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ON BRANCHES OF POSITIVE SOLUTIONS FOR p-LAPLACIAN PROBLEMS AT THE EXTREME VALUE OF THE NEHARI MANIFOLD METHOD

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Abstract. This paper is concerned with variational continuation of branches of solutions for nonlinear boundary value problems, which involve the p-Laplacian, an indefinite nonlinearity, and depend on a real parameter λ. A special focus is given to the extreme value λ^* of the Nehari manifold that determines the threshold of applicability of the Nehari manifold method. In the main result the existence of two branches of positive solutions for the cases where the parameter λ lies above the threshold λ^* is obtained.

1. Introduction

We study the following p-Laplacian problem with indefinite nonlinearity:

$$\begin{cases} -\Delta_p u = \lambda|u|^{p-2}u + f|u|^{\gamma-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here Ω denotes a bounded domain in $\mathbb{R}^N$ with C^1-boundary $\partial\Omega$, λ is a real parameter, $1 < p < \gamma < p^*$, where p^* is the critical Sobolev exponent, $f \in L^d(\Omega)$ where $d \geq p^*/(p^*-\gamma)$ if $p < N$, and $d > 1$ if $p \geq N$. We suppose that (1.1) has an indefinite nonlinearity, i.e., f changes sign in Ω. By a solution of (1.1) we mean a critical point $u \in W := W_0^{1,p}(\Omega)$ of the energy functional

$$\Phi_\lambda(u) = \frac{1}{p}\int |\nabla u|^p dx - \frac{\lambda}{p}\int |u|^p dx - \frac{1}{\gamma}\int f|u|^\gamma dx,$$

where $W_0^{1,p}(\Omega)$ is the standard Sobolev space.

The problems with the indefinite nonliearity of type (1.1) have been intesively studied; see, e.g., Alama & Tarantello [1], Berestycki, Capuzzo–Dolcetta & Nirenberg [3], and Ouyang [14]. One of the fruitful approaches in the study of such problems is the Nehari manifold method [13] where solutions are obtained through the constrained minimization problem

$$\min\{\Phi_\lambda(u) : \ u \in \mathcal{N}_\lambda\} \tag{1.2}$$

with the Nehari manifold $\mathcal{N}_\lambda := \{u \in W \setminus 0 : \ D_u\Phi_\lambda(u)(u) = 0\}$ (see, e.g., Drábek & Pohozaev [7], Il'yasov [8,10], and Ouyang [14]).

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The applicability of the Nehari manifold method to (1.1) depends on the parameter λ. Indeed, (1.1) possesses the so-called *extreme value of the Nehari manifold method* [10]

$$\lambda^* = \inf\left\{\frac{\int |\nabla u|^p dx}{\int |u|^p dx} : \int f|u|^\gamma dx \geq 0,\ u \in W \setminus 0\right\}, \tag{1.3}$$

which was known to be the first found by Ouyang [14]. A feature of λ^* is that it defines a threshold for the applicability of the Nehari manifold method so that for any $\lambda < \lambda^*$ the set $\mathcal{N}_\lambda$ is a C^1-manifold of codimension 1 in W wherein for any $\lambda \geq \lambda^*$ there is $u \in \mathcal{N}_\lambda$ such that $\Phi''_\lambda(u) := D^2_{uu}\Phi_\lambda(u)(u,u) = 0$. Moreover, Φ_λ is unbounded from below over $\mathcal{N}_\lambda$ if $\lambda \geq \lambda^*$ (see, e.g., [10]). It is remarkable that once the extreme value (1.3) is detected, one is able to directly find solutions for (1.1) as $\lambda < \lambda^*$, by means of the Nehari minimization problems (1.2) (see, e.g., [14] for $p = 2$ and [7,8] for $1 < p < +\infty$).

A natural question which arises from this is whether there are any positive solutions of (1.1) for $\lambda > \lambda^*$. An answer for this question, in the case $p = 2$, follows from the works of Alama & Tarantello [1] and Ouyang [14], where the authors proved that (1.1) possesses a branch of minimal positive solution for λ belonging to the whole interval $(-\infty, \Lambda]$, for some $\Lambda > \lambda^*$, and does not admit any positive solutions for $\lambda > \Lambda$. However, the approach used in [1, 14] is based on the application of the local continuation method [4], which essentially involves an analysis of the corresponding linearized problems.

The main aim of the present paper is to give a contribution in the investigation of the branches of solutions for the problems where the application of local continuation methods can cause difficulty. Our approach is based on the development of the Nehari manifold method where we focus also on obtaining a new knowledge on the extreme value of the Nehari manifold method.

Let us state our main results. Denote

$$\Omega^+ = \{x \in \Omega :\ f^+(x) \neq 0\},\ \Omega^- = \{x \in \Omega :\ f^-(x) \neq 0\},$$

and $\Omega^0 = \Omega \setminus (\overline{\Omega^+} \cup \overline{\Omega^-})$. We write $U \neq \emptyset$ if the interior $\mathrm{int}(U)$ of a set $U \subset \mathbb{R}^n$ is nonempty. We denote $(\lambda_1(\mathrm{int}(U)), \phi_1(\mathrm{int}(U)))$ the first eigenpair of $-\Delta_p$ on $\mathrm{int}(U)$ with zero Dirichlet boundary conditions. It is known that $\lambda_1(\mathrm{int}(U))$ is positive, simple and isolated, and $\phi_1(\mathrm{int}(U))$ is positive [12]. To simplify notations, we write $\lambda_1 := \lambda_1(\Omega)$, $\phi_1 := \phi_1(\Omega)$.

Throughout the paper, we assume that $\Omega^+ \neq \emptyset$. Furthermore, we shall need the following assumption:

(f_1): If $\Omega^0 \neq \emptyset$, then $\lambda_1(\mathrm{int}(\Omega^0 \cup \Omega^+)) < \lambda_1(\mathrm{int}(\Omega^0))$.

Notice that if $\Omega^0 \cup \Omega^+ \neq \emptyset$, then $\lambda^* < +\infty$ and for sufficiently large $\overline{\lambda} > \lambda_1$, problem (1.1) has no positive solutions for any $\lambda > \overline{\lambda}$ (see, e.g., [9]).

Our main result is the following

Theorem 1.1. *Let $1 < p < \gamma < p^*$ and suppose that $\Omega^+ \neq \emptyset$, $F(\phi_1) := \int f|\phi_1|^\gamma dx < 0$ and (f_1) is satisfied. Then there exists $\Lambda \in (\lambda^*, \overline{\lambda}]$ such that for all $\lambda \in (\lambda^*, \Lambda)$ problem* (1.1) *admits at least two positive weak solutions $u_\lambda, \overline{u}_\lambda$. Moreover,*

- **(i):** $\Phi''_\lambda(u_\lambda) > 0, \Phi''_\lambda(\overline{u}_\lambda) > 0$ *and* $\Phi_\lambda(u_\lambda) < \Phi_\lambda(\overline{u}_\lambda) < 0$ *for any* $\lambda \in (\lambda^*, \Lambda)$*;*
- **(ii):** $\Phi_\lambda(u_\lambda) \uparrow \Phi_{\lambda^*}(u_{\lambda^*})$ *as* $\lambda \downarrow \lambda^*$*.*

This paper is organized as follows. Section 2 contains preliminaries. In Section 3, we show the existence of solutions u_λ. In Section 4, we show the existence of solutions $\overline{u}_\lambda$ and conclude the proof of Theorem 1.1. In the Appendix, we provide some technical and auxiliary results.

2. Preliminaries

In what follows, the norm in W we will denote by $\|\cdot\|$. Write

$$H_\lambda(u) = \int |\nabla u|^p\,dx - \lambda \int |u|^p\,dx,\ \ F(u) = \int f(x)|u|^\gamma,\ \ u \in W.$$

Then

$$\Phi_\lambda(u) = \frac{1}{p}H_\lambda(u) - \frac{1}{\gamma}F(u),\ \mathcal{N}_\lambda = \{u \in W \setminus 0 :\ H_\lambda(u) - F(u) = 0\}.$$

To our aims, it is sufficient to use the following Nehari submanifold:

$$\mathcal{N}_\lambda^+ := \{u \in \mathcal{N}_\lambda :\ D_{uu}\Phi_\lambda(u)(u,u) > 0\}$$

which we shall use in the fibering representation [8, 15]:

$$\mathcal{N}_\lambda^+ = \{u = sv :\ s = s_\lambda^+(v),\ v \in \Theta_\lambda^+\},$$

where $\Theta_\lambda^+ = \{v \in W \setminus 0 :\ H_\lambda(v) < 0,\ F(v) < 0\}$ and

$$s_\lambda^+(v) = \left(\frac{H_\lambda(v)}{F(v)}\right)^{1/(\gamma-p)}. \tag{2.1}$$

Thus we are able to introduce

$$J_\lambda^+(v) =: \Phi_\lambda(s_\lambda^+(v)v) = -c_{p,\gamma}\frac{|H_\lambda(v)|^{\gamma/(\gamma-p)}}{|F(v)|^{p/(\gamma-p)}},\ v \in \Theta_\lambda^+, \tag{2.2}$$

where $c_{p,\gamma} = (\gamma - p)/p\gamma$.

Observe, J_λ^+ is the 0-homogeneous functional on $\mathcal{N}_\lambda^+$, i.e., $J_\lambda^+(su) = J_\lambda^+(u)$ for any $s > 0$, $u \in \mathcal{N}_\lambda^+$. It is worth pointing out that (1.3) implies

$$\Theta_\lambda^+ = \{v \in W \setminus 0 :\ H_\lambda(v) < 0\}$$

for any $\lambda \in (\lambda_1, \lambda^*)$. In what follows, we write $\partial\Theta_\lambda^+ = \{v \in W \setminus 0 :\ H_\lambda(v) = 0\}$ and the closure of Θ_λ^+ we denote by $\overline{\Theta}_\lambda^+$.

It is not hard to prove (see, e.g., [8])

Proposition 2.1. *If $D_v J_\lambda^+(v)(\eta) = 0$ for any $\eta \in W \setminus 0$, then $s_\lambda^+(v)v$ weakly satisfies* (1.1)*.*

Consider the following minimization problem:

$$\hat{J}_\lambda^+ := \min\{J_\lambda^+(v) |\ v \in \Theta_\lambda^+\}. \tag{2.3}$$

Lemma 2.2. *Suppose the assumptions of Theorem* 1.1 *are satisfied. Then*

- **(a):** $\lambda_1 < \lambda^* < \infty$.
- **(b):** *There exists a minimizer ϕ_1^* of the problem* (1.3) *such that $\phi_1^* > 0$ and $\phi_1^* \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. Moreover, any minimizer ϕ_1^* of* (1.3) *weakly satisfies, up to a scalar multiplier, to* (1.1) *for $\lambda = \lambda^*$ and $H_{\lambda^*}(\phi_1^*) = F(\phi_1^*) = 0$.*
- **(c):** *$\hat{J}_{\lambda^*}^+ > -\infty$ and there exists a minimizer $v_{\lambda^*} \in \Theta_{\lambda^*}^+$ of $J_{\lambda^*}^+(v_{\lambda^*})$ so that $u_{\lambda^*} := s_2(v_{\lambda^*})v_{\lambda^*}$ satisfies* (1.1) *and $u_{\lambda^*} > 0$.*

Proof. The proof of **(a)** can be found in [8,9]. Furthermore, by [8], there exists a nonzero minimizer ϕ_1^* of (1.3) such that $\phi_1^* \geq 0$. Hence by the Lagrange multiplier rule there exist $\mu_0, \mu_1 \geq 0$, $|\mu_0| + |\mu_1| \neq 0$ such that

$$\mu_0 D_v H_{\lambda^*}(\phi_1^*) = \mu_1 D_v F(\phi_1^*). \tag{2.4}$$

Since ϕ_1^* is a minimizer of (1.3), then $H_{\lambda^*}(\phi_1^*) = 0$ and therefore $\mu_1 F(\phi_1^*) = 0$.

Suppose $\mu_0 = 0$; then $f|\phi_1^*|^{\gamma-2}\phi_1^* = 0$ a.e. in Ω. This is possible only if supp $\phi_1^* \subset \Omega^0$. Thus if $\Omega^0 = \emptyset$, then we get a contradiction. Assume that $\Omega^0 \neq \emptyset$. Then there exist eigenpairs $(\lambda_1(\mathrm{int}(\Omega^0)), \phi_1(\mathrm{int}(\Omega^0)))$ and $(\lambda_1(\mathrm{int}(\Omega^0 \cup \Omega^+)), \phi_1(\mathrm{int}(\Omega^0 \cup \Omega^+)))$. Since $\phi_1(\mathrm{int}(\Omega^0)) \in W_0^{1,p}(\mathrm{int}(\Omega^0))$ and $H_{\lambda^*}(\phi_1^*) = 0$, $\lambda_1(\mathrm{int}(\Omega^0)) \leq \lambda^*$. On the other hand, the assumption (f_1) entails the strong inequality $\lambda_1(\mathrm{int}(\Omega^0 \cup \Omega^+)) < \lambda_1(\mathrm{int}(\Omega^0))$. Hence we get a contradiction because $\lambda^* \leq \bar{\lambda} \leq \lambda_1(\mathrm{int}(\Omega^0 \cup \Omega^+))$ (see [9]). Thus $\mu_0 \neq 0$.

Suppose $\mu_1 = 0$; then $D_u H_{\lambda^*}(\phi_1^*) = 0$. By the Harnack inequality (see [17]) we have $\phi_1^* > 0$ in Ω. But this is possible only if $\lambda^* = \lambda_1$, $\phi_1^* = \phi_1$ (see [12]). However, by (1.3), $F(\phi_1^*) \geq 0$ which contradicts the assumption $F(\phi_1) < 0$. Hence $\mu_1 > 0$ and therefore $F(\phi_1^*) = 0$ and there exists $t(\mu) > 0$ such that $t(\mu)\phi_1^*$ satisfies (1.1). The Harnack inequality and regularity of solutions for the p-Laplacian equation yields that $\phi_1^* > 0$ and $\phi_1^* \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. Thus we have proved **(b)**.

Let us prove **(c)**. By [8] there is a limit

$$\hat{J}_\lambda^+ \to \bar{J}^+(\lambda^*) \geq -\infty \text{ as } \lambda \uparrow \lambda^*, \tag{2.5}$$

and there exists a weak positive solution u_{λ^*} of (1.1) such that $\bar{J}^+(\lambda^*) = J_{\lambda^*}^+(u_{\lambda^*})$. It is clear that $\hat{J}_{\lambda^*}^+ \leq \bar{J}^+(\lambda^*)$. Thus, we will obtain the proof if we show that $\hat{J}_{\lambda^*}^+ = \bar{J}^+(\lambda^*)$. Suppose, contrary to our claim, that $\hat{J}_{\lambda^*}^+ < \bar{J}^+(\lambda^*)$. We prove that this is impossible if $\hat{J}_{\lambda^*}^+ = -\infty$. The proof in the other case is similar.

Since $\hat{J}_{\lambda^*}^+ = -\infty$, for every $K > 0$, one can find $v_K \in \Theta_{\lambda^*}^+$ such that $J_{\lambda^*}^+(v_K) < \bar{J}^+(\lambda^*) - K$. Since $J_\lambda^+(v_K) \to J_{\lambda^*}^+(v_K)$, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|J_\lambda^+(v_K) - J_{\lambda^*}^+(v_K)| < \varepsilon$ as $|\lambda - \lambda^*| < \delta$. In view of (2.5), we may assume that there holds also $|\hat{J}_\lambda^+ - \bar{J}^+(\lambda^*)| < \varepsilon$ if $|\lambda - \lambda^*| < \delta$. Then

$$\bar{J}^+(\lambda^*) - \varepsilon < \hat{J}_\lambda^+ \leq J_\lambda^+(v_K) < J_{\lambda^*}^+(v_K) + \varepsilon < \bar{J}^+(\lambda^*) - K + \varepsilon.$$

Since $K > 0$, $\varepsilon > 0$ may be chosen arbitrarily, we get a contradiction. □

We need also

Corollary 2.3. *There exists $\mu_0 \in (\lambda_1, \lambda^*)$ such that any minimizer w_{λ^*} of (2.3) for $\lambda = \lambda^*$ satisfies $H_{\mu_0}(w_{\lambda^*}) < 0$.*

Proof. Suppose the assertion of the corollary is false. Then there exists a sequence $w_n \in \Theta_{\lambda^*}^+$ such that $\hat{J}_{\lambda^*}^+ = J_{\lambda^*}^+(w_n)$ and $H_{\lambda^*}(w_n) \to 0$ as $n \to \infty$. By homogeneity of $J_{\lambda^*}^+(v)$ we may assume that $||w_n|| = 1$, $n = 1, 2, ...$ Hence by reflexivity of W and the embedding theorems there exists a subsequence, which we still denote by (w_n), such that $w_n \rightharpoonup w$ weakly in W and $w_n \to w$ strongly in $L^q(\Omega)$, $1 < q < p^*$ for some $w \in W$. Since $H_{\lambda^*}(w_n) < 0$, $n = 1, 2, ...$, it follows that $w \neq 0$. Observe

$$J_{\lambda^*}^+(w_n) = -c_{p,\gamma}\frac{|H_{\lambda^*}(w_n)|^{\gamma/(\gamma-p)}}{|F(w_n)|^{p/(\gamma-p)}} = -c_{p,\gamma} s_{\lambda^*}^+(w_n)^p |H_{\lambda^*}(w_n)|.$$

From this and since $J^+_{\lambda^*}(w_n) = \hat{J}^+_{\lambda^*} < 0$, it follows that $s^+_{\lambda^*}(w_n) \to \infty$ and $F(w_n) \to 0$. Hence $F(w) = 0$ and therefore $H_{\lambda^*}(w) = 0$ which implies by **(b)**, Lemma 2.2 that $w = \phi_1^* > 0$. Note that

$$-\Delta_p w_n - \lambda^* |w_n|^{p-2} w_n - s^+_{\lambda^*}(w_n)^{\gamma-p} f |w_n|^{\gamma-2} w_n = 0, \ n = 1, 2, \dots .$$

Thus $s^+_{\lambda^*}(w_n) \to \infty$ implies $f(\phi_1^*)^{\gamma-1} = 0$ a.e. in Ω, which is absurd. □

Corollary 2.4. *For each $\mu \in (\lambda_1, \lambda^*)$, there is $c_\mu < 0$ such that $F(v) \le c_\mu$ $\forall v \in \overline{\Theta}^+_\mu \cap S^1$, where $S^1 := \{u \in W : \ \|u\| = 1\}$.*

Proof. Let $\mu < \lambda^*$ and assume contrary to our claim that there exists a sequence $v_n \in \overline{\Theta}^+_\mu \cap S^1$ such that $F(v_n) \to 0$ as $n \to \infty$. Then by reflexivity of W and the embedding theorems we may assume that $v_n \rightharpoonup v$ weakly in W and $v_n \to v$ strongly in $L^q(\Omega)$ for $1 < q < p^*$. Since $H_\mu(v_n) \le 0$, $n = 1, 2, \dots$, we conclude $v \ne 0$. Thus, by the weakly lower-semicontinuity of $\int |\nabla v|^p dx$ we conclude that

$$H_{\lambda^*}(v) \le \liminf_{n\to\infty} H_{\lambda^*}(v_n) = \liminf_{n\to\infty} (H_\mu(v_n) + (\mu - \lambda^*) \int |v_n|^p \, dx) < 0.$$

But this contradicts the definition of λ^* since $F(v) = 0$. □

3. Local minima solution

In this section, we show the existence of local minima type solutions u_λ for (1.1). Let us consider the following family of constrained minimization problems:

$$\hat{J}^+_\lambda(\mu) = \inf\{J^+_\lambda(v) : \ v \in \Theta^+_\mu\} \tag{3.1}$$

parametrized by $\lambda \ge \lambda^*$ and $\mu \in (\lambda_1, \lambda^*)$.

Proposition 3.1. *For each $\lambda \ge \lambda^*$ and $\mu \in (\lambda_1, \lambda^*)$ there holds*

- **(a):** *$\hat{J}^+_\lambda(\mu) > -\infty$;*
- **(b):** *there exists a minimizer $v_\lambda(\mu)$ of (3.1).*

Proof. **(a)** follows immediately from Corollary 2.4. Let us prove **(b)**. Take a minimizing sequence $v_n \in \Theta^+_\mu$ of (3.1), that is, $J^+_\lambda(v_n) \to \hat{J}^+_\lambda(\mu) > -\infty$ as $n \to \infty$. Arguing as in the proof of Corollary 2.3 one may infer that there exist $v \in W \setminus 0$ and a subsequence, which we still denote by (v_n), such that $v_n \rightharpoonup v$ weakly in W and $v_n \to v$ strongly in $L^q(\Omega)$ for $1 < q < p^*$. Hence, $H_\mu(v) \le \liminf_{n\to\infty} H_\mu(v_n) \le 0$, $F(v) = \lim_{n\to\infty} F(v_n) < 0$ and therefore $v \in \overline{\Theta}^+_\mu$, $J^+_\lambda(v) \le \liminf_{n\to\infty} J^+_\lambda(v_n) = \hat{J}^+_\lambda(\mu)$. In view of (3.1), this is possible only if $J^+_\lambda(v) = \hat{J}^+_\lambda(\mu)$, that is, v is a minimizer of (3.1). □

We denote the set of minimizers for (3.1) by $\mathcal{S}_\lambda(\mu) = \{v \in \overline{\Theta}^+_\mu : \ J^+_\lambda(v) = \hat{J}^+_\lambda(\mu)\}$ and let $\mathcal{S}^\partial_\lambda(\mu) = \{v \in \mathcal{S}_\lambda(\mu) : \ H_\mu(v) = 0\}$.

Lemma 3.2. *Let $\lambda_0 \ge \lambda^*$ and $\mu \in (\lambda_1, \lambda^*)$ be such that $\mathcal{S}^\partial_{\lambda_0}(\mu) = \emptyset$. Then there exists $\varepsilon > 0$ such that $\mathcal{S}^\partial_\lambda(\mu) = \emptyset$ for each $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$.*

Proof. Suppose the lemma is false. Then we can find a sequence $\lambda_n \to \lambda_0$ such that $v_n := v^+_{\lambda_n}(\mu) \in \partial\Theta^+_\mu$, $n = 1, 2, \dots$. Arguing as above one may infer that there exist $v \in W \setminus 0$ and a subsequence, which we still denote by (v_n), such that $v_n \rightharpoonup v$ weakly in W and $v_n \to v$ strongly in $L^q(\Omega)$ for $1 < q < p^*$. This implies

$H_\mu(v) \le \liminf_{n\to\infty} H_\mu(v_n) = 0,\ F(v) = \lim_{n\to\infty} F(v_n) < 0$ and therefore $v \in \overline{\Theta}^+_\mu$. Furthermore,

$$J^+_{\lambda_0}(v) \le \liminf_{n\to\infty} J^+_{\lambda_n}(v_n) =: \tilde{J} < +\infty. \tag{3.2}$$

From the Poincaré inequality and Corollary 2.4, we have that for all $w \in \overline{\Theta}^+_\mu$ and $\lambda \ge \lambda_1$

$$|(-J^+_\lambda(w))^{\frac{\gamma-p}{\gamma}} - (-J^+_{\lambda_0}(w))^{\frac{\gamma-p}{\gamma}}| = \frac{|\lambda-\lambda_0|G(w)}{|F(w)|^{p/\gamma}} \le \frac{|\lambda-\lambda_0|}{\lambda_1}\frac{1}{|c_\mu|^{p/\gamma}}.$$

Thus, $J^+_{\lambda_n}(w) \to J^+_{\lambda_0}(w)$ uniformly on $w \in \overline{\Theta}^+_\mu$ as $n \to \infty$ and therefore $\tilde{J} = \hat{J}^+_{\lambda_0}(\mu)$. Hence if $J^+_{\lambda_0}(v) < \hat{J}^+_{\lambda_0}(\mu)$, we obtain a contradiction since $v \in \overline{\Theta}^+_\mu$. On the other hand, if $J^+_{\lambda_0}(v) = \hat{J}^+_{\lambda_0}(\mu)$, then $H_{\mu_0}(v) = 0$ and $v = v_{\lambda_0}(\mu)$. Consequently $v_{\lambda_0}(\mu) \in \mathcal{S}^\partial_{\lambda_0}(\mu)$ which contradicts the assumption $\mathcal{S}^\partial_{\lambda_0}(\mu) = \emptyset$. ☐

Let us prove the existence of the solution u_λ in Theorem 1.1.

Lemma 3.3. *Let $1 < p < \gamma < p^*$ and suppose that $\Omega^+ \neq \emptyset$, $F(\phi_1) < 0$ and (f_1) is satisfied. Then there exists $\Lambda > \lambda^*$ such that for all $\lambda \in (\lambda^*, \Lambda)$ problem (1.1) admits a positive weak solution u_λ such that*

- **(li):** $\Phi''_\lambda(u_\lambda) > 0$ *and* $\Phi_\lambda(u_\lambda) < 0$;
- **(lii):** $\Phi_\lambda(u_\lambda) \uparrow \Phi_{\lambda^*}(u_{\lambda^*})$ *as* $\lambda \downarrow \lambda^*$.

Proof. By Corollary 2.3, there exists $\mu_0 \in (\lambda_1, \lambda^*)$ such that $\mathcal{S}^\partial_{\lambda^*}(\mu_0) = \emptyset$. Thus Lemma 3.2 implies that there exists $\Lambda > \lambda^*$ such that $\mathcal{S}^\partial_\lambda(\mu_0) = \emptyset$ for all $\lambda \in (\lambda^*, \Lambda)$. Since by Proposition 3.1, $\mathcal{S}_\lambda(\mu) \neq \emptyset$ for $\lambda \ge \lambda^*$, we conclude that for every $\lambda \in (\lambda^*, \Lambda)$ there exists a minimizer $v_\lambda(\mu_0)$ of (3.1) such that $v_\lambda(\mu_0) \in \Theta^+_{\mu_0}$. This and Proposition 2.1 yield that $u_\lambda = s^+_\lambda(v_\lambda(\mu_0))v_\lambda(\mu_0)$ is a weak solution of (1.1) for $\lambda \in (\lambda^*, \Lambda)$.

By virtue of $J^+_\lambda(v) = J^+_\lambda(|v|)$ and $|v| \in \Theta^+_{\mu_0}$ for any $v \in \Theta^+_{\mu_0}$, we may assume that $u_\lambda \ge 0$ in Ω. Now by the Harnack inequality (see [17]) we conclude that $u^+_\lambda > 0$ in Ω.

Since $v^+_\lambda \in \Theta^+_\lambda$, we get **(li)**. Let us prove assertion **(lii)**. Notice that from (3.1) it follows that $\Phi_{\lambda^*}(u_{\lambda^*}) = \hat{J}^+_{\lambda^*}(\mu_0) \ge \hat{J}^+_\lambda(\mu_0)$ for any $\lambda > \lambda^*$. Thus if we suppose that assertion **(lii)** is false, then we could find a sequence $\lambda_n \downarrow \lambda^*$ such that $J^+_{\lambda_n}(v_{\lambda_n}(\mu_0)) \to J^+_{\lambda^*} < \hat{J}^+_{\lambda^*}(\mu_0)$. Then as above, we may assume that $v_{\lambda_n}(\mu_0) \rightharpoonup v$ weakly in W and $v_n \to v$ strongly in $L^p(\Omega)$, $L^\gamma(\Omega)$ as $n \to \infty$ with $v \neq 0$. This implies that $v \in \overline{\Theta}^+_{\mu_0}$ and $J^+_{\lambda^*}(v) < \hat{J}^+_{\lambda^*}(\mu_0)$. Thus we get a contradiction. ☐

From the proof of Lemma 3.3 we see that the solution u_λ may depend on the parameter $\mu \in (\lambda_1, \lambda^*)$. However, we can prove that at least locally with respect to μ there is no such dependence.

Corollary 3.4. *Let $\lambda \ge \lambda^*$ and $\mu_0 \in (\lambda_1, \lambda^*)$. Suppose that $\mathcal{S}^\partial_\lambda(\mu_0) = \emptyset$. Then there exists $\varepsilon > 0$ such that $\mathcal{S}_\lambda(\mu_0) = \mathcal{S}_\lambda(\mu)$ for all $\mu \in (\mu_0 - \varepsilon, \mu_0 + \varepsilon)$.*

Proof. Conversely, suppose that there is (μ_n) such that $\mu_n \to \mu_0$ and $\exists v_n \in \mathcal{S}_\lambda(\mu_n) \setminus \mathcal{S}_\lambda(\mu_0)$. Then $J^+_\lambda(v_n) < \hat{J}^+_\lambda(\mu_0)$ and $v_n \in \Theta^+_{\mu_n} \setminus \Theta^+_{\mu_0}$. Arguing as in the proof of Proposition 3.1 it can be shown that there exists a subsequence (which we denote again (v_n)) such that $v_n \to v$ strongly in $W^{1,2}$. Hence, $J^+_\lambda(v) = \hat{J}^+_\lambda(\mu_0)$ and $v \in \partial\Theta^+_{\mu_0}$, that is, $v \in \mathcal{S}^\partial_\lambda(\mu_0)$ which is a contradiction. ☐

4. Mountain pass solution

In this section, for $\lambda \in (\lambda^*, \Lambda)$, we will find the second positive solution $\bar{u}_\lambda$ for (1.1) of a mountain pass type.

Fix $\mu_0 \in (\lambda_1, \lambda^*)$ such that $H_{\mu_0}(w_{\lambda^*}) < 0$ for any minimizer w_{λ^*} of (2.3) with $\lambda = \lambda^*$. The existence of μ_0 follows from Corollary 2.3.

Let $\lambda \in (\lambda^*, \Lambda)$. Define

$$\mu^\lambda = \sup\{\mu \in (\mu_0, \lambda^*) : \ \hat{J}^+_\lambda(\mu) = \hat{J}^+_\lambda(\mu_0)\}. \tag{4.1}$$

Proposition 4.1. *For each $\lambda \in (\lambda^*, \Lambda)$ there holds*

- **(a):** $\mu_0 < \mu^\lambda < \lambda^*$;
- **(b):** $\hat{J}^+_\lambda(\mu^\lambda) = \hat{J}^+_\lambda(\mu_0)$ *and* $\mathcal{S}^\partial_\lambda(\mu^\lambda) \neq \emptyset$.

Proof. **(a)** By Corollary 2.3, $\mathcal{S}^\partial_{\lambda^*}(\mu_0) = \emptyset$ and by Corollary 3.4, $\mathcal{S}_\lambda(\mu_0) = \mathcal{S}_\lambda(\mu)$ for $\mu \in (\mu_0, \mu_0 + \varepsilon)$ and some $\varepsilon > 0$. Hence, $\mu_0 < \mu^\lambda$. Notice that by Proposition 5.1 from the Appendix, the function $\hat{J}^+_\lambda(\mu)$ is continuous with respect to $\mu \in (\mu_0, \lambda^*)$. Hence and since $\hat{J}^+_\lambda(\mu) \to -\infty$ as $\mu \to \lambda^*$, there is $\mu' \in (\mu_0, \lambda^*)$ such that $\hat{J}^+_\lambda(\mu_0) > \hat{J}^+_\lambda(\mu)$ for each $\mu \in (\mu', \lambda^*)$. Thus $\mu^\lambda \leq \mu' < \lambda^* < +\infty$.

(b) Continuity of $\hat{J}^+_\lambda(\cdot)$ and (4.1) yield $\hat{J}^+_\lambda(\mu^\lambda) = \hat{J}^+_\lambda(\mu_0)$. Suppose, contrary to our claim, that $\mathcal{S}^\partial_\lambda(\mu^\lambda) = \emptyset$. Then by Corollary 3.4, there is $\varepsilon' > 0$ such that for $\mu \in (\mu^\lambda, \mu^\lambda + \varepsilon')$, $\mathcal{S}_\lambda(\mu^\lambda) = \mathcal{S}_\lambda(\mu)$ and consequently $\hat{J}^+_\lambda(\mu) = \hat{J}^+_\lambda(\mu^\lambda) = \hat{J}^+_\lambda(\mu_0)$, which is a contradiction. □

Observe, for any $\lambda \in (\lambda^*, \Lambda)$ and $\mu \in (\lambda_1, \lambda^*)$, if $w \in \mathcal{S}^\partial_\lambda(\mu^\lambda)$; then $|w| \in \mathcal{S}^\partial_\lambda(\mu^\lambda)$.

For each $\lambda \in (\lambda^*, \Lambda)$, fix $0 \leq w_\lambda \in \mathcal{S}^\partial_\lambda(\mu^\lambda)$ and let $0 < u_\lambda \in \Theta^+_{\mu_0}$ be the solution that has been found in Lemma 3.3. Define

$$c_\lambda = \inf_{\eta \in \Gamma_\lambda} \max_{t \in [0,1]} \Phi_\lambda(\eta(t)), \tag{4.2}$$

where

$$\Gamma_\lambda = \{\eta \in C([0,1], W) : \eta(0) = u_\lambda, \ \eta(1) = w_\lambda\}.$$

Proposition 4.2. *For each $\lambda \in (\lambda^*, \Lambda)$ there exists j_λ such that*

$$\Phi_\lambda(u) \geq j_\lambda > \hat{J}^+_\lambda(\mu_0) \ \ \forall\ u \in \partial\Theta^+_{\mu_0}.$$

Proof. Evidently, $\mathcal{S}^\partial_\lambda(\mu_0) = \emptyset$ implies

$$j_\lambda := \inf\{J^+_\lambda(v) : \ v \in \partial\Theta^+_\mu\} > \hat{J}^+_\lambda(\mu_0), \ \forall\ \lambda \in (\lambda^*, \Lambda).$$

Thus for any $u \in \partial\Theta^+_{\mu_0}$, one has

$$\Phi_\lambda(u) \geq \Phi_\lambda(s^+_\lambda(u)u) = J^+_\lambda(u) \geq j_\lambda > \hat{J}^+_\lambda(\mu_0).$$

□

Let us show that every path from Γ_λ intersects $\partial\Theta^+_{\mu_0}$.

Proposition 4.3. *Let $\lambda \in (\lambda^*, \Lambda)$. Then for any $\eta \in \Gamma_\lambda$ there exists $t_0 \in (0,1)$ such that $\eta(t_0) \in \partial\Theta^+_{\mu_0}$.*

Proof. Notice $H_{\mu_0}(\eta(0)) = H_{\mu_0}(u_\lambda) < 0$ while $H_{\mu_0}(\eta(1)) = H_{\mu_0}(w_\lambda) > 0$ because $w_\lambda \in \Theta^+_{\lambda^*} \setminus \overline{\Theta}^+_{\mu_0}$. Thus by the continuity of $H_{\mu_0}(\eta(\cdot))$, there is $t_0 \in (0,1)$ such that $H_{\mu_0}(\eta(t_0)) = 0$. □

Using [5] we are able to prove

Proposition 4.4. *For each $\lambda \in (\lambda^*, \Lambda)$, there is $\overline{\eta} \in \Gamma_\lambda$ such that $H_{\lambda^*}(\overline{\eta}(t)) < c < 0$ for all $t \in [0,1]$*

Proof. Consider the path $\overline{\eta}(t) = [(1-t)u_\lambda^p + tw_\lambda^p]^{1/p}$, $t \in [0,1]$. Once $u_\lambda > 0$, $\{x \in \Omega : \ u_\lambda(x) = w_\lambda(x) = 0\} = \emptyset$. Hence we may apply Proposition 5.2 from the Appendix and thus $\overline{\eta} \in C([0,1], W)$ and for $t \in [0,1]$ we have

$$\begin{aligned}
H_{\lambda^*}(\overline{\eta}(t)) &= \int |\nabla \overline{\eta}(t)|^p - \lambda^* \int |\overline{\eta}(t)|^p \\
&\leq (1-t)\int |\nabla u_\lambda|^p + t\int |\nabla w_\lambda|^p - \lambda^* \left((1-t)\int |u_\lambda|^p + t\int |w_\lambda|^p\right) \\
&= (1-t)H_{\lambda^*}(u_\lambda) + tH_{\lambda^*}(w_\lambda) < 0 \\
&\leq \max\{H_{\lambda^*}(u_\lambda), H_{\lambda^*}(w_\lambda)\},
\end{aligned}$$

since $H_{\lambda^*}(u_\lambda) < 0$, $H_{\lambda^*}(w_\lambda) < 0$. □

Corollary 4.5. *For all $\lambda \in (\lambda^*, \Lambda)$ there holds*

$$\hat{J}_\lambda^+(\mu_0) < c_\lambda < 0. \tag{4.3}$$

Proof. Let us start with the first inequality. Take any $\eta \in \Gamma_\lambda$. From Proposition 4.3, there is $t_0 \in (0,1)$ such that $\eta(t_0) \in \partial\Theta_{\mu_0}^+$, therefore by Proposition 4.2, $\max_{t\in[0,1]} \Phi_\lambda(\eta(t)) \geq \Phi_\lambda(\eta(t_0)) > \hat{J}_\lambda^+(\mu_0)$ and consequently $\hat{J}_\lambda^+(\mu_0) < c_\lambda$. Let $\overline{\eta}$ be given by Proposition 4.4. Then

$$\Phi_\lambda(\overline{\eta}(t)) = J_\lambda^+(\overline{\eta}(t)) < 0 \ \ \forall\, t \in [0,1],$$

which implies that $c_\lambda < 0$. □

Now we are able to find the second solution $\bar{u}_\lambda$.

Lemma 4.6. *For each $\lambda \in (\lambda^*, \Lambda)$, c_λ is a critical value of Φ_λ. Furthermore, there exists $\overline{u}_\lambda$ such that $\Phi_\lambda(s_\lambda^+(\overline{u}_\lambda) = c_\lambda$, $\overline{u}_\lambda$ is a weak solution of (1.1) and $\overline{u}_\lambda > 0$ in Ω.*

Proof. Since $\Phi_\lambda(u) = \Phi_\lambda(|u|)$ for all $u \in W$, then by (4.2) there is a sequence of paths $\eta_n \geq 0$ in Ω such that

$$\lim_{n\to\infty} \max_{t\in[0,1]} \Phi(\eta_n(t)) = c_\lambda.$$

Now, following [11], let us introduce for each $\epsilon > 0$

$$\eta_{n,\epsilon} = \{u \in W : \inf_{t\in[0,1]} \|u - \eta_n(t)\| \leq \epsilon\} \cap K_{c_\lambda, 2\epsilon},$$

where $K_{c_\lambda,2\epsilon} = \{u \in W : |\Phi_\lambda(u) - c_\lambda| \leq 2\epsilon\}$. Then by Theorem E.5 from [11], there is a sequence $u_n \in W$ satisfying

$$\Phi_\lambda(u_n) \to c_\lambda, \ D_u\Phi_\lambda(u_n) \to 0, \tag{4.4}$$

and

$$\inf_{t\in[0,1]} \|u_n - \eta_n(t)\| \to 0. \tag{4.5}$$

By Corollary 4.5 we know that $c_\lambda < 0$. Thus, in view of (4.4) we can apply Proposition 5.3 to conclude that $u_n \to \overline{u}_\lambda \in W \setminus 0$ so that $\Phi_\lambda(\overline{u}_\lambda) = c_\lambda$ and

$D_u\Phi_\lambda(\overline{u}_\lambda) = 0$. Moreover, once (4.5) is satisfied, we also have that $\overline{u}_\lambda \geq 0$. Now applying the Harnack inequality [17] we deduce that $\overline{u}_\lambda > 0$ in Ω. □

Conclusion of the proof of Theorem 1.1. Let $\Lambda > \lambda^*$ be given by Lemma 3.3. Then Lemma 3.3 and Lemma 4.6 yield the existence of positive weak solutions $u_\lambda, \overline{u}_\lambda$. Since $c_\lambda < 0$, $\Phi_\lambda(\overline{u}_\lambda) < 0$. Thus, by virtue that $\overline{u}_\lambda$ is a critical value of Φ_λ, we conclude that $\Phi''_\lambda(\overline{u}_\lambda) > 0$. Corollary 4.5 and Lemma 4.6 imply that $\Phi_\lambda(u_\lambda) < \Phi_\lambda(\overline{u}_\lambda) < 0$ for any $\lambda \in (\lambda^*, \Lambda)$. Hence and by **(li)**, Lemma 3.3 we get assertion **(i)** of the theorem. The proof of **(ii)** follows from **(lii)**, Lemma 3.3. □

5. Appendix

Proposition 5.1. *For any $\lambda \geq \lambda^*$, the function $\mu \mapsto \hat{J}^+_\lambda(\mu)$ is continuous over the interval (λ_1, λ^*).*

Proof. Let $\mu \in (\lambda_1, \lambda^*)$. Suppose, contrary to our claim, that there is $\mu_n \to \mu$ and $r > 0$ such that $|\hat{J}^+_\lambda(\mu_n) - \hat{J}^+_\lambda(\mu)| > r$ for all n, or equivalently

$$\hat{J}^+_\lambda(\mu_n) > \hat{J}^+_\lambda(\mu) + r \text{ or } \hat{J}^+_\lambda(\mu) > \hat{J}^+_\lambda(\mu_n) + r, \tag{5.1}$$

for sufficiently large n. Suppose the first inequality is true, i.e., $\hat{J}^+_\lambda(\mu_n) > \hat{J}^2_\lambda(\mu) + r$. From (3.1) this is possible only if $\mu_n < \mu$. Moreover, we can assume without loss of generality that μ_n is monotone increasing and consequently $\hat{J}^2_\lambda(\mu_n)$ is decreasing. Thus $\hat{J}^+_\lambda(\mu_n) \to I > \hat{J}^+_\lambda(\mu)$.

By Proposition 3.1, there is $v \in \mathcal{S}_\lambda(\mu)$, that is, $J^2_\lambda(v) = \hat{J}^+_\lambda(\mu)$. Suppose $v \in \mathcal{S}_\lambda(\mu) \setminus \mathcal{S}^\partial_\lambda(\mu)$; then convergence $\mu_n \to \mu$ entails that there is n such that $v \in \Theta^2_{\mu_n}$. However $J^+_\lambda(v) \geq \hat{J}^+_\lambda(\mu_n)$ which contradicts $J^+_\lambda(v) = \hat{J}^+_\lambda(\mu) < I \leq \hat{J}^+_\lambda(\mu_n)$. Suppose now that $v \in \mathcal{S}^\partial_\lambda(\mu)$. Then, taking into account the continuity of $J^+_\lambda(u)$ on Θ^+_μ, we can choose $w \in \Theta^+_\mu$ such that $J^+_\lambda(v) \leq J^+_\lambda(w) < I$. However, there is n such that $w \in \Theta^+_{\mu_n}$. This implies $\hat{J}^+_\lambda(\mu_n) \leq J^+_\lambda(w) < I$ which is absurd.

Now suppose the second inequality in (5.1) is true. Then $\mu < \mu_n$ and we may assume that μ_n is decreasing. Consequently $\hat{J}^+_\lambda(\mu_n)$ is increasing and $\hat{J}^+_\lambda(\mu_n) \to I < \hat{J}^+_\lambda(\mu)$. From Proposition 3.1, there is v_n such that $v_n \in \mathcal{S}_\lambda(\mu_n)$. If $v_n \in \Theta^+_\mu$ for some n, then $\hat{J}^+_\lambda(\mu) \leq J^+_\lambda(v_n) = \hat{J}^+_\lambda(\mu_n)$ which contradicts the assumption $\hat{J}^+_\lambda(\mu) > \hat{J}^+_\lambda(\mu_n)$. Thus it is only possible that $v_n \in \overline{\Theta}^+_{\mu_n} \setminus \overline{\Theta}^+_\mu$ for all $n = 1, 2, \ldots$. Arguing as above one may infer that there exist $v \in W \setminus 0$ and a subsequence, which we still denote by (v_n), such that $v_n \rightharpoonup v$ weakly in W and $v_n \to v$ strongly in $L^q(\Omega)$ for $1 < q < p^*$. Then

$$H_\mu(v) \leq \liminf_{n\to\infty} H_{\mu_n}(v_n) \leq 0, \; F(v) = \lim_{n\to\infty} F(v_n) < 0,$$

which implies that $v \in \overline{\Theta}^+_\mu$ and

$$\hat{J}^+_\lambda(\mu) \leq J^+_\lambda(v) \leq \liminf_{n\to\infty} J^+_{\lambda_n}(v_n) = I,$$

which is absurd because $I < \hat{J}^+_\lambda(\mu)$. □

The next result can be found in the paper of Díaz & Saá [5]. We give a proof of it for the reader's convenience.

Proposition 5.2. *Let $u, v \in W \setminus 0$, $u, v \geq 0$ in Ω and define $\overline{\eta}(t) = [(1-t)u^p + tv^p]^{1/p}$ for $t \in [0,1]$. Suppose that the set $\{x \in \Omega : \ u(x) = v(x) = 0\}$ has zero Lebesgue measure. Then*

$$|\nabla \overline{\eta}(t)|^p \leq (1-t)|\nabla u|^p + t|\nabla v|^p \ \ \forall\ t \in [0,1], \ \textit{a.e. in } \Omega$$

and $\overline{\eta} \in C([0,1], W)$.

Proof. First note that $\nabla \eta(t) = [(1-t)u^p + tv^p]^{(1-p)/p}[(1-t)u^{p-1}\nabla u + tv^{p-1}\nabla v]$. Let p' be the conjugate exponent of p, i.e., $1/p + 1/p' = 1$. From the Hölder inequality, we have
(5.2)

$$\begin{aligned}
|\nabla \overline{\eta}(t)| &\leq [(1-t)u^p + tv^p]^{(1-p)/p}[(1-t)u^{p-1}|\nabla u| + tv^{p-1}|\nabla v|] \\
&= [(1-t)u^p + tv^p]^{(1-p)/p}[(1-t)^{1/p'}u^{p-1}(1-t)^{1/p}|\nabla u| + t^{1/p'}v^{p-1}t^{1/p}|\nabla v|] \\
&\leq [(1-t)u^p + tv^p]^{(1-p)/p}[(1-t)u^p + tv^p]^{1/p'}[(1-t)|\nabla u|^p + t|\nabla v|^p]^{1/p} \text{ a.e. in } \Omega
\end{aligned}$$

for all $t \in [0,1]$. Once $\{x \in \Omega : \ u(x) = v(x) = 0\}$ has zero Lebesgue measure, we have that $(1-t)u^p + tv^p > 0$ a.e. in Ω, for all $t \in (0,1)$ and therefore, from (5.2), we conclude that

$$|\nabla \overline{\eta}(t)| \leq [(1-t)|\nabla u|^p + t|\nabla v|^p]^{1/p} \ \ \forall\ t \in [0,1], \text{ a.e. in } \Omega$$

which implies

$$|\nabla \overline{\eta}(t)|^p \leq (1-t)|\nabla u|^p + t|\nabla v|^p, \ \forall\ t \in [0,1], \text{ a.e. in } \Omega.$$

Consequently $\overline{\eta} \in W$ for all $t \in [0,1]$. The continuity of η follows by a standard application of the Lebesgue theorem. □

The proof of the next proposition is standard (see Ambrosetti & Rabinowitz [2] as well as the related papers of Alama & Tarantello [1], and Pucci & Rădulescu [16]). We include its sketch for the convenience of the reader.

Proposition 5.3. *Suppose that $u_n \in W \setminus 0$ is a (P.-S.) sequence at level $c < 0$, i.e.,*

$$\Phi_\lambda(u_n) \to c < 0, \ D_u\Phi(u_n) \to 0.$$

Then u_n has a strong convergent subsequence with nonzero limit point $u \in W \setminus 0$ satisfying $\Phi_\lambda(u) = c$ and $D_u\Phi(u) = 0$.

Proof. The assumption $D_u\Phi_\lambda(u_n) \to 0$ entails $(H_\lambda(u_n) - F(u_n))/\|u_n\|^p = o(1)$. On the other hand, from the limit $\Phi_\lambda(u_n) \to c$ as $n \to \infty$, we also have

$$H_\lambda(u_n) = \frac{p}{\gamma}F(u_n) + pc + o(1), \ n \to \infty.$$

Thus

$$0 = \lim_{n\to\infty} \frac{H_\lambda(u_n) - F(u_n)}{\|u_n\|^p} = \lim_{n\to\infty} \left[\left(\frac{p}{\gamma} - 1\right) F(u_n)/\|u_n\|^p + \frac{pc + o(1)}{\|u_n\|}\right].$$

From this it is not hard to show that $\|u_n\|$ is bounded and $||u_n|| \geq \delta > 0$ for some constance δ. Thus we may assume $u_n \rightharpoonup u$ in W, $u_n \to u$ in $L^p(\Omega)$ and $L^\gamma(\Omega)$ and $u \neq 0$. Hence and since $D_u\Phi(u_n) \to 0$ as $n \to \infty$, we have

$$\limsup_{n\to\infty} \langle -\Delta_p u_n, u_n - u\rangle = 0.$$

Thus by the S^+ property of the p-Laplacian operator (see [6]) we derive that $u_n \to u$ strongly in W. □

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