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ON SOME POLYNOMIALS AND SERIES OF BLOCH–PÓLYA TYPE

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Abstract. We will show that $(1-q)(1-q^2)\dots(1-q^m)$ is a polynomial in q with coefficients from $\{-1,0,1\}$ iff $m = 1,\ 2,\ 3$, or 5 and explore some interesting consequences of this result. We find explicit formulas for the q-series coefficients of $(1-q^2)(1-q^3)(1-q^4)(1-q^5)\dots$ and $(1-q^3)(1-q^4)(1-q^5)(1-q^6)\dots$. In doing so, we extend certain observations made by Sudler in 1964. We also discuss the classification of the products $(1-q)(1-q^2)\dots(1-q^m)$ and some related series with respect to their absolute largest coefficients.

1. Introduction and background

Polynomials with coefficients from the set $\{-1,0,1\}$ were first studied by Bloch and Pólya [4]. Their study sparked interest especially about the roots of these polynomials. An interested reader may refer to these prominent results (listed in order of publication) [9], [5], [8], and [6]. The first two references focus on *Littlewood polynomials*, i.e., polynomials where all coefficients are ± 1. In the same spirit, we would like to call polynomials (and series) with integer coefficients from the set $\{-1,0,1\}$ *Bloch–Pólya type* polynomials (and series, resp.).

We start by defining a *q-Pochhammer symbol* or a *rising q-factorial.* For variables $a,\ q$, and a non-negative integer L, we define

$$(a;q)_L = \prod_{i=0}^{L-1}(1-aq^i),$$
$$(a;q)_\infty = \lim_{L\to\infty}(a;q)_L,\ \text{for } |q|<1.$$

The rising q-factorials have been studied extensively; one notable example is Euler's Pentagonal Number Theorem [2, Cor. 1.7, pg. 11].

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Theorem 1.1 (Euler's Pentagonal Number Theorem, 1750)**.** *We have the identity*

(1.1)
$$
\begin{aligned}
(q;q)_\infty &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}\\
&= 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - q^{35} - q^{40} + q^{51} + q^{57} - q^{70} \dots .
\end{aligned}
$$

The identity (1.1) shows that the q-Pochhammer symbol $(q;q)_\infty$ can be represented as Bloch–Pólya type series.

We will also require the q-binomial theorem [2, Thm. 2.1, p. 17]:

Theorem 1.2 (q-Binomial Theorem)**.** *We have*

$$
\sum_{n\geq 0} \frac{(a;q)_n}{(q;q)_n} t^n = \frac{(ta;q)_\infty}{(t;q)_\infty}. \tag{1.2}
$$

Next, we define the following family of sums:

$$
F_{k,M}(q) := \sum_{j=0}^{M} q^{kj} (q;q)_j, \tag{1.3}
$$

$$
F_k(q) := \lim_{M\to\infty} F_{k,M}(q) = \sum_{j\geq 0} q^{kj}(q;q)_j, \tag{1.4}
$$

for positive integers k and M. The series $F_k(q)$ can be made convergent by picking $|q|<1$.

These series were introduced by Eden, and they are closely related to the theory of partitions. Eden observed that

$$
(-q)^k F_k(q) = \sum_{n,m>0} (-1)^m p_k(n,m) q^n,
$$

where $p_k(n,m)$ is the number of non-empty partitions of n into exactly m parts where the largest part appears k times and all the other parts appear distinctly [7]. Moreover, the series $F_1(q)$ plays an instrumental role in Euler's original proof of Theorem 1.1 [1]. Recently $F_{i,M}(q)$ for $i = 1,\ 2$, and 3 arose naturally in our studies of partitions with bounded gaps between largest and smallest parts [3].

In the following sections, among other observations, we will prove the next two theorems.

Theorem 1.3. *For $m \in \mathbb{Z}_{\geq 0}$, $(q;q)_m$ is of Bloch–Pólya type iff $m = 0,\ 1,\ 2,\ 3$, or 5.*

Theorem 1.4. *For $k \geq 7$, there is no polynomial $f(q)$ such that $F_k(q) + f(q)$ is of Bloch–Pólya type.*

C. Sudler, in his 1964 papers [10,11], studied the maximum coefficient a^*_m of the power series expansion of $(q;q)_m$. He noted that a^*_m is unbounded as m gets larger using a special case of Theorem 1.2 and that $\log a^*_m \sim Km$ where $K>0$ using Cauchy's integral formula in the respective papers. The unboundedness of a^*_m was shown by observing that $(q^3;q)_\infty$ has unbounded q-series coefficients.

In Section 2, we start with some observations about pentagonal numbers. After that we give explicit formulas for the q^j's coefficient in the power series of both $(q^2;q)_\infty$ and $(q^3;q)_\infty$, for any $j \in \mathbb{Z}_{\geq 0}$. This section is finalized with a proof of Theorem 1.3. Section 3 starts by proving a recurrence relation for $F_{k,M}(q)$ polynomials, and the rest of the section deals with Bloch–Pólya properties of $F_k(q)$ series. We discuss the classification of the polynomials $(q;q)_m$ and the series $F_k(q)$ with respect to their coefficients in a broader perspective than the Bloch–Pólya property in Section 4.

2. q-series coefficients of $(q;q)_m$, $(q^2;q)_\infty$, and $(q^3;q)_\infty$. Proof of Theorem 1.3

We start by observing that the minimum gap between pentagonal numbers increase. An alternative way of writing Theorem 1.1 is

$$(q;q)_\infty = 1 + \sum_{n=1}^{\infty}(-1)^n(q^{n(3n-1)/2} + q^{n(3n+1)/2}).$$

Let

$$p_1(n) := \frac{n(3n-1)}{2} \quad \text{and} \quad p_2(n) := \frac{n(3n+1)}{2} \tag{2.1}$$

be the two families of pentagonal numbers for $n \geq 0$. Observe that there is a natural order between these families:

$$\begin{aligned} 0 = p_1(0) = p_2(0) < 1 = p_1(1) < 2 = p_2(1) < 5 \\ = p_1(2) < \cdots < p_1(n) < p_2(n) < p_1(n+1) < \cdots \end{aligned} \tag{2.2}$$

for any $n \geq 1$. Also note that

$$p_2(n) - p_1(n) = n \quad \text{and} \quad p_1(n+1) - p_2(n) = 2n+1. \tag{2.3}$$

This proves that

Lemma 2.1. *For any $M > 0$ the gap between successive pentagonal numbers is*

$$p_2(n) - p_1(n) > M \quad \textit{and} \quad p_1(n+1) - p_2(n) > M,$$

for all $n > M$.

We will use Lemma 2.1 to find a suitable separation point for the tail of the series (1.1) in the following theorems.

Theorem 2.2. *The power series of*

$$(q^2;q)_\infty := \sum_{j\geq 0} a_j q^j$$

is of Bloch–Pólya type. Furthermore, for any $j \in \mathbb{Z}_{\geq 0}$ there exist a unique $n \in \mathbb{Z}_{\geq 0}$ such that

$$p_2(2n) \leq j < p_2(2n+2)$$

and

$$a_j = \begin{cases} 1, & \textit{if } \ p_2(2n) \leq j < p_1(2n+1), \\ -1, & \textit{if } \ p_2(2n+1) \leq j < p_1(2n+2), \\ 0, & \textit{otherwise.} \end{cases} \tag{2.4}$$

Proof. We start with

$$(q^2;q)_\infty = \frac{(q;q)_\infty}{1-q} = (1+q+q^2+q^3+\cdots)(1-q-q^2+q^5+q^7-q^{12}-q^{15}+\cdots), \tag{2.5}$$

which is clear by geometric series and Theorem 1.1. Openly evaluating the latter product is enough to demonstrate this result:

$$\begin{aligned}
(q^2;q)_\infty &= (1+q+q^2+q^3+q^4+q^5+\dots)(1-q-q^2+q^5+q^7-q^{12}-q^{15}+q^{22}+q^{26}-q^{35}-\dots)\\
&= 1+q+q^2+q^3+q^4+q^5+q^6+q^7+q^8+q^9+q^{10}+q^{11}+q^{12}+q^{13}+q^{14}+q^{15}+q^{16}+q^{17}\dots\\
&\quad -q-q^2-q^3-q^4-q^5-q^6-q^7-q^8-q^9-q^{10}-q^{11}-q^{12}-q^{13}-q^{14}-q^{15}-q^{16}-q^{17}\dots\\
&\quad -q^2-q^3-q^4-q^5-q^6-q^7-q^8-q^9-q^{10}-q^{11}-q^{12}-q^{13}-q^{14}-q^{15}-q^{16}-q^{17}\dots\\
&\quad +q^5+q^6+q^7+q^8+q^9+q^{10}+q^{11}+q^{12}+q^{13}+q^{14}+q^{15}+q^{16}+q^{17}\dots\\
&\quad +q^7+q^8+q^9+q^{10}+q^{11}+q^{12}+q^{13}+q^{14}+q^{15}+q^{16}+q^{17}\dots\\
&\quad -q^{12}-q^{13}-q^{14}-q^{15}-q^{16}-q^{17}\dots\\
&\quad -q^{15}-q^{16}-q^{17}\dots\\
&= 1 \quad -q^2-q^3-q^4 \quad +q^7+q^8+q^9+q^{10}+q^{11} \quad -q^{15}-q^{16}-q^{17}\dots .
\end{aligned}$$

The sign changes at every other non-zero coefficient term in the pentagonal numbers series (1.1) make sure that the coefficients of (2.5) are in the set $\{-1,0,1\}$ when $(q;q)_\infty$ is divided by $(1-q)$.

With the 0 coefficients explicitly written, we have

$$(q^2;q)_\infty = 1+0\,q-q^2-q^3-q^4+0\,q^5+0\,q^6+q^7+q^8+q^9+q^{10}+q^{11}+0\,q^{12}\dots . \tag{2.6}$$

For $n\geq 1$, it is clear that the n-th non-zero coefficient block starts at $p_2(n-1)$. This block is of the size $2n-1$, and its coefficients are all $(-1)^{n+1}$. Moreover, a zero coefficient block of size n follows the n-th non-zero coefficient block. □

Next we observe that for $n\geq 0$ and $r=0$, 1,

$$\sum_{2j+r\in\mathcal{I}_n} a_{2j+r} = -1, \tag{2.7}$$

where $\mathcal{I}_n := \{p_2(2n),\dots,p_1(2n+2)-1\}$ and a_j is as in Theorem 2.2. It is clear that the number of -1 coefficients a_j in $p_2(2n)$ to $p_1(2n+2)-1$ is 2 more than the $+1$ coefficients. This observation is true due to the number of zero coefficients in this interval being an odd number. Another way of seeing this is to observe $p_2(2n)$ and $p_2(2n+1)$ having the same parity, for $n\geq 0$.

Let

$$(q^3;q)_\infty = \sum_{i\geq 0} b_i q^i \tag{2.8}$$

be the power series representation. Since $(q^3;q)_\infty = (q^2;q)_\infty/(1-q^2)$, it is clear that

$$b_i = \sum_{\substack{j\geq 0\\ i-2j\geq 0}} a_{i-2j},$$

where a_j is as in Theorem 2.2. With that, one can prove the following.

Theorem 2.3. *For every integer $i\geq 0$, let b_i be as in* (2.8). *There exists a unique integer $n\geq 0$ such that*

$$p_1(2n)-2\leq i\leq p_1(2n+2)-3.$$

Then, the power series coefficients b_i of $(q^3;q)_\infty$ are given explicitly by

$$b_i = \begin{cases} -n, & \text{if } p_1(2n)-2 \le i \le p_2(2n)-1, \\ 1-n+\lfloor \frac{i-p_2(2n)}{2} \rfloor, & \text{if } p_2(2n) \le i \le p_1(2n+1)-2, \\ 1+n, & \text{if } p_1(2n+1)-1 \le i \le p_2(2n+1)-2 \\ & \quad \text{and } i \equiv p_2(2n) \pmod 2, \\ n, & \text{if } p_1(2n+1)-1 \le i \le p_2(2n+1)-2 \\ & \quad \text{and } i \not\equiv p_2(2n) \pmod 2, \\ n-\lceil \frac{i-p_2(2n+1)}{2} \rceil, & \text{if } p_2(2n+1)-1 \le i \le p_1(2n+2)-3, \end{cases}$$

where $\lfloor x \rfloor$ is the greatest integer $\le x$, and $\lceil x \rceil$ is the smallest integer $\ge x$.

As an example, if $i = 10^{100}$, then

$$n = 408248290463863016366214012450981898660991246776 11.$$

Moreover, for this particular i the second case of the formula above applies. Hence, after the addition of three numbers we get

$$b_i = -19888090251390639910818356938628130689602741018379.$$

Theorem 2.3 already says that series expansion of $(q^3;q)_\infty$ is not of Bloch–Pólya type. Moreover, *every integer occurs as a coefficient of $(q^3;q)_\infty$ infinitely many times.* For illustrative purposes, some first appearances of non-zero coefficient sizes are

(2.9)
$$(q^3;q)_\infty = 1+\cdots+2q^{11}+\cdots+3q^{34}+\cdots+4q^{69}+\cdots+5q^{116}+\cdots+6q^{175}+\cdots+7q^{246}+\cdots.$$

We remark that Euler's Pentagonal Number Theorem (Theorem 1.1) has a partition-theoretic interpretation; that is, a finite sequence $\pi = (\lambda_1, \lambda_2, \dots)$ of non-increasing positive integers is called a *partition.* Let $\mathcal{D}$ be the set of partitions into distinct parts (i.e., $\lambda_i \ne \lambda_j$ for $i \ne j$), let $\nu(\pi)$ be the number of parts of partition π, and let $|\pi|$ be the sum of all the parts of π. The empty sequence is the unique partition of zero. Then (1.1) can be interpreted as

$$(q;q)_\infty = \sum_{n\ge 0} \left(\sum_{\substack{\pi\in\mathcal{D} \\ |\pi|=n}} (-1)^{\nu(\pi)} \right) q^n.$$

Similar to this interpretation one can also interpret Theorems 2.2 and 2.3 as partition theorems.

Theorem 2.4. *We have*

$$a_j = \sum_{\substack{\pi\in\mathcal{D} \\ |\pi|=j \\ s(\pi)>1}} (-1)^{\nu(\pi)} \quad \text{and} \quad b_i = \sum_{\substack{\pi\in\mathcal{D} \\ |\pi|=i \\ s(\pi)>2}} (-1)^{\nu(\pi)},$$

where $s(\pi)$ is the smallest part of partition π, and a_j and b_i are defined as in Theorems 2.2 *and* 2.3.

Now we can give an easy proof of Theorem 1.3. To this end, we will define $\llbracket q^i \rrbracket f(q)$ to be the q^i's coefficient in the power series expansion of $f(q)$.

Proof. Proof of Theorem 1.3. Initial cases of q-factorials can easily be checked to be Bloch–Pólya type polynomials for $m = 0$ to 5, except for $m = 4$:

$$\begin{aligned}
(q;q)_0 &= 1,\\
(q;q)_1 &= 1-q,\\
(q;q)_2 &= 1-q-q^2+q^3,\\
(q;q)_3 &= 1-q-q^2+q^4+q^5-q^6,\\
(q;q)_4 &= 1-q-q^2+2\,q^5-q^8-q^9+q^{10},\\
(q;q)_5 &= 1-q-q^2+q^5+q^6+q^7-q^8-q^9-q^{10}+q^{13}+q^{14}-q^{15}.
\end{aligned}$$

It is clear that $\llbracket q^5\rrbracket (q;q)_4 = 2$. Also observe that

$$\begin{aligned}
\llbracket q^7\rrbracket (q;q)_6 &= 2,\\
\llbracket q^{12}\rrbracket (q;q)_m &= -2, \text{ for } \ 7\le m\le 9,\\
\llbracket q^{15}\rrbracket (q;q)_{10} &= -2,\\
-3\le \llbracket q^{2m+22}\rrbracket (q;q)_m &\le -2, \text{ for } \ 11\le m\le 20.
\end{aligned}$$

Another example that will come in handy is

$$\llbracket q^{51}\rrbracket (q;q)_{41} = 2. \tag{2.10}$$

For some $m \geq 1$, choose $(a, q, t) = (0, q, q^m)$ in the q-binomial theorem (1.2). Multiplying both sides of this special case with $(q;q)_\infty$, one can easily show that

$$(q;q)_{m-1} = \sum_{i\geq 0} q^{mi}(q^{i+1};q)_\infty = (q;q)_\infty + q^m(q^2;q)_\infty + q^{2m}(q^3;q)_\infty \dots. \tag{2.11}$$

Note that for any $2m \leq n < 3m$ the contribution for the coefficient of q^n of $(q;q)_{m-1}$ comes only from the first three terms of the expansion in (2.11). Keeping (2.9) in mind, for $m > 69$ one can deduce that

$$2 \leq \llbracket q^{2m+69}\rrbracket (q;q)_{m-1} \leq 6,$$

since the coefficients of q^{2m+69} and q^{m+69} of $(q;q)_\infty$ and $(q^2;q)_\infty$ are in the set $\{-1, 0, 1\}$ by Theorem 1.1 and Theorem 2.2, respectively. We can now directly verify the claim for the intermediate interval of unchecked m values ($m \geq 22$ and the inequality $2m \leq 2m + 69 < 3m$ does not hold) that

$$2 \leq \llbracket q^{2m+69}\rrbracket (q;q)_{m-1} \leq 12 \text{ for } 22 \leq m \leq 69 \text{ and } m \neq 42.$$

Recall that the $m = 42$ case was handled in (2.10) above. □

3. Recurrence relations for $F_{k,M}(q)$ and a proof of Theorem 1.4

We start by the recurrence relations for the $F_{k,M}(q)$ functions.

Lemma 3.1. *For $k \geq 1$,*

$$q^{k+1}F_{k+1,M}(q) = 1 + (q^k - 1)F_{k,M}(q) - q^{k(M+1)}(q;q)_{M+1}.$$

Proof. Observe that

$$\begin{aligned}(q^k-1)F_{k,M}(q) &= \sum_{n=1}^{M+1} q^{kn}(q;q)_{n-1} - \sum_{n=0}^{M} q^{kn}(q;q)_n \\ &= q^{k+1}F_{k+1,M-1}(q) + q^{k(M+1)}(q;q)_M - 1.\end{aligned}$$

Adding $q^{(k+1)(M+1)}(q;q)_M$, we get the term $q^{k+1}F_{k+1,M}(q)$ on the right side of the equation. Isolating this term yields the result. □

Let $\mathcal{D}_{1,k}$ be the set of non-empty partitions into distinct parts congruent to 1 modulo k, and let $\nu(\pi)$ be the number of parts of the partition π. We note that

$$\sum_{\substack{\pi\in\mathcal{D}_{1,k}\\ l(\pi)\le kM+1}} (-1)^{\nu(\pi)+1}q^{|\pi|} = \sum_{j=0}^{M} q^{kj+1}(q;q^k)_j = 1-(q;q^k)_{M+1}, \tag{3.1}$$

where $l(\pi)$ is the largest part of π.

The q-factorial $(q;q^k)_{M+1}$ is the generating function for the partitions π^* into distinct parts $\le kM+1$, each 1 modulo k, counted with the weights $(-1)^{\nu(\pi^*)}$. The summand of the middle term of (3.1) is the generating function for partitions into distinct parts, each 1 modulo k counted with the weight $(-1)^{\nu(\pi)+1}$ where the largest part is $jk+1$. Summing from $j=0$ to M, we get the generating function for the number of partitions into distinct parts $\le kM+1$, each 1 modulo k. This justifies the first equality of (3.1). The second equality can also be clarified in the same manner. We multiply $(q;q^k)_{M+1}$ by -1 to match the weight $(-1)^{\nu(\pi)+1}$ and add 1 to remove the empty partition from our calculations.

We will later refer to this special case of (3.1), where k is 1:

$$qF_{1,M}(q) = \sum_{j=0}^{M} q^{j+1}(q;q)_j = 1-(q;q)_{M+1}. \tag{3.2}$$

This special case also appears in [1, (5)].

Now we can prove some results about the coefficients of $F_i(q)$ functions.

Theorem 3.2.

(i) *$F_1(q)$ is of Bloch–Pólya type.*
(ii) *$F_2(q)$ is of Bloch–Pólya type.*
(iii) *$F_3(q)-q^9$ and $F_4(q)-q^{16}+q^{18}+q^{30}-q^{31}$ are both of Bloch–Pólya type.*
(iv) *$F_5(q)$ is not a Bloch–Pólya type series, and there is no polynomial $f(q)$ such that $F_5(q)+f(q)$ is one.*
(v) *$F_6(q)-f_6(q)$ is a Bloch–Pólya type series, where*

$$\begin{aligned}f_6(q) :=\, &q^{29}-q^{32}+q^{36}-q^{38}+q^{43}-q^{45}+q^{50}-q^{56}+q^{57}+q^{58}-q^{62}-q^{63}+q^{64}\\ &+q^{71}-q^{80}-q^{81}+q^{84}+q^{85}+q^{106}-q^{110}-q^{239}+q^{241}+q^{280}-q^{281}.\end{aligned}$$

(vi) *And for $k\ge 7$, there is no polynomial $f(q)$ such that $F_k(q)+f(q)$ is of Bloch–Pólya type.*

Item (vi) of Theorem 3.2 is the earlier highlighted Theorem 1.4.

Proof. (i) Taking the limit $M \to \infty$ on the extreme sides of (3.2), combined with (1.1), we have

$$qF_1(q) = 1 - (q;q)_\infty = q + q^2 - q^5 - q^7 + q^{12} + q^{15} \dots . \tag{3.3}$$

This is enough to show that $F_1(q)$ is of Bloch–Pólya type.

The proofs of cases (ii) and (iii) will rely on a combination of Theorem 1.1, Theorem 1.3, Lemma 2.1, and Lemma 3.1 with $M \to \infty$. The combination of Lemma 3.1 with $M \to \infty$, together with (3.3), yields

$$q^{k(k+1)/2}F_k(q) = \sum_{i=0}^{k-1}(-1)^i(q^{k-i};q)_i q^{(k-1-i)(k-i)/2} + (-1)^k(q;q)_{k-1}(q;q)_\infty, \tag{3.4}$$

for $k \geq 1$.

(ii) The difference between successive pentagonal numbers (which appear in the exponent of q) is greater than 1 for exponents of q greater than or equal to $p_1(2) = 5$, by Lemma 2.1. Therefore, the series $q^5 + q^7 - q^{12} - q^{15} \dots$ and $q(q^5 + q^7 - q^{12} - q^{15} \dots) = q^6 + q^8 - q^{13} - q^{16} \dots$ do not share any common exponents of q. Hence, their difference $(1-q)(q^5 + q^7 - q^{12} - q^{15} \dots)$ has all the exponents of q greater than or equal to 5, and it remains a Bloch–Pólya type series.

Using (3.4), we get

$$q^3F_2(q) = -1 + 2q + (1-q)(q;q)_\infty.$$

Using (1.1) once again

$$\begin{aligned} q^3F_2(q) &= -1 + 2q + (1-q)(1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} \dots) \\ &= -1 + 2q + (1-q)(1 - q - q^2) + (1-q)(q^5 + q^7 - q^{12} - q^{15} \dots) \\ &= q^3 + (1-q)(q^5 + q^7 - q^{12} - q^{15} \dots). \end{aligned}$$

The above lines with the previous observation show that $q^3F_2(q)$ is of Bloch–Pólya type. One can divide the series expansions of $q^3F_2(q)$ with q^3 to show the claimed results.

(iii) For $k \geq 3$, one gets power series coefficients with modulus ≥ 2 in the expansion of $F_k(q)$, but in the initial cases there are only finitely many exceptions which can be corrected. Using (3.4), we get

$$\begin{aligned} q^6F_3(q) &= q^3 - q(1-q^2) + (q;q)_2 - (q;q)_2(q;q)_\infty \\ &= 1 - 2q - q^2 + 3q^3 - (1 - q - q^2 + q^3)(q;q)_\infty, \\ q^{10}F_4(q) &= q^6 - q^3(1-q^3) + q(1-q^2)(1-q^3) - (q;q)_3 + (q;q)_3(q;q)_\infty \\ &= -1 + 2q + q^2 - 2q^3 - 2q^4 - q^5 + 4q^6 + (1 - q - q^2 + q^4 + q^5 - q^6)(q;q)_\infty. \end{aligned}$$

The q-factorials $(q;q)_2$ and $(q;q)_3$ have degrees 3 and 6, respectively. Differences between the pentagonal numbers are larger than 3 and 6 starting from the pentagonal numbers 22 and 70 by Lemma 2.1. Using Theorem 1.1

and splitting the series at these pentagonal numbers we get

(3.5)
$$q^6F_3(q) = q^6 + q^9 - q^{10} + q^{12} - q^{13} - q^{14} + 2q^{15} - q^{16} - q^{17} + q^{18} - (q;q)_2(q^{22} + q^{26} - q^{35} - q^{40} + q^{51}\dots),$$

(3.6)
$$\begin{aligned} q^{10}F_4(q) =& q^{10} + q^{14} - q^{15} + q^{18} - q^{19} - q^{20} + q^{21} + q^{22} - q^{23} - q^{24} + 2q^{26} - 2q^{28} \\ &+ q^{30} + q^{31} - q^{32} - q^{35} + q^{36} + q^{37} - q^{39} - 2q^{40} + 2q^{41} + q^{42} - q^{44} \\ &- q^{45} + q^{46} + q^{51} - q^{52} - q^{53} + q^{55} + q^{56} - q^{58} - q^{59} + q^{61} + q^{62} - q^{63} \\ &- (q;q)_3(q^{70} + q^{77} - q^{92} - q^{100} + q^{117}\dots). \end{aligned}$$

The fact that $(q;q)_2$ and $(q;q)_3$ are of Bloch–Pólya type ensures that the tail ends of (3.5) and (3.6) are of Bloch–Pólya type. The explicit equations (3.5) and (3.6) are enough to prove the claims. All one needs to do is to divide both sides of these equations with q^6 and q^{10}, respectively, and add in the claimed correction factors.

(iv) The argument for non-zero coefficients being ± 1 for the tail end can be used in the opposite direction as well. By (3.4) we get $q^{15}F_5(q) = \cdots - (q;q)_4(q;q)_\infty$. This implies that

$$F_5(q) = P(q) + (q;q)_4(q^{161} + q^{172} - q^{195} - q^{207}\dots), \tag{3.7}$$

where $P(q)$ is a polynomial of degree 150. Since $(q;q)_4 = 1 - q - q^2 + 2q^5 - q^8 - q^9 + q^{10}$, the tail of (3.7) cannot be of Bloch–Pólya type. Hence, $F_5(q)$ is neither Bloch–Pólya type series nor can it be made to be one by adding a polynomial correction term.

(v) Similar to cases (i)-(iii), as $(q;q)_5$ is a Bloch–Pólya polynomial, one can conclude that $F_6(q)$, subject to a polynomial correction term $f_6(q)$, can be made a Bloch–Pólya type series. More precisely,

$$F_6(q) - f_6(q) = Q(q) + (q;q)_5(q^{355} + q^{371} - q^{404} - q^{421}\dots),$$

where $Q(q)$ is a Bloch–Pólya polynomial of degree 339.

(vi) (Proof of Theorem 1.4.) By Theorem 1.3 we know that $(q;q)_{k-1}$ is not of Bloch–Pólya type for $k \geq 7$. Hence, following the steps of case (iv), the tail of $F_k(q)$ for $k \geq 7$ cannot be of Bloch–Pólya type. That implies that for $k \geq 7$, $F_k(q)$ is neither itself Bloch–Pólya nor can it be corrected to be one by an addition of a polynomial.

□

4. Further observations

Another topic to address is the classification of $(q;q)_m$ polynomials with coefficients from the set $\{-h, -h+1, \dots, h-1, h\}$, for any positive integer h. Let S_h be the set of all the m values such that the coefficients of $(q;q)_m$ lie in between $-h$ and h, where at least one coefficient has the absolute value h. We already proved that $S_1 = \{0, 1, 2, 3, 5\}$ using (2.11). This argument can be repeated to find all S_h for $h \geq 2$. As an example, from (2.9), it is easy to see that for all $m > 116$,

$$3 \leq [\![q^{2m+116}]\!](q;q)_{m-1} \leq 7.$$

Therefore, $m = 116$ is the cut-off point for S_2, and one only needs to check $m \leq 116$ manually to find all m values in S_2. The general formula for the cut-off points for S_h are

$$p_1(2h+5) - 1 = (h+2)(6h+17),$$

where $p_1(n)$ is defined as in (2.1).

We display more sets, which are confirmed, and their related cut-off points in Table 1.

TABLE 1. List of S_h for $h = 1 \dots 40$ with the cut-off values of m.

h	S_h	Cut-off	h	S_h	Cut-off	h	S_h	Cut-off	h	S_h	Cut-off
1	$\{0,1,2,3,5\}$	69	11	$\{23\}$	1079	21	$\{27\}$	3289	31	$\emptyset$	6699
2	$\{4,6,7,8,9,11\}$	116	12	$\emptyset$	1246	22	$\emptyset$	3576	32	$\emptyset$	7106
3	$\{10,13,14\}$	175	13	$\emptyset$	1425	23	$\emptyset$	3875	33	$\emptyset$	7525
4	$\{12,15\}$	246	14	$\emptyset$	1616	24	$\emptyset$	4186	34	$\{30\}$	7956
5	$\{17\}$	329	15	$\emptyset$	1819	25	$\emptyset$	4509	35	$\emptyset$	8399
6	$\{16,18\}$	424	16	$\{24,25\}$	2034	26	$\emptyset$	4844	36	$\emptyset$	8854
7	$\{19\}$	531	17	$\emptyset$	2261	27	$\emptyset$	5191	37	$\emptyset$	9321
8	$\{20,21\}$	650	18	$\emptyset$	2500	28	$\{28\}$	5550	38	$\emptyset$	9800
9	$\emptyset$	781	19	$\{26\}$	2751	29	$\{29\}$	5921	39	$\emptyset$	10291
10	$\{22\}$	924	20	$\emptyset$	3014	30	$\emptyset$	6304	40	$\emptyset$	10794

The data in Table 1 is consistent with the following.

Conjecture 4.1. *Either*

$$S_h = \emptyset \text{ or } S_h = \{i(h)\},$$

for $h > 16$, where $i(h)$ is a positive integer, and

$$i(h_1) > i(h_2) \text{ when } h_1 > h_2 > 16.$$

Moreover, for $h > 5$, the set

$$S_1 \cup S_2 \cup \cdots \cup S_h = \{0, 1, 2, \dots, M(h)\}$$

consists of all consecutive integers from 0 up to some positive $M(h)$.

Similarly, one can also define the set $\hat{S}_h$ for the series $F_k(q)$. Let $\hat{S}_h$ be the set of positive integers k such that $F_k(q)$ has its coefficients from the set $\{-h, \dots, h\}$, where at least one coefficient has the absolute value h. Theorem 3.2 shows that $1, 2 \in \hat{S}_1$ and $3, 4, 6 \in \hat{S}_2$. Moreover, similarly to the proof of Theorem 3.2, using Lemma 2.1, we can easily find the cut-off points, making sure that the pentagonal numbers are farther apart from the degree of $(q;q)_{k-1}$. Using this cut-off point, one can identify which $\hat{S}_h$ set $F_k(q)$ lies in by looking at the initial coefficients of $F_k(q)$ and the coefficients of $(q;q)_{k-1}$. For example, recall (3.7),

$$F_5(q) = P(q) + (q;q)_4(q^{161} + q^{172} - q^{195} - q^{207} \dots),$$

where

$$P(q) = 1 + q^5 + \cdots - 2q^{21} + \cdots + 3q^{30} + \cdots + q^{150}.$$

The polynomial $P(q)$ has all of its coefficients between -2 and 3. There is more than a 10 difference between all the exponents of q with non-zero coefficients in the Bloch–Pólya type series $q^{161} + q^{172} - q^{195} - q^{207} \dots$. The polynomial $(q;q)_4$ has degree 10, and its largest absolute coefficient is 2. Hence, $(q;q)_4(q^{161} + q^{172} - q^{195} - q^{207} \dots)$ is a series with all its coefficients from the set $\{-2, \dots, 2\}$. Comparing the $P(q)$ polynomial and the tail end of $F_5(q)$ we deduce that the maximum absolute coefficient of $F_5(q)$ is 3. Therefore, $5 \in \hat{S}_3$. In general, it is sufficient to check the coefficients of $F_k(q)$ until the exponent

$$p_1(k(k-1)/2+1)-k=\frac{(k-1)(3k^3-3k^2+10k-8)}{8}$$

to classify its respective $\hat{S}_h$ set, where $p_1(n)$ is defined as in (2.1). This bound is used in comparison with the coefficients of $(q;q)_{k-1}$, which appear repeatedly as shifted copies in the tail end of $F_k(q)$, with the initial coefficients of $F_k(q)$.

We give a list of confirmed $\hat{S}_h$ sets in Table 2.

TABLE 2. List of $\hat{S}_h$ for $h = 1\dots 42$.

h	$\hat{S}_h$	h	$\hat{S}_h$	h	$\hat{S}_h$	h	$\hat{S}_h$	h	$\hat{S}_h$	h	$\hat{S}_h$	h	$\hat{S}_h$
1	$\{1,2\}$	7	$\{11,14\}$	13	$\emptyset$	19	$\emptyset$	25	$\emptyset$	31	$\emptyset$	37	$\{25\}$
2	$\{3,4,6\}$	8	$\{13,15\}$	14	$\{18\}$	20	$\emptyset$	26	$\emptyset$	32	$\emptyset$	38	$\emptyset$
3	$\{5,8\}$	9	$\emptyset$	15	$\{19\}$	21	$\emptyset$	27	$\emptyset$	33	$\emptyset$	39	$\emptyset$
4	$\{7,9\}$	10	$\emptyset$	16	$\emptyset$	22	$\emptyset$	28	$\emptyset$	34	$\emptyset$	40	$\emptyset$
5	$\emptyset$	11	$\{16\}$	17	$\{20\}$	23	$\emptyset$	29	$\{23\}$	35	$\emptyset$	41	$\emptyset$
6	$\{10,12\}$	12	$\{17\}$	18	$\{21\}$	24	$\{22\}$	30	$\{24\}$	36	$\emptyset$	42	$\{26\}$

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