SOME NEW COMPUTABLE STRUCTURES OF HIGH RANK

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ABSTRACT. We give several new examples of computable structures of high Scott rank. For earlier known computable structures of Scott rank ω_1^{CK}, the computable infinitary theory is $\aleph_0$-categorical. Millar and Sacks asked whether this was always the case. We answer this question by constructing an example whose computable infinitary theory has non-isomorphic countable models. The standard known computable structures of Scott rank $\omega_1^{CK} + 1$ have infinite indiscernible sequences. We give two constructions with no indiscernible ordered triple.

1. INTRODUCTION

Our main result answers an open problem posed by Millar and Sacks [12]. They asked whether every computable structure of Scott rank ω_1^{CK} is completely determined by the computable sentences it satisfies. We give a negative answer by building a computable structure of Scott rank ω_1^{CK} whose computable infinitary theory is *not* $\aleph_0$-categorical. This is a new model of high Scott rank which is fundamentally different from all previously constructed models.

The Scott rank of a structure measures the internal complexity. We give one definition below. There are other definitions, which assign slightly different ordinals, but the important distinctions are the same for all definitions in current use. In particular, if one definition assigns Scott rank $\omega_1^{CK} + 1$ or ω_1^{CK} to a particular computable structure, then the other definitions do the same.

Let $\mathcal{A}$ be a countable structure for a computable language. Our definition of Scott rank is based on a family of equivalence relations $\sim^\alpha$, for countable ordinals α. Scott's original definition [14] was based on a slightly different family of equivalence relations.

Definition 1.1. Let $\bar{a}$ and $\bar{b}$ be tuples in $\mathcal{A}$ of the same finite length. Then

  (1) $\bar{a} \sim^0 \bar{b}$ if $\bar{a}$ and $\bar{b}$ satisfy the same atomic formulas.
  (2) For $\alpha > 0$, $\bar{a} \sim^\alpha \bar{b}$ if:
    • for each $\beta < \alpha$ and $\bar{c}$, there exists $\bar{d}$ such that $\bar{a}, \bar{c} \sim^\beta \bar{b}, \bar{d}$, and
    • for each $\beta < \alpha$ and $\bar{d}$, there exists $\bar{c}$ such that $\bar{a}, \bar{c} \sim^\beta \bar{b}, \bar{d}$.

For later use, we extend the definition $\sim^\alpha$ to allow tuples from different structures.

Definition 1.2. Let $\mathcal{A}$ and $\mathcal{B}$ be structures for the same language, and let $\bar{a}$ and $\bar{b}$ be tuples of the same length in $\mathcal{A}$, $\mathcal{B}$, respectively.

   (1) $(\mathcal{A}, \bar{a}) \sim^0 (\mathcal{B}, \bar{b})$ if $\bar{a}$ and $\bar{b}$ satisfy the same atomic formulas in their respective structures.

   (2) For $\alpha > 0$, $(\mathcal{A}, \bar{a}) \sim^\alpha (\mathcal{B}, \bar{b})$ if:
- for each $\beta < \alpha$ and $\bar{c}$ in $\mathcal{A}$, there exists $\bar{d}$ in $\mathcal{B}$ such that $(\mathcal{A}, \bar{a}, \bar{c}) \sim^\beta (\mathcal{B}, \bar{b}, \bar{d})$, and
- for each $\beta < \alpha$ and $\bar{d}$ in $\mathcal{B}$, there exists $\bar{c}$ in $\mathcal{A}$ such that $(\mathcal{A}, \bar{a}, \bar{c}) \sim^\beta (\mathcal{B}, \bar{b}, \bar{d})$.

Remark. If $(\mathcal{A}, \bar{a}) \sim^\alpha (\mathcal{B}, \bar{b})$, then for any Σ_α formula $\varphi(\bar{x})$ of $L_{\omega_1\omega}$, $\mathcal{A} \models \varphi(\bar{a})$ iff $\mathcal{B} \models \varphi(\bar{b})$.

We define Scott rank, first for a tuple in a structure $\mathcal{A}$, and then for the structure itself.

Definition 1.3.

   (1) The *Scott rank of a tuple* $\bar{a}$ is the least α such that for all $\bar{b}$, if $\bar{a} \sim^\alpha \bar{b}$, then for all $\gamma > \alpha$, $\bar{a} \sim^\gamma \bar{b}$.

   (2) The *Scott rank of the structure* $\mathcal{A}$ is the least ordinal greater than the Scott ranks of all tuples in $\mathcal{A}$.

Nadel [13] observed that for a computable structure $\mathcal{A}$, two tuples are automorphic just in case they satisfy the same computable infinitary formulas. This implies that the Scott rank of $\mathcal{A}$ is at most $\omega_1^{CK} + 1$. The following is well-known.

Fact. Let $\mathcal{A}$ be a computable structure.

   (1) $\mathcal{A}$ has computable Scott rank iff there is a computable ordinal α such that for all tuples $\bar{a}$ in $\mathcal{A}$, the orbit of $\bar{a}$ is defined by a computable Σ_α formula.

   (2) $\mathcal{A}$ has Scott rank ω_1^{CK} iff for each tuple $\bar{a}$, the orbit is defined by a computable infinitary formula, but for each computable ordinal α, there is a tuple $\bar{a}$ whose orbit is not defined by a computable Σ_α formula.

   (3) $\mathcal{A}$ has Scott rank $\omega_1^{CK} + 1$ iff there is a tuple $\bar{a}$ whose orbit is not defined by a computable infinitary formula.

There are familiar examples of computable structures having various computable ordinal ranks. The canonical example of a computable structure of Scott rank $\omega_1^{CK} + 1$ is the *Harrison ordering*, a linear order with order type $\omega_1^{CK}(1 + \eta)$ (see [5]). Producing a computable structure of Scott rank ω_1^{CK} took longer. Makkai gave an example of an arithmetical structure of Scott rank ω_1^{CK} [11], a "group-tree". In [10], Makkai's construction is re-worked to give a computable structure. In [2], there is a simpler example, a computable tree of Scott rank ω_1^{CK}. In [1], the tree is used to produce further structures in familiar classes—a field, a group, etc.[1]

These examples of computable structures of Scott rank ω_1^{CK} all have the feature that the computable infinitary theory is $\aleph_0$-categorical. The conjunction of the sentences in the computable infinitary theory forms a Scott sentence. In [12], Millar and Sacks produced a structure $\mathcal{A}$ of Scott rank ω_1^{CK} such that the computable infinitary theory of $\mathcal{A}$ is not $\aleph_0$-categorical. The structure $\mathcal{A}$ is not computable;

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[1] Although [10] was not published until 2011, it was written before [2], which was published in 2006, and [1], which was published in 2009.

it is not even hyperarithmetical, but it has the feature that $\omega_1^{\mathcal{A}} = \omega_1^{CK}$. This means that $\mathcal{A}$ lives in a fattening of the admissible set $L_{\omega_1^{CK}}$.

In [12], Millar and Sacks asked whether there is a computable structure of Scott rank ω_1^{CK} whose computable infinitary theory is not $\aleph_0$-categorical. The question is asked again in [1]. Millar and Sacks also asked whether there are similar examples for other countable admissible ordinals. In [4], Freer proved the analog of the result of Millar and Sacks, producing, for an arbitrary countable admissible ordinal α, a structure $\mathcal{A}$ with $\omega_1^{\mathcal{A}} = \alpha$, such that the theory of $\mathcal{A}$ in the admissible fragment L_α is not $\aleph_0$-categorical. Freer's structure is not in L_α, but in a fattening of L_α. The main result of the present paper is an answer to the question of Millar and Sacks mentioned above: there is a computable structure of Scott rank ω_1^{CK} whose computable infinitary theory is not $\aleph_0$-categorical. The construction appears in Section 2. The essential idea of the construction is that the types over the model we build will encode ascending sequences through the Harrison linear order, but this coding will be "hidden" in the sense that the coding of an element a from the Harrison linear order, where a is identified with an ordinal α, will require on the order of α jumps. The non-principal type will be the type of a sequence that is cofinal in the well-founded part of the Harrison linear order.

By an "indiscernible sequence", we mean an infinite sequence that is indiscernible for $L_{\omega_1\omega}$ formulas.

Definition 1.4. Fix a structure $\mathcal{A}$. An *indiscernible sequence* in $\mathcal{A}$ is a sequence $(a_i)_{i\in\omega}$ of elements of $\mathcal{A}$ such that any two finite subsequences $a_{i_1}, \ldots, a_{i_n}$ and $b_{j_1}, \ldots, b_{j_n}$ (with $i_1 < i_2 < \cdots < i_n$ and $j_1 < j_2 < \cdots < j_n$) satisfy the same $L_{\omega_1\omega}$ formulas.

The Harrison ordering obviously has an infinite indiscernible sequence. Other examples of computable structures of Scott rank $\omega_1^{CK} + 1$ share this feature. Goncharov and Knight (unpublished) asked whether every computable structure of Scott rank $\omega_1^{CK} + 1$ has an infinite indiscernible sequence. At the same time, they also noticed that the structures of Scott rank ω_1^{CK} constructed by Makkai [11] and Knight and Millar [10] did not have an infinite indiscernible sequence.

In Section 3, we describe two constructions producing computable structures of Scott rank $\omega_1^{CK} + 1$ with not even an indiscernible ordered triple. The first is produced by taking a Fraïssé limit with infinitely many infinite equivalence classes and putting the structure of the Harrison ordering on the equivalence classes. Although this structure has no indiscernible triples, it is effectively bi-interpretable with the Harrison ordering [6], and hence it has an infinite indiscernible sequence of imaginaries.

The second example is a modified version of Makkai's construction [11]. Makkai gave a "computable operator" taking an input tree T to a group-tree $\mathcal{A}(T)$—the group-tree $\mathcal{A}(T)$ is computable uniformly in the input tree T. The structure $\mathcal{A}(T)$ is built by putting a group structure on each level of the tree (the language of the structure $\mathcal{A}(T)$ does not include the group operation, but, instead, has a collection of unary functions). Makkai constructed a Δ_2^0 "thin" tree T such that $\mathcal{A}(T)$ had Scott rank ω_1^{CK}. Knight and Millar [10] modified the construction to make the input tree (and hence the output group-tree) computable. All of the elements of $\mathcal{A}(T)$ are definable from a collection of parameters g_n, one at each level. Hence, $\mathcal{A}(T)$ does not have an indiscernible ordered triple. We will show that if our input tree T is the sequence of descending sequences in the Harrison ordering, then the

resulting group-tree $\mathcal{A}(T)$ has Scott rank $\omega_1^{CK} + 1$, but it still does not have an indiscernible triple (or even an indiscernible triple of imaginaries).

2. A STRUCTURE OF SCOTT RANK ω_1^{CK} WHOSE INFINITARY THEORY IS NOT $\aleph_0$-CATEGORICAL

In this section, our goal is to prove the following.

Theorem 2.1. *There is a computable structure $\mathcal{M}$ with Scott rank ω_1^{CK} such that the computable infinitary theory of $\mathcal{M}$ is not $\aleph_0$-categorical.*

We will use some material on trees from [2]. The trees are isomorphic to subtrees of $\omega^{<\omega}$. We use a language with a successor relation. Here are the facts that we need. There is a computable tree T^* of Scott rank ω_1^{CK}. In addition, for a fixed Π_1^1 path P through $\mathcal{O}$, there is a family of approximating trees $(T^a)_{a \in P}$ of computable Scott rank. For $a \in P$ such that $|a| = \alpha$, T^a has tree rank at most $\omega(\alpha + 1)$, and $T^* \sim^\alpha T^a$. The family $(T^a)_{a \in P}$ is computable uniformly in a, and the tree ranks of the nodes of T^a are also computable, uniformly in a, in the sense that we can effectively label the nodes of T^a by pairs (b, n), where $\sigma \in T^a$ has label (b, n) for $b \in P$ just in case σ has tree rank $\omega \cdot \beta + n$ for $|b| = \beta$. The tree T^* and the approximations T^a are all "rank-homogeneous".

The computable structure $\mathcal{M}$ we build to prove the theorem will essentially be of the following form. We will identify the well-founded part of the Harrison ordering with a path P through $\mathcal{O}$, so that we can assign to each element a of the Harrison ordering a tree T^a (with $T^a = T^*$ if a is in the non-well-founded part of $\mathcal{O}$). The elements of $\mathcal{M}$ will represent finite ascending sequences in the Harrison ordering. Fix an element x of $\mathcal{M}$ representing the finite ascending sequence σ. For each finite ascending sequence τ through the Harrison ordering, with last element a, we will code whether or not $\tau \preceq \sigma$. If $\tau \preceq \sigma$, we will attach a copy of T^a to x in a certain way, and if not, then we will attach a copy of T^* to x; this will be the only way in which x codes σ. Note that if a is not in the well-founded part of the Harrison ordering, then $T^a = T^*$ and so there is no distinction made. If a is in the well-founded part of the Harrison ordering, say corresponding to an ordinal α, then we will have $T^a \sim^\alpha T^*$, and so if α is large, it is difficult to tell whether $\tau \preceq \sigma$. In a computable model, no element x can represent an infinite ascending sequence σ which is cofinal in the well-founded part of the Harrison ordering. On the other hand, one can see by a compactness argument that the infinitary theory cannot forbid such a sequence, as with computable formulas of bounded rank one cannot tell the difference between T^a and T^* for large enough a in the standard part of the Harrison ordering. So this gives a non-principal type which is omitted from $\mathcal{M}$. Since there is a model realizing this type, the infinitary theory of $\mathcal{M}$ is not $\aleph_0$-categorical.

Definition 2.2. A tree T is *rank-homogeneous* provided that for each node x at level n, if x has tree rank α and there is a node at level $n + 1$ (not necessarily a successor of x) of tree rank $\beta < \alpha$, then x has infinitely many successors of tree rank β. Also, for each node x at level n, if x has infinite rank, then it has infinitely many successors of infinite rank, in addition to infinitely many of each ordinal rank β that occurs at level $n + 1$.

A rank-homogeneous tree can be completely described by saying which ordinal ranks appear at each level. Moreover, the following definition and lemma show

how this is stratified: to describe a rank-homogeneous tree up to the αth level of back-and-forth relations, one needs to say which ordinal ranks $\leq \alpha$ appear at each level.

Definition 2.3. For rank-homogeneous trees $\mathcal{A}$ and $\mathcal{B}$, with $\bar{a}$ in $\mathcal{A}$ and $\bar{b}$ in $\mathcal{B}$, we write $(\mathcal{A}, \bar{a}) \approx^\alpha (\mathcal{B}, \bar{b})$ provided that

(1) for all n, the tree ranks less than $\omega\alpha$ of nodes at level n are the same in $\mathcal{A}$ and $\mathcal{B}$,

(2) the subtree of $\mathcal{A}$ "generated" by $\bar{a}$ (by closing under predecessors) is isomorphic to the subtree of $\mathcal{B}$ generated by $\bar{b}$, with an isomorphism taking $\bar{a}$ to $\bar{b}$,

(3) for corresponding elements x in the subtree of $\mathcal{A}$ generated by $\bar{a}$ and x' in the subtree of $\mathcal{B}$ generated by $\bar{b}$, either the tree ranks of x and x' match or else both are at least $\omega \cdot \alpha$.

Lemma 2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ each be one of our trees T^* or T^a. If $(\mathcal{A}, \bar{a}) \approx^\alpha (\mathcal{B}, \bar{b})$, then $(\mathcal{A}, \bar{a}) \sim^\alpha (\mathcal{B}, \bar{b})$.*

Proof. The statement is clear for $\alpha = 0$. Also, if α is a limit ordinal and the statement holds for all $\beta < \alpha$, then it holds for α. Supposing that it holds for α, we prove it for $\alpha + 1$. For simplicity, suppose that $\bar{a}$ and $\bar{b}$ are subtrees. Let a be an element of $\bar{a}$ that has a new successor c at the top of a finite subtree $\bar{c}$. Let b be the element corresponding to a. We need d and $\bar{d}$ matching c and $\bar{c}$. If the tree ranks of a and b match and are less than $\omega(\alpha + 1)$, then we can choose d and $\bar{d}$ with tree ranks matching the corresponding elements of c and $\bar{c}$. If the tree ranks of a and b are at least $\omega(\alpha + 1)$ and the tree rank of c and the elements of $\bar{c}$ are at least $\omega \cdot \alpha$, then we choose d and $\bar{d}$ also with tree ranks at least $\omega \cdot \alpha$. We can choose d of rank $\omega \cdot \alpha + n$ for n sufficiently large to leave room for choosing the rest of $\bar{d}$. If some of the elements of $\bar{c}$ have tree ranks less than $\omega \cdot \alpha$, then we choose the corresponding elements of $\bar{d}$ with matching tree ranks. $\qquad \square$

Given a, we can effectively find a Scott sentence for T^a. We give the tree rank of the top node, and for each n, we say what are the tree ranks of nodes at level n. Finally, we say that the tree is "rank-homogeneous"; i.e., for all x at level n of tree rank β and all $\gamma < \beta$ such that there is a node of tree rank γ at level $n + 1$, x has infinitely many successors of tree rank γ. This is effective since we have, uniformly in a, a function giving the ranks of the nodes of T^a.

We want a computable copy $(U, <_U, S_U)$ of the Harrison ordering with the successor relation, with a family of trees $(T_u)_{u \in U}$, uniformly computable in u, such that if $pred(u)$ has order type α with notation $a \in P$, then $T_u \cong T^a$, and if $pred(u)$ is not well-ordered, then $T_u \cong T^*$.

Lemma 2.5. *There is a computable structure $\mathcal{A}$ with universe the union of disjoint sets U and V, with an ordering $<$ of type $\omega_1^{CK}(1+\eta)$ and successor relation S on U and with a function Q from V to U such that for each $u \in U$, $Q^{-1}(u)$ is an infinite set, with a tree structure T_u. If $pred(u)$ has order type α with notation $a \in P$, then $T_u \cong T^a$. If $pred(u)$ is not well-ordered, then $T_u \cong T^*$.*

Proof. To prove the lemma, we use Barwise-Kreisel Compactness. Let Γ be a Π_1^1 set of computable infinitary sentences in the language of $\mathcal{A}$ saying that U and V are disjoint, U is linearly ordered by $<_U$, S_U is the successor relation on the ordering, Q maps V onto U such that for each $u \in U$, $Q^{-1}(u)$ is infinite, and S_T is the union

of successor relations putting a tree structure T_u on the set $Q^{-1}(u)$, with further axioms guaranteeing the following:

(1) for each computable ordinal α, the ordering $(U, <_U)$ has an initial segment of type α,

(2) for each computable ordinal α, each $u \in U$ is the left endpoint of an interval of type ω^α,

(3) the ordering $(U, <_U)$ has no infinite hyperarithmetical decreasing sequence,

(4) for each $u \in U$, if $pred(u)$ has order type α, where $a \in P$ is the notation for α, then $T_u \cong T^a$,

(5) for a computable ordinal α, if $u <_U v$, where $pred(u)$ has order type α, then T_u and T_v satisfy the same computable Σ_α sentences.

For a hyperarithmetical set $\Gamma' \subseteq \Gamma$, there is a computable ordinal γ bounding the ordinals α corresponding to sentences in Γ' of types (1), (2), (4), and (5). Then we get a model of Γ' as follows. We fix computable sets U and V in advance. Let c be the notation for γ in P. Since $<_O -pred(c)$ is c.e., we have a computable function f from U $1-1$ onto $<_O -pred(c)$. For $x, y \in U$, $x <_U y$ iff $f(x) <_O f(y)$. Let $S_U(x, y)$ iff $f(y) = 2^{f(x)}$. If $f(x) = a$, then $T_x \cong T^a$. Since every hyperarithmetical subset of Γ has a model, the whole set does. In this model, $(U, <_U)$ has order type $\omega_1^{CK}(1 + \eta)$. For $u \in U$ such that $pred(u)$ is not well-ordered, T_u satisfies the computable infinitary sentences true in T^*. Since T_u is computable, it must be isomorphic to T^*. □

Lemma 2.6. *Let I be the well-ordered initial segment of U of order type ω_1^{CK}. There is a uniformly computable sequence $(R_n)_{n \in \omega}$ of infinite subsets of U with the following properties:*

(1) *R_0 contains some element of I,*

(2) *for each n, there exists $u \in I$ that is an upper bound on $R_n \cap I$,*

(3) *for each $u \in R_n$, $u + 1 \in R_{n+1}$,*

(4) *$\bigcup_n R_n$ is unbounded in the well-ordered initial segment of U.*

Proof. Fix $u_0 \notin I$, $u_1 \in I$. Let R_0 consist of all elements of $u \geq u_0$, plus u_1. Given R_m for $m \leq n$, let R_{n+1} consist of the successors in U of each $u \in R_n$, plus the ω-first element of U not in $\bigcup_{m \leq n} R_m$. □

Let T be the tree of finite sequences $(u_1, \ldots, u_n)$ that are increasing in $(U, <_U)$, with $u_n \in R_n$. We define a function H that takes each non-empty sequence $\sigma \in T$ to its last term u_n. This function is obviously computable. We are about to describe the structure $\mathcal{M}$. The language includes unary predicates A and B, a binary relation S, and for each $\tau \in T$, a binary relation C_τ. In $\mathcal{M}$, we will interpret A and B as disjoint sets. We will think of C_τ as associating to each $x \in A$ a subset $T_{(\tau,x)}$ of B, where for distinct pairs (τ, x) and (τ', x'), the sets $T_{(\tau,x)}$ and $T_{(\tau',x')}$ are disjoint. The relation S will be interpreted as a successor relation that puts a tree structure on each set $T_{(\tau,x)}$.

For the structure $\mathcal{M}$, we let $A^{\mathcal{M}}$ consist of (codes for) the elements of T. For each $\sigma \in A$ and $\tau \in T$, we define $S^{\mathcal{M}}$ so that

$$T_{(\tau,\sigma)} \cong \begin{cases} T_u & \text{if } \tau \preceq \sigma \ \& \ H(\tau) = u, \\ T^* & \text{if } \tau \npreceq \sigma. \end{cases}$$

The structure $\mathcal{M}$ is computable. We note that for $u \in T$, if $pred(u)$ is well-ordered, then T_u is isomorphic to the appropriate T^a, while if $pred(u)$ is not well-ordered, then T_u is isomorphic to T^*.

To show that the computable infinitary theory of $\mathcal{M}$ is not $\aleph_0$-categorical, we produce a second model $\mathcal{N}$, not isomorphic to $\mathcal{M}$. We want a path through T with special features.

Lemma 2.7. *There is a path π through T such that for all n, $\pi(n) \in R_n$ and $ran(\pi)$ is cofinal in I.*

Proof. Let $(u_n)_{n \in \omega}$ be a list of the elements of I. Let $\pi(0) \in R_0 \cap I$. Given $\pi(n)$, take the first k such that $u_k > \pi(n)$. We choose $\pi(n+1)$ to be some $v > \pi(n)$ in $R_{n+1} \cap I$. If possible, we take $v \geq u_k$. Since I is cofinal in $\bigcup_n R_n$, there is some m such that $R_m \cap I$ has an element $v \geq u_k$, and then the same is true for all $m' \geq m$. So, for each k, we will come to m such that we can choose $\pi(m) \geq u_k$. $\square$

Let $\mathcal{N}$ be the extension of $\mathcal{M}$ with an additional element of $A^{\mathcal{N}}$ representing the path π. We define $S^{\mathcal{N}}$ on $T_{(\tau, \pi)}$ so that

$$T_{(\tau,\pi)} \cong \begin{cases} T_u & \text{if } H(\tau) = u \text{ and } \tau \preceq \pi, \\ T^* & \text{otherwise.} \end{cases}$$

Lemma 2.8. $\mathcal{M}$ *and* $\mathcal{N}$ *are not isomorphic.*

Proof. For a fixed $\sigma \in T$, there are only finitely many τ such that $\tau \preceq \sigma$. In $\mathcal{M}$, for a fixed σ, all but finitely many of the trees $T_{(\tau,\sigma)}$ are isomorphic to T^*. On the other hand, in $\mathcal{N}$, there are infinitely many initial segments τ of π with $T_{(\tau,\pi)}$ isomorphic to some $T_u \not\cong T^*$. Thus, no element of $\mathcal{M}$ can be mapped isomorphically to $\pi \in \mathcal{N}$. $\square$

Definition 2.9. We write $\mathcal{A} \preceq_\infty \mathcal{B}$ if for any computable infinitary formula $\varphi(\bar{x})$ and any $\bar{a}$ in $\mathcal{A}$, $\mathcal{A} \models \varphi(\bar{a})$ iff $\mathcal{B} \models \varphi(\bar{a})$.

To show that $\mathcal{N}$ satisfies the computable infinitary theory of $\mathcal{M}$, we show that $\mathcal{M} \preceq_\infty \mathcal{N}$. For this, it is enough to show that for any computable ordinal α and any tuple $\bar{a} \in \mathcal{M}$, $(\mathcal{M}, \bar{a}) \sim^\alpha (\mathcal{N}, \bar{a})$.

Lemma 2.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be structures, each isomorphic to one of $\mathcal{M}$ or $\mathcal{N}$. Let $\bar{a} = (a_1, \ldots, a_n)$ be a tuple in $\mathcal{A}$, and let $\bar{b} = (b_1, \ldots, b_n)$ be a tuple in $\mathcal{B}$ of the same length. Suppose that:*

    (1) *$\bar{a}$ and $\bar{b}$ satisfy the same atomic formulas,*
    (2) *for each a_i and the corresponding b_i in the predicate A (and with u being the element of U with $pred(u)$ having order type α), for each n, if one of $a_i(n)$ or $b_i(n)$ is defined and $\leq_U u$, then $a_i(n) = b_i(n)$, and*
    (3) *for each a_i and corresponding b_i, both in A, and for each $\tau \in T$, we have $(T_{(\tau,a_i)}, \bar{c}) \sim^\alpha (T_{(\tau,b_i)}, \bar{d})$, where $\bar{c}$ consists of the elements from $\bar{a}$ that are in $T_{(\tau,a_i)}$ and $\bar{d}$ consists of the corresponding elements from $\bar{b}$.*

(We assume that for each element a of the tuple $\bar{a}$ that is in the predicate B, the corresponding element a' of the predicate A with $a \in T_{(\tau,a')}$ is also present in the tuple $\bar{a}$. We make a similar assumption about $\bar{b}$.) Then $(\mathcal{A}, \bar{a}) \sim^\alpha (\mathcal{B}, \bar{b})$.

Note that if $\bar{a}$ and $\bar{b}$ both consist solely of elements from the predicate A, then (1) and (2) imply (3).

Proof. We argue by induction on α. Suppose that $\bar{a}$ and $\bar{b}$ satisfy the conditions above for α. Given $\beta < \alpha$ and $\bar{a}'$ a tuple in $\mathcal{A}$, we will find a tuple $\bar{b}'$ in $\mathcal{B}$ such that $\bar{a}, \bar{a}'$ and $\bar{b}, \bar{b}'$ satisfy the conditions above for β. It suffices to assume about $\bar{a}'$ that for each element a of the tuple $\bar{a}'$ that is in the predicate B, the corresponding element a' of the predicate A with $a \in T_{(\tau, a')}$ is present in the tuple $\bar{a}, \bar{a}'$.

First, for each $a_i' \in A$, we choose b_i' such that for all $\tau \in T$, $T_{(\tau, a_i')} \sim^\alpha T_{(\tau, b_i')}$. If $a_i' \in T$, then we choose $b_i' = a_i'$, if possible. However, it may be that a_i' is already in $\bar{b}$. So, instead, let n be the length of a_i', and choose $b_i' = a_i' \hat{\ } \langle v \rangle$ for some sufficiently large $v \in R_n$ with $pred(v)$ ill-founded. For each τ, if $\tau \preceq a_i'$, then $\tau \preceq b_i'$, so that $T_{(\tau, a_i')} \cong T_u \cong T_{(\tau, b_i')}$ for some u. If $\tau \npreceq a_i'$, then either $\tau \npreceq b_i'$ or $\tau = b_i'$ (whence $H(\tau) = v$). Either way, $T_{(\tau, a_i')} \cong T^* \cong T_{(\tau, b_i')}$ by choice of v.

If, instead, $a_i' = \pi$, let $u \in U$ be such that $pred(u)$ is well-founded with order type α. Let σ be the initial segment of π consisting of all of the entries v of π with $v \leq_U u$. Let b_i' be a code for $\sigma \hat{\ } \langle v \rangle$ for some $v \in R_n$ with $pred(v)$ ill-founded, which is sufficiently large that b_i' codes a new element. Then for all $\tau \preceq \sigma$, we have $\tau \preceq \pi$ and so $T_{(\tau, a_i')} \cong T_{(\tau, b_i')}$. For all $\tau \npreceq \sigma$, either $\tau \npreceq \pi$, in which case $T_{(\tau, a_i')} \cong T^* \cong T_{(\tau, b_i')}$, or $\tau \preceq \pi$, in which case $T_{(\tau, a_i')} \cong T_v \sim^\alpha T^* \cong T_{(\tau, b_i')}$ for some $v >_U u$. In this manner, we may reduce to the case where $\bar{a}_i'$ contains only elements of the predicate B.

For each element a from $\bar{a}$ in the predicate A, let b be the corresponding element from $\bar{b}$. Fix $\tau \in T$. Let $\bar{c}$ consist of the elements from $\bar{a}$, and let $\bar{c}'$ consist of the elements from $\bar{a}'$ that are in $T_{(\tau, a)}$. Similarly, let $\bar{d}$ consist of the elements from $\bar{b}$ that are in $T_{(\tau, b)}$. By assumption, $(T_{(\tau, a)}, \bar{c}) \sim^\alpha (T_{(\tau, b)}, \bar{d})$. Thus, there is $\bar{d}'$ such that $(T_{(\tau, a)}, \bar{c}, \bar{c}') \sim^\beta (T_{(\tau, b)}, \bar{d}, \bar{d}')$. The tuple $\bar{b}'$ consists of the elements of the tuples $\bar{d}'$ for each a and τ. $\qquad \square$

Lemma 2.11. *Let* $\mathcal{M} \preceq_\infty \mathcal{N}$.

Proof. Let $\bar{a}$ be a tuple in $\mathcal{M}$. Then by the previous lemma, $(\mathcal{M}, \bar{a}) \sim^\alpha (\mathcal{N}, \bar{a})$. It follows that for any Σ_α formula $\varphi(\bar{x})$, if $\mathcal{M} \models \varphi(\bar{a})$, then $\mathcal{N} \models \varphi(\bar{a})$. This proves the lemma. $\qquad \square$

Lemma 2.12. *$\mathcal{M}$ has Scott rank* ω_1^{CK}.

Proof. First, note that there is an automorphism of $\mathcal{M}$ taking $\sigma_1 \in A$ to $\sigma_2 \in A$ if and only if for each $\tau \in T$, $T_{(\tau, \sigma_1)}$ is isomorphic to $T_{(\tau, \sigma_2)}$. This is the case if and only if for each n, if $pred(\sigma_1(n))$ or $pred(\sigma_2(n))$ is well-founded, then $\sigma_1(n) = \sigma_2(n)$.

Fix $\sigma \in A$. We define the orbit of σ by saying, for the finitely many $\tau \preceq \sigma$ with $H(\tau) = u$ and $pred(u)$ well-founded, that $T_{(\tau, \sigma)}$ is isomorphic to T_u, and for each other τ, that $T_{(\tau, \sigma)} \cong T^*$. We can express the former by a computable formula using the Scott sentences of the T_u. For the latter, note that it suffices to say that $T_{(\tau, \sigma)}$ is isomorphic to T^* for only those τ of length at most $n + 1$, where n is the length of σ. We cannot express this directly by a computable infinitary formula, but there is a computable infinitary formula that is satisfied in $\mathcal{M}$ exactly by such elements of A. Let $\alpha < \omega_1^{CK}$ be large enough that for all elements v of $R_0, \ldots, R_n$ with $pred(v)$ well-founded (recalling that such v are not cofinal in the initial segment of U of order type ω_1^{CK}), $T_v \sim^\alpha T^*$. Then, for τ of length at most $n + 1$, $T_{(\tau, \sigma)}$ is isomorphic to T^* if and only if $T_{(\tau, \sigma)} \sim^\alpha T^*$. This can be expressed by a computable infinitary formula.

The Scott rank of a tuple $\bar{b}$ in $T_{(\tau,\sigma)}$ is not greater than the Scott rank of $\bar{b}$ in $\mathcal{M}$. Therefore, the Scott rank of $\mathcal{M}$ is at least ω_1^{CK}, since there are many τ and σ such that $T_{\tau,\sigma}$ is isomorphic to T^*, which has Scott rank ω_1^{CK}. The Scott rank of $\mathcal{M}$ is at most ω_1^{CK}, since we can define the orbit of any tuple by a computable infinitary formula. For a tuple $\bar{u}, \bar{v}$ in $\mathcal{M}$, where $\bar{u} \in A$ and $\bar{v} \in B$, we can define the orbit as follows: for each element $\sigma \in A$ in the tuple $\bar{u}$, we give a definition as above, and if $\bar{b}$ is the part of the tuple $\bar{v}$ in a particular tree $T_{(\tau,\sigma)}$, then we say what the orbit of $\bar{b}$ is in $T_{(\tau,\sigma)}$. Here we use the fact that each of the trees $T_{(\tau,\sigma)}$ itself has Scott rank at most ω_1^{CK}. □

3. A STRUCTURE OF SCOTT RANK $\omega_1^{CK} + 1$ WITHOUT INDISCERNIBLES

We begin this section by proving the following:

Theorem 3.1. *There is a computable structure of Scott rank $\omega_1^{CK} + 1$ with no indiscernible ordered triple.*

As our structure will be a Fraïssé limit, we begin by recalling some of the definitions and results required. Recall that the *age* of a structure is the class K of finitely generated substructures. If K is the age of a structure, then it satisfies the following two conditions:

Hereditary property (HP): If $\mathcal{A} \in K$ and $\mathcal{B}$ is a finitely generated substructure of $\mathcal{A}$, then $\mathcal{B} \in K$.

Joint embedding property (JEP): If $\mathcal{A}$ and $\mathcal{B}$ are in K, then there is $\mathcal{C}$ in K such that both $\mathcal{A}$ and $\mathcal{B}$ embed in $\mathcal{C}$.

Fraïssé showed that the converse is true: any non-empty countable set K of finitely generated structures satisfying these two conditions is the age of a countable structure. Moreover, suppose that K further satisfies the following condition:

Amalgamation property (AP): For any structures $\mathcal{A}$, $\mathcal{B}_1$, and $\mathcal{B}_2$ in K and embeddings i_1 and i_2 of $\mathcal{A}$ into $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively, there is a structure $\mathcal{C}$ in K and embeddings j_1 and j_2 of $\mathcal{B}_1$ and $\mathcal{B}_2$ into $\mathcal{C}$ so that the images $j_1 \circ i_1(\mathcal{A})$ and $j_2 \circ i_2(\mathcal{A})$ coincide.

In this case, there is a unique homogeneous countable structure whose age is K, where by *homogeneous* we mean that any isomorphism between countable substructures extends to an automorphism of the whole structure. We call this homogeneous structure the Fraïssé limit of the class K.

For a discussion of Fraïssé limits from the point of view of computability, see [3]. In that paper it is shown that if a class K of *finite* structures satisfies the HP, the JEP, and the AP, then there is a computable presentation of the Fraïssé limit. (If the structures are not finite, then we need computable versions of the JEP and the AP.) We note that Henson [7] gave an example of a homogeneous triangle-free graph.

Proof of Theorem 3.1. We define a class K of finite structures with signature consisting of binary relations E and $(C_i)_{i \in \omega}$. We view the relations C_i as "colors" (in the sense of Ramsey theory) with which we color (unordered) pairs of vertices. A finite structure $\mathcal{A}$ will be in K if E is an equivalence relation, the C_i color the unordered pairs of vertices (i.e., xC_iy if and only if yC_ix) with exactly one color per edge, and there are no *monochromatic triangles*; i.e., there are no $i \in \omega$ and $x, y, z \in \mathcal{A}$ with xC_iy, yC_iz, and zC_ix.

Claim 1. K satisfies the hereditary property (HP), amalgamation property (AP), and the joint embedding property (JEP).

Proof of Claim 1. The HP is clear. To see that K has the AP, suppose that $\mathcal{A} \subseteq \mathcal{B}, \mathcal{C}$ are structures in K. We define a structure $\mathcal{D} \in K$ extending $\mathcal{B}$ and $\mathcal{C}$. We can extend the equivalence relation to $\mathcal{D}$. Since there are only finitely many elements of $\mathcal{A}$ and $\mathcal{B}$, only finitely many colors have been used so far. To color edges (x, y), where $x \in \mathcal{B} - \mathcal{A}$ and $y \in \mathcal{C} - \mathcal{A}$, simply choose i that has not colored any edge yet, and color (x, y) with i. This cannot introduce any monochromatic triangles. A similar argument, omitting $\mathcal{A}$, shows that K has the JEP. □

Note that we can effectively list the structures of K. Thus, K has a computable Fraïssé limit $\mathcal{M}$, and $\mathcal{M}$ has, as its Scott sentence, a computable infinitary sentence φ. Models of φ have infinitely many equivalence classes, all of which have infinitely many elements. Now, let $\mathcal{N}$ be an expansion of $\mathcal{M}$ with a linear order $\leq$ of order type $\omega_1^{CK}(1+\eta)$ on the equivalence classes. The structure $\mathcal{N}$ has a computable copy, since we can find an effective labeling of the equivalence classes by elements of ω and use the Harrison ordering. Let $R : \mathcal{A} \to \omega_1^{CK}(1+\eta)$ be the resulting effective order-preserving map, which respects equivalence classes and which induces a bijection between the equivalence classes and elements of $\omega_1^{CK}(1+\eta)$.

Claim 2. For $\alpha \geq 1$ and tuples $\bar{x}, \bar{y}$ in $\mathcal{N}$, $\bar{x} \sim^\alpha \bar{y}$ in $\mathcal{N}$ if and only if $R(\bar{x}) \sim^\alpha R(\bar{y})$ in $\omega_1^{CK}(1+\eta)$ and $\bar{x} \equiv_{at} \bar{y}$, where $\bar{x} \equiv_{at} \bar{y}$ if $\bar{x}$ and $\bar{y}$ satisfy the same atomic formulas in $\mathcal{N}$.

Proof of Claim 2. Suppose that $R(\bar{x}) \sim^\alpha R(\bar{y})$ and $\bar{x} \equiv_{at} \bar{y}$. Then we will show that $\bar{x} \sim^\alpha \bar{y}$. Take $\beta < \alpha$ and $\bar{x}'$ a new tuple of elements. Then there is $\bar{a}$ in $\omega_1^{CK}(1+\eta)$ such that $R(\bar{x})R(\bar{x}') \sim^\beta R(\bar{y})\bar{a}$. Choose $\bar{z}$ such that $R(\bar{z}) = \bar{a}$. Let $\bar{y}'$ be a tuple of new symbols of the same length as $\bar{z}$. Consider the finite structure, in the signature of K, defined on the elements $\bar{y}, \bar{y}', \bar{z}$ as follows. The relations E and C_i are defined on $\bar{y}$ and $\bar{z}$ as in $\mathcal{N}$. We set $y_i' E z_i$, and the equivalence classes are completely determined by this. Define $y_i C_k y_j'$ if and only if $x_i C_k x_j'$ (and $y_i' C_k y_j'$ if and only if $x_i' C_k x_j'$). There are no monochromatic triangles among $\bar{y}, \bar{y}'$, or among $\bar{y}, \bar{z}$. Since we have only used finitely many colors so far, we can color the remaining pairs so that there are no monochromatic triangles.

The finite structure we have defined is in the class K, so we can find a realization of $\bar{y}'$ in $\mathcal{N}$. Then $R(\bar{y}') = \bar{a}$, so that $R(\bar{x}), R(\bar{x}') \sim^\beta R(\bar{y}), R(\bar{y}')$. Also, $\bar{x}, \bar{x}' \equiv_{at} \bar{y}, \bar{y}'$. Thus, $\bar{x}, \bar{x}' \sim^\beta \bar{y}, \bar{y}'$ by the inductive hypothesis (or, for $\beta = 0$, because $\bar{x}, \bar{x}' \sim^\beta \bar{y}, \bar{y}'$). So, we have shown that $\bar{x} \sim^\alpha \bar{y}$. On the other hand, if $\bar{x} \not\equiv_{at} \bar{y}$, then it is immediate that $\bar{x} \not\sim^\alpha \bar{y}$. If $R(\bar{x}) \not\sim^\alpha R(\bar{y})$, then it is not hard to see that $\bar{x} \not\sim^\alpha \bar{y}$. □

Claim 3. Let $SR(\mathcal{N}) = \omega_1^{CK} + 1$.

Proof of Claim 3. Let $x \in \mathcal{N}$ be such that $R(x)$ is in the ill-founded part of $\omega_1^{CK}(1+\eta)$. We claim that $SR(x) = \omega_1^{CK}$. Fix $\alpha < \omega_1^{CK}$. Let y be such that $pred(R(y))$ is well-founded and $R(x) \sim^\alpha R(y)$. Now, there is no automorphism of the Harrison ordering taking $R(y)$ to $R(x)$, so there is no automorphism of $\mathcal{N}$ taking y to x. Thus, x and y are in different automorphism orbits. Since x and y are singletons, $x \equiv_{at} y$. Thus, by the previous claim, $x \sim^\alpha y$. Since α was arbitrary, $SR(x) = \omega_1^{CK}$, completing the proof of the claim. □

Claim 4. $\mathcal{N}$ has no indiscernible ordered triple.

Proof of Claim 4. It suffices to show that no three singleton elements are order indiscernible. Given x, y, and z, let i be such that xC_iy. Since $\mathcal{N}$ has no monochromatic triangles, it cannot be the case that yC_iz and xC_iz. Thus x, y, and z are not indiscernible. $\qquad\square$

This completes the proof of Theorem 3.1. $\qquad\square$

Note that this construction is, in some sense, cheating. The structure $\mathcal{N}$ is effectively bi-interpretable with the Harrison ordering (see [6]): the Harrison ordering lives inside $\mathcal{N}$ as the definable quotient modulo the definable equivalence relation E. The indiscernible sequence of the Harrison ordering becomes an indiscernible sequence of imaginaries in $\mathcal{N}$.

Definition 3.2. Fix a structure $\mathcal{A}$. An *indiscernible sequence of imaginaries of* $\mathcal{A}$ is a sequence $(E_i)_{i\in\omega}$ of equivalence classes of $\mathcal{A}$, modulo some $L_{\omega_1\omega}$-definable equivalence relation, such that for any two finite subsequences $E_{i_1},\ldots,E_{i_n}$ and $E_{j_1},\ldots,E_{j_n}$ (with $i_1 < i_2 < \cdots < i_n$ and $j_1 < j_2 < \cdots < j_n$) there is an automorphism of $\mathcal{A}$ mapping E_{i_k} to E_{j_k}.

Proposition 3.3. *Let $\mathcal{N}$ be the structure from Theorem* 3.1. *Then $\mathcal{N}$ has an indiscernible sequence of imaginaries.*

Proof. The map R defined just before Claim 2 above induced a bijection between the E-equivalence classes and the elements of $\omega_1^{CK}(1+\eta)$. We claim that each automorphism of $\omega_1^{CK}(1+\eta)$ induces an automorphism of $\mathcal{N}$. Then, since $\omega_1^{CK}(1+\eta)$ has an indiscernible sequence, $\mathcal{N}$ will have an indiscernible sequence of imaginaries, namely the E-equivalence classes in bijection to the indiscernible sequence of $\omega_1^{CK}(1+\eta)$.

It suffices to see that in the Fraïssé limit $\mathcal{M}$, any permutation π of the E-equivalence classes extends to an automorphism of $\mathcal{M}$. Let $\bar{a}$ and $\bar{b}$ be tuples of elements of $\mathcal{M}$ of the same length satisfying the same atomic formulas, and such that if a_i is in the jth equivalence class, then b_i is in the $\pi(j)$th equivalence class. Let c be an additional element of $\mathcal{M}$. We can find an element d of $\mathcal{M}$ such that d is in the $\pi(j)$th equivalence class (if c was in the jth equivalence class) and such that $\bar{b}, d$ is colored in the same way as $\bar{a}, c$. We can do this since $\bar{a}, c$ has no monochromatic triangles. This lets us construct the desired automorphism using a back-and-forth construction. $\qquad\square$

A construction similar to that of Theorem 3.1 allows us to turn any structure $\mathcal{M}$ into a structure $\mathcal{M}^*$ that is effectively bi-interpretable with $\mathcal{M}$, but has no indiscernible triples. Two structures which are effectively bi-interpretable have many of the same computability-theoretic properties; for example, they have the same computable dimension (see [8]). In light of this, we want not just a structure $\mathcal{M}$ with Scott rank $\omega_1^{CK} + 1$ and no indiscernible sequence, but a structure $\mathcal{M}$ with Scott rank $\omega_1^{CK} + 1$ and no indiscernible sequence of imaginaries. To produce such a structure, we use a construction originally due to Makkai [11] and refined by Knight and Millar [10] (see also [9]). Makkai used this construction to produce an arithmetical structure of Scott rank ω_1^{CK}, and Knight and Millar used it to give the first example of a computable structure of Scott rank ω_1^{CK}.

Let $T \subseteq \omega^{<\omega}$ be a tree. We will define a new structure $\mathcal{A}(T)$. Let T_n be the set of nodes at the nth level of T. For each n, we define $G_n = \mathcal{P}_{<\omega}(T_n)$ to be the collection of finite subsets of T_n. Now, G_n forms an abelian group under symmetric difference Δ. The identity element of G_n is the empty set, which we denote by id_n. Let $G = \bigcup_n G_n$. The tree structure on T induces a tree structure on G, which we will define using a predecessor relation p. Given $a \in G_{n+1}$, write $a = \{t_1, \ldots, t_n\}$. Then set $p(a)$ to be the sum of the predecessors of $t_1, \ldots, t_n$. An element t^* is in $p(a)$ if and only if the number of successors of t^* in a is odd. We have $p(\mathrm{id}_{n+1}) = \mathrm{id}_n$. Note that p is a homomorphism from G_{n+1} to G_n. For each $a \in G$, we will define a unary function f_a. If $a \in G_m$ and $b \in G_n$, let $k = \min(m, n)$. Let a^* and b^* be the p-predecessors of a and b that are in G_k: $a^* = p^{m-k}(a)$ and $b^* = p^{n-k}(b)$. Then set $f_a(b) = a^* \Delta b^*$, noting that $f_a(b) = f_b(a)$. Let $\mathcal{A}(T)$ be the structure $(T, (f_a)_{a \in G})$.

We note the following facts from [10]. Since $a \in G_n$ if and only if $f_{\mathrm{id}_n}(a) = a$ and for all $m < n$, $f_{\mathrm{id}_m}(a) \neq a$, G_n is preserved under automorphisms of $\mathcal{A}(T)$. Also, p is preserved under automorphisms as for $a \in G_{n+1}$ and $b \in G_n$, $p(a) = b$ if and only if $f_{\mathrm{id}_n}(a) = b$. Finally, for any $a \in G_n$, $a = f_a(\mathrm{id}_n)$, and so any automorphism of $\mathcal{A}(T)$ is determined by the images of the elements id_n.

Lemma 3.4 (Lemma 3.3 of [10]). *Let $a \in G_n$, with $a \neq \mathrm{id}_n$. Then the tree rank of a is the minimum of the tree ranks of t for $t \in a$.*

Lemma 3.5 (Lemma 3.6 of [10]). *For $a \in G_n$, $a \equiv^\beta \mathrm{id}_n$ if and only if the tree rank of a is at least $\omega \cdot \beta$.*

Theorem 3.6 (Theorem 3.7 and Lemma 4.3 of [10]). *There is a computable tree T such that $SR(\mathcal{A}(T)) = \omega_1^{CK}$.*

Lemma 3.7. *Let T be a tree. Then $\mathcal{A}(T)$ does not have an indiscernible ordered triple of imaginaries.*

Proof. Suppose to the contrary that there is a definable equivalence relation with three equivalence classes E_1, E_2, and E_3 that form an indiscernible triple. Fix $a \in E_1$. Let n be such that $a \in G_n$. There are automorphisms of $\mathcal{A}(T)$, one taking E_1 to E_2, and one taking E_1 to E_3. So, E_2 and E_3 both contain elements of G_n. Let g be an automorphism of $\mathcal{A}(T)$ fixing E_1. Then for each $b \in G_n$, $b = f_{b\Delta a}(a)$, and so $g(b) = f_{b\Delta a}(g(a))$. Hence, the action of g on G_n is entirely determined by where g sends a. Thus, there cannot be two automorphisms of $\mathcal{A}(T)$, one fixing E_1, E_2, and E_3 and the other fixing E_1 and mapping E_2 to E_3. This contradicts the order-indiscerniblity of E_1, E_2, and E_3. $\square$

Theorem 3.8. *There is a computable tree T such that $\mathcal{A}(T)$ has Scott rank $\omega_1^{CK}+1$.*

Proof. Let T be the tree of finite decreasing sequences in the Harrison ordering. We claim that $\mathcal{A}(T)$ has Scott rank $\omega_1^{CK} + 1$. Note that at each level of T, there are elements of every computable tree rank, and there are elements with infinite tree rank. Let G be the tree defined from T as above. Then by Lemma 3.4, at each level of G, there are elements of every computable tree rank, and there are elements with infinite tree rank. Fix n. Given $\beta < \omega_1^{CK}$, there is $a \in G_n$ with computable tree rank at least $\omega \cdot \beta$. By Lemma 3.5, $a \equiv^\beta \mathrm{id}_n$, but $a \not\equiv^\gamma \mathrm{id}_n$ for some $\gamma > \beta$. Hence $SR(\mathrm{id}_n) = \omega_1^{CK}$. It follows that $SR(\mathcal{A}(T)) = \omega_1^{CK} + 1$. $\square$

Corollary 3.9. *There are computable structures M of Scott rank ω_1^{CK} and of $\omega_1^{CK} + 1$ that have no indiscernible ordered triples of imaginaries.*

Proof. This follows from Lemma 3.7 and Theorems 3.6 and 3.8. □

We end with an open question.

Question. Is there a structure of Scott rank ω_1^{CK} that is computably approximable and has no indiscernible sequences of imaginaries?

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