l u) \right|^{1-\frac{2}{q}}$$

for all levels $k > -\mu(R_0)$ and radii $r' < r < R$. Now if

$$\mu(R) \geq \chi^{\frac{1}{p}} R^{\left(1-\frac{4}{q}\right)\frac{2}{p}}, \tag{5.1}$$

denoting

$$\chi' = C_p q^{\frac{12}{p}} (\delta^2 + \mu(R_0)^2) \left(\frac{R}{R_0}\right)^{2\left(1-\frac{4}{q}\right)\frac{2}{p}} \ R^{2\left(\frac{4}{q}-1\right)}$$

we get that $X_l u \in DG^+(B_{R_0}, C_p, \chi', q)$ for all levels $k > -\mu(R_0)$ and radii $r' < r < R$.

Analogously from (4.2) we get also that $-X_l u \in DG^+(B_{R_0}, C_p, \chi', q)$ for all levels $k < \mu(R_0)$ and radii $r' < r < R$. Hence we can apply the oscillation estimate in Theorem 3.5 to get for any $\lambda_1 \in (0,1)$ the existence of $A = A(p, \lambda_1) \in (0,1)$ such that for every $l \in \{1, 2\}$ we have

$$\operatorname{osc}_{B_{\lambda_1 R}}(X_l u) \leq A \operatorname{osc}_{B_R}(X_l u) + B' R^{1-\frac{4}{q}},$$

where $4(1-A)B' = (\chi')^{\frac{1}{2}}$. By the definition of χ' and combining with the case when (5.1) does not hold, we get the result. $\square$

We now consider the interesting case when the equation degenerates, namely $\delta < \lambda(R)$. Here we face an alternative: either the maximum $\mu(R)$ has the right 'Hölder decay' or the horizontal gradient $\nabla_{\mathbb{H}} u$ is bounded away from zero, and hence the equation behaves like the nondegenerate case in Remark 5.1. More precisely we have:

Proposition 5.2. *Let $u \in HW^{1,p}(\Omega)$ be a weak solution of the nondegenerate equation* (1.2) *for $p > 4$ and consider $B_R \subset B_{\frac{R_0}{2}}$. Then there exist $\theta = \theta(p) \in (0,1)$ and $A = A(p) \in (0,1)$ such that:*

Case 1. If for some $l \in \{1,2\}$ we have either

$$\left| B_R \cap \left\{ X_l \geq \frac{1}{2}\mu(R) \right\} \right| \geq \theta |B_R| \tag{5.2}$$

or

$$\left| B_R \cap \left\{ X_l \leq -\frac{1}{2}\mu(R) \right\} \right| \geq \theta |B_R|, \tag{5.3}$$

then

$$\text{either} \quad \mu(R) \leq c_p \chi^{\frac{1}{p}} R^{\left(1-\frac{4}{q}\right)\frac{2}{p}} \quad \text{or} \quad |X_l u| \geq \frac{1}{32}\mu(R) \quad \text{a.e. in } B_{R/2},$$

where $c_p = 2(4/3)^{\frac{2}{p}}$.

Case 2. If for every $l \in \{1,2\}$ neither (5.2) nor (5.3) holds, then

$$\mu(R/2) \leq A\mu(R) + BR^{\alpha}, \tag{5.4}$$

where

$$B = \frac{C_p q^{\frac{6}{p}}}{2(1-A)} \frac{\mu(R_0)}{R_0^{\alpha}} \quad \text{and} \quad \alpha = \left(1 - \frac{4}{q}\right)\frac{2}{p}.$$

Proof. Case 1: Consider (5.3). We will show that it implies $X_l u \leq -\frac{1}{32}\mu(R)$ provided $\mu(R) \geq c_p \chi^{\frac{1}{p}} R^{\left(1-\frac{4}{q}\right)\frac{2}{p}}$. Define the auxiliary function

$$V_l = |X_l u|^{\frac{p}{2}} \operatorname{sign}(X_l u).$$

Observe that $|V_l| \leq (2\lambda(R))^{\frac{p}{2}}$ on B_R. Also

$$|\nabla_{\mathbb{H}} V_l|^2 = \frac{p^2}{4} |X_l u|^{p-2} |\nabla_{\mathbb{H}} X_l u|^2. \tag{5.5}$$

Denote $h = |k|^{\frac{p}{2}} \operatorname{sign}(k) = g(k)$ and note that $\{X_l > k\} = \{V_l > h\}$. Therefore (4.1) becomes

$$\begin{aligned} \int_{A^+_{h,r'}(V_l)} |\nabla_{\mathbb{H}} V_l|^2 \, \mathrm{d}x &\leq C_p \int_{B_{r'}} w^{\frac{p-2}{2}} |\nabla_{\mathbb{H}} v_l|^2 \, \mathrm{d}x \\ &\leq \frac{C_p}{(r-r')^2} (\mu(r) - k)^2 (\delta^2 + \mu(r)^2)^{\frac{p-2}{2}} |A^+_{h,r}(V_l)| \\ &\quad + \chi |A^+_{h,r}(V_l)|^{1-\frac{2}{q}} \\ &\leq \frac{C_p}{(r-r')^2} (\lambda(R))^p |A^+_{h,r}(V_l)| + \chi |A^+_{h,r}(V_l)|^{1-\frac{2}{q}} \end{aligned} \tag{5.6}$$

for $k > -\lambda(R)$ and $r' < r \leq R$. Denoting $H = H(R) = (\lambda(R))^{\frac{p}{2}}$ the inequality (5.6) rewrites as

$$\int_{A^+_{h,r'}(V_l)} |\nabla_{\mathbb{H}} V_l|^2 \, \mathrm{d}x \leq \frac{C_p}{(r-r')^2} (H(R))^2 |A^+_{h,r}(V_l)| + \chi |A^+_{h,r}(V_l)|^{1-\frac{2}{q}} \tag{5.7}$$

for levels $h > -H(R)$ and radii $r' < r \le R$. Now denote $M(\frac{R}{2}) = \sup_{B_{R/2}} V_l$.

Case a. $M(\frac{R}{2}) < -\frac{H(R)}{4}$.

This means $X_l u < 0$ in $B_{R/2}$ and after some algebraic manipulations

$$X_l u < -\frac{\mu(R)}{32} \quad \text{on } B_{R/2}.$$

Case b. $M(\frac{R}{2}) \ge -\frac{H(R)}{4}$.

For levels $h \in [-H(R), -H(R)/2]$ we have $\sup_{B_{R/2}} (V_l - h)^+ \ge \frac{H(R)}{4}$; therefore

$$\frac{(H(R))^2}{16} \le \sup_{B_r} |(V_l - h)^+|^2 \quad \text{for } r \in [R/2, R].$$

Hence, from (5.7) we get that $V_l \in DG^+(B_{R_0}, C_p, \chi, q)$ for levels $h \in [-H(R), -H(R)/2]$ and radii $r' < r \in [R/2, R]$. Apply Lemma 3.3 with $k = -H(R)$, $\lambda_0 = \frac{2^{\frac{p}{2}+1/2}}{2^{\frac{p}{2}+1}}$, $\lambda_1 = 1/2$ to get the existence of $\theta_1 = \theta_1(p) \in (0,1)$ such that $|A^+_{-H,R}(V_l)| \le \theta_1 |B_R|$ implies that $V_l \le -\frac{H(R)}{2}$ on $B_{R/2}$, provided $M(R) + H(R) \ge \chi^{\frac{1}{2}} R^{1-\frac{4}{q}}$. This is true if

$$\mu(R) \ge 2\left(\frac{4}{3}\right)^{\frac{2}{p}} \chi^{\frac{1}{p}} R^{\left(1-\frac{4}{q}\right)\frac{2}{p}}. \tag{5.8}$$

Then as in Case a, we obtain

$$X_l u < -\frac{\mu(R)}{8} \quad \text{in } B_{R/2}.$$

Observe that $\{V_l > -H(R)\} = \{X_l u > -\lambda(R)\} = \{X_l u > -\mu(R)/2\}$. Passing to the complements we have proved that there exists $\theta = 1 - \theta_1$ such that (5.3) implies that $X_l u \le -\mu(R)/32$ on $B_{R/2}$, provided (5.8).

If (5.2) holds, then we proceed similarly and we get the conclusion of Case 1.

Case 2: If for the θ found in Case 1 neither (5.2) nor (5.3) holds for any $l \in \{1, 2\}$, then there exist $\frac{1}{2} < \lambda_1 < 1$ and $0 < C_0 < 1$ such that

$$\left| B_{\lambda_1 R} \cap \left\{ X_l \ge \frac{1}{2}\mu(R) \right\} \right| \le C_0 |B_{\lambda_1 R}| \tag{5.9}$$

and

$$\left| B_{\lambda_1 R} \cap \left\{ X_l \le -\frac{1}{2}\mu(R) \right\} \right| \le C_0 |B_{\lambda_1 R}| \tag{5.10}$$

are satisfied for every $l \in \{1, 2\}$. Considering levels $k \in [\frac{\mu(R)}{2}, \mu(R)]$, on $\{X_l u > k\}$ we have

$$k^{p-2} \le w^{\frac{p-2}{2}} \le C_p k^{p-2}.$$

Therefore in (4.1) we can get rid of the weight:

$$\int_{B_{r'}} |\nabla_{\mathbb{H}} v_l|^2 \,\mathrm{d}x \le \frac{C}{(r-r')^2} \int_{B_r} v_l^2 \,\mathrm{d}x + \frac{2^{p-2}\chi}{\mu(R)^{p-2}} \left| A^+_{k,r}(X_l u) \right|^{1-\frac{2}{q}},$$

and proceeding as in Remark 5.1, if

$$\mu(R) \ge \chi^{\frac{1}{p}} R^{\left(1-\frac{4}{q}\right)\frac{2}{p}} \tag{5.11}$$

we get that $X_l u \in DG^+(B_{R_0}, C_p, \chi', q)$ for levels $k \in [\frac{\mu(R)}{2}, \mu(R)]$, radii $r' < r < R$ with

$$\chi' = C_p q^{\frac{12}{p}} \lambda(R_0)^2 \left(\frac{R}{R_0}\right)^{2\left(1-\frac{4}{q}\right)\frac{2}{p}} R^{2\left(\frac{4}{q}-1\right)}.$$

Apply Lemma 3.4 with λ_1 and C_0 as in (5.9), $k = \frac{\mu(R)}{2}$ to conclude that given $\theta_0 \in (0,1)$ there exists a natural number $s = s(p, \lambda_1, C_0, \theta_0)$ such that either

$$\mu(R) \leq 2^{s+1} (\chi')^{\frac{1}{2}} R^{1-\frac{4}{q}} \tag{5.12}$$

or

$$|A^+_{k_s, \lambda_1 R}| \leq \theta |B_{\lambda_1 R}|, \tag{5.13}$$

where $k_s = \mu(R)(1 - 2^{-s-1})$.

Now in the case (5.13) we want to use Lemma 3.3 for radii $r' < r \in [R/2, \lambda_1 R]$, $k = k_s = (1 - 2^{-s-1})\mu(R)$, $\lambda_0 = 1/2$. This can be applied if

$$k_s < \sup_{B_{\lambda_1 R}} (X_l u). \tag{5.14}$$

Then we would conclude that either

$$\begin{aligned} X_l u &\leq \frac{1}{2} k_s + \frac{1}{2}\mu(\lambda_1 R) \leq \left(1 - \frac{1}{2^{s+1}}\right)\frac{1}{2}\mu(R) + \frac{1}{2}\mu(R) \\ &= \mu(R)\left(1 - \frac{1}{2^{s+2}}\right) \quad \text{a.e. in} \quad B_{\frac{R}{2}} \end{aligned} \tag{5.15}$$

or

$$\sup_{\lambda_1 R}(X_l u) \leq \left(1 - \frac{1}{2^{s+1}}\right)\mu(R) + (\chi')^{\frac{1}{2}} R^{1-\frac{4}{q}}. \tag{5.16}$$

If (5.14) is not true, then we get

$$\sup_{B_{R/2}} (X_l u) \leq \sup_{\lambda_1 R}(X_l u) \leq k_s = \left(1 - \frac{1}{2^{s+1}}\right)\mu(R). \tag{5.17}$$

Repeating the same steps for $-X_l u$ using assumption (5.10) and the estimate (4.2), we will find the same alternatives, except instead of (5.15)-(5.17) we will have

$$X_l u \geq -\mu(R)\left(1 - \frac{1}{2^{s+2}}\right) - (\chi')^{\frac{1}{2}} R^{1-\frac{4}{q}} \quad \text{a.e. in} \quad B_{\frac{R}{2}}. \tag{5.18}$$

In conclusion, combining all the cases we get

$$\mu(R/2) \leq \left(1 - \frac{1}{2^{s+2}}\right)\mu(R) + c_p q^{\frac{6}{p}} 2^{s+1} \lambda(R_0) \left(\frac{R}{R_0}\right)^{\left(1-\frac{4}{q}\right)\frac{2}{p}}.$$

□

Now we need the following technical lemma, adapted from Lemma 7.3 in [12]:

Lemma 5.3. *Let $0 < A, \lambda, \alpha < 1$ with $A \neq \lambda^\alpha$ and $B, R_0 \geq 0$. Let $\varphi : [0, +\infty[\longrightarrow [0, +\infty[$ be an increasing function such that*

$$\varphi(\lambda R) \leq A\varphi(R) + BR^\alpha \quad \textit{for all} \quad R \leq R_0. \tag{5.19}$$

Then for every $R \leq R_0$ we have

$$\varphi(r) \leq \frac{1}{A}\left(\frac{r}{R}\right)^{\min\{\log_\lambda A, \alpha\}} \left[\varphi(R) + \frac{BR^\alpha}{|A - \lambda^\alpha|}\right] \quad \textit{for all} \quad r \leq R. \tag{5.20}$$

We finally prove Theorem 1.1.

Proof of Theorem 1.1. We prove the result for $u \in HW^{1,p}(\Omega)$, the weak solution of the nondegenerate equation (1.2). Then we can obtain the estimate for solutions to the degenerate equation (1.1) by an approximation argument as in [24, Theorem 5.3].

The alternatives in Proposition 5.2 can be combined in either

$$\mu(R/2) \leq A\mu(R) + BR^{\alpha} \tag{5.21}$$

or

$$|X_l u| \geq \frac{1}{32}\mu(R) \quad \text{a.e. in } B_{R/2}. \tag{5.22}$$

In this last case we have

$$w^{\frac{p-2}{2}} \geq \left(\frac{1}{32}\right)^{p-2} \mu(R)^{p-2} \quad \text{a.e. in} \quad B_{\frac{R}{2}}.$$

Since also

$$w^{\frac{p-2}{2}} \leq \left(\delta^2 + \mu(R)^2\right)^{\frac{p-2}{2}} \leq C_p \mu(R)^{p-2} \quad \text{in} \quad B_R,$$

from the estimate (4.1) we get

$$\int_{B_{r'}} |\nabla_{\mathbb{H}} v_l|^2 \, \mathrm{d}x \leq \frac{C}{(r-r')^2} \int_{B_r} v_l^2 \, \mathrm{d}x + \frac{\chi}{\mu(R)^{p-2}} \left| A^+_{k,r}(X_l u)\right|^{1-\frac{2}{q}}$$

for every $r' < r \leq R/2$ and for every level $k > -\mu(R_0)$. Now as before, if

$$\mu(R) \geq \chi^{\frac{1}{p}} R^{\left(1-\frac{4}{q}\right)\frac{2}{p}} \tag{5.23}$$

we get $v_l \in DG^+(B_{R_0}, C_p, \chi', q)$ for every $r' < r \leq R/2$ and for every level $k > -\mu(R_0)$. The same is true for $-v_l$, with levels $k < \mu(R_0)$, so proceeding as in the proof of Remark 5.1, we are in the position to apply the oscillation Lemma 3.5 to conclude there exists $A = A(p) \in (0,1)$ such that

$$\operatorname{osc}_{B_{R/4}}(X_l u) \leq A\operatorname{osc}_{B_{R/2}}(X_l u) + BR^{\alpha} \leq A\operatorname{osc}_{B_R}(X_l u) + BR^{\alpha} \quad \text{for all} \quad R \leq \frac{R_0}{2}, \tag{5.24}$$

where B and α are as in Proposition 5.2.

Now apply Lemma 5.3 to (5.21) and (5.24) with $\lambda = 1/4$, and A and B as given in (5.2). Noting that $\operatorname{osc}_{B_r}(X_l u) \leq 2\mu(r)$ we can combine all the estimates, and hence the theorem is proved with $\beta = \min\{-\log_4(A)\,,\,\alpha\}$. □

Remark 5.4. From the explicit expression of β and B we see that the estimate blows up when q goes to infinity; hence the Hölder exponent found with this proof satisfies the constraint $0 < \beta < \frac{2}{p}$.

6. Appendix: A proof of Lemma 2.2

We use the following estimates of Zhong [26] (see also [24], Lemmas 5.3 and 5.4).

Lemma 6.1. *Let $q \geq 4$ and $\xi \in C_0^{\infty}(\Omega)$. Then*

$$\int_{\Omega} \xi^q \, w^{\frac{p-2}{2}} \, |Tu|^{q-2} \, |\nabla^2_{\mathbb{H}} u|^2 \, dx \leq C_p^{\frac{q-2}{2}} (q-1)^{q-2} \, \|\nabla_{\mathbb{H}}\xi\|^{q-2}_{L^\infty} \int_{\Omega} \xi^2 \, w^{\frac{p+q-4}{2}} \, |\nabla^2_{\mathbb{H}} u|^2 \, dx \,. \tag{6.1}$$

Lemma 6.2. *Let $q \geq 4$ and $\xi \in C_0^\infty(\Omega)$. Then*

$$\int_\Omega \xi^2\, w^{\frac{p+q-4}{2}}\, |\nabla^2_{\mathbb{H}} u|^2\, dx \leq C_p\, \Big(\|\nabla_{\mathbb{H}}\xi\|^2_{L^\infty} + \xi\, \|T\xi\|_{L^\infty}\Big)\, (q-1)^{10} \int_{\mathrm{supp}(\xi)} w^{\frac{p+q-2}{2}}\, dx.$$

Lemma 6.1 follows by using $\phi = \xi^q |Tu|^{q-2} X_i u$ as test functions in equations (2.6) and (2.7), while Lemma 6.2 follows by using $\phi = \xi^2 w^{\frac{q-2}{2}} X_i u$ and the estimate in Lemma 6.1.

Proof of Lemma 2.2. Using $|Tu| \leq 2|\nabla^2_{\mathbb{H}} u|$ and Lemmas 6.1 and 6.2 we have for $q \geq 4$,

(6.2)

$$\begin{aligned}
\int_\Omega \xi^q\, w^{\frac{p-2}{2}}\, |Tu|^q\, \mathrm{d}x &\leq 2 \int_\Omega \xi^q\, w^{\frac{p-2}{2}}\, |Tu|^{q-2}\, |\nabla^2_{\mathbb{H}} u|^2\, \mathrm{d}x \\
&\leq C_p^{\frac{q-2}{2}}\, (q-1)^{q-2}\, \|\nabla_{\mathbb{H}}\xi\|^{q-2}_{L^\infty} \int_\Omega \xi^2\, w^{\frac{p+q-4}{2}}\, |\nabla^2_{\mathbb{H}} u|^2\, \mathrm{d}x \\
&\leq C^{\frac{q-2}{q}}\, (q)^{q+8} \Big(\|\nabla_{\mathbb{H}}\xi\|^2_{L^\infty} + \xi\, \|T\xi\|_{L^\infty}\Big)^{\frac{q}{2}} \int_{\mathrm{supp}(\xi)} w^{\frac{p+q-2}{2}}\, \mathrm{d}x.
\end{aligned}$$

□

Note 6.3. While working on this paper the January 2017 preprint [4] was brought to my attention. This manuscript contains a general statement valid for all $p > 2$ and generalizes the regularity results proved above. The proof in [4] is based on the proof put forward by Zhong in [26], which I have credited throughout. The proof in this manuscript is different in the way it handles the term that is here denoted by J_5 in the proof of Proposition 4.1. In [4] additional iterations are used to control this term, while here we use a double integration by parts and the structure of the equation. We also point out the recent preprint [27] which addresses the case $1 < p < 2$.

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