

FRACTIONAL DIFFERENTIABILITY FOR SOLUTIONS OF THE INHOMOGENEOUS p -LAPLACE SYSTEM

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(Communicated by Joachim Krieger)

ABSTRACT. It is shown that if $p \geq 3$ and $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ solves the inhomogeneous p -Laplace system

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = f, \quad f \in W^{1,p'}(\Omega, \mathbb{R}^N),$$

then locally the gradient ∇u lies in the fractional Nikol'skiĭ space $\mathcal{N}^{\theta,2/\theta}$ with any $\theta \in [\frac{2}{p}, \frac{2}{p-1})$. To the author's knowledge, this result is new even in the case of p -harmonic functions, slightly improving known $\mathcal{N}^{2/p,p}$ estimates. The method used here is an extension of the one used by A. Cellina in the case $2 \leq p < 3$ to show $W^{1,2}$ regularity.

1. INTRODUCTION

Recall that for $p > 1$, a p -harmonic function is a minimizer of the Dirichlet p -energy functional $\frac{1}{p} \int_{\Omega} |\nabla u|^p$ in the class $W^{1,p}(\Omega)$ with fixed Dirichlet boundary conditions. It is also a solution of the Euler-Lagrange equation $\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$. To the author's knowledge, some of the best known local regularity results for the gradient of a p -harmonic function $u \in W^{1,p}$ are:

- $\nabla u \in C^{0,\alpha}$ for $1 < p < \infty$ (Ural'tseva [18] for $p \geq 2$; see also [6–9, 16, 17]),
- $\nabla u \in W^{1,2}$ for $1 < p < 3 + \frac{2}{n-2}$ (Manfredi, Weitsman [11], see also Cellina [5], Sciunzi [14] for the inhomogeneous equation in the case $2 \leq p < 3$),
- $\nabla u \in \mathcal{N}^{2/p,p}$ for $p \geq 2$ (Mingione [12]).

It is worth noting that most of them were obtained for more general second order operators, non-trivial source terms, or in case of systems of equations. The Nikol'skiĭ space $\mathcal{N}^{\theta,q}$ mentioned in the last result is a variant of fractional Sobolev spaces (see Definition 2.1), and it appears naturally in this context. The main result of this paper holds for solutions of the inhomogeneous p -Laplace system, but to the author's knowledge it is new also in the case of p -harmonic functions.

Received by the editors August 2, 2017, and, in revised form, October 9, 2017.

2010 *Mathematics Subject Classification*. Primary 35B65, 35J92.

Key words and phrases. p -Laplacian, degenerate elliptic systems, fractional order Nikol'skiĭ spaces.

The author's research was supported by the NCN grant no. 2012/05/E/ST1/03232 (years 2013–2017).

Theorem 1.1. *Let $p \geq 3$ and assume that $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ solves the system*

$$(1) \quad \operatorname{div}(|\nabla u|^{p-2} \nabla u^\alpha) = f^\alpha \quad \text{in } \Omega \text{ for } \alpha = 1, \dots, N,$$

where $f \in W^{1,p'}(\Omega, \mathbb{R}^N)$. Then $\nabla u \in \mathcal{N}_{\text{loc}}^{\theta, 2/\theta}(\Omega, \mathbb{R}^{nN})$ for every $\theta \in [\frac{2}{p}, \frac{2}{p-1})$ with

$$\|\nabla u\|_{\mathcal{N}^{\theta, 2/\theta}(\Omega')} \leq C \left(\|u\|_{W^{1,p}(\Omega)} + \|f\|_{W^{1,p'}(\Omega)}^{\frac{1}{p-1}} \right) \quad \text{for } \Omega' \Subset \Omega.$$

Here and in the sequel, the constant C may depend on the domains Ω', Ω , the dimensions n, N , and the parameters p, θ , but not the functions involved.

Remark 1.2. A well-known example (discussed in Section 6) shows the endpoint estimate $\mathcal{N}^{\frac{2}{p-1}, p-1}$ to be sharp: there is a solution of (1) satisfying $\nabla u \in \mathcal{N}^{\frac{2}{p-1}, p-1}$, but $\nabla u \notin \mathcal{N}^{\frac{2}{p-1}, q}$ for $q > p - 1$ and $\nabla u \notin \mathcal{N}^{\theta, p-1}$ for $\theta > \frac{2}{p-1}$.

Remark 1.3. Regularity of the source term f is only needed for the estimate (3). A closer look reveals that for fixed θ it is enough to assume $f \in L^{p'}$ and $\nabla f \in L^r$ with $r = \frac{p}{2p - \frac{2}{\theta} - 1}$. Note that $r \searrow 1$ when $\theta \nearrow \frac{2}{p-1}$, so the assumptions are actually weaker for θ close to optimal. For $\theta = \frac{2}{p}$, optimal assumptions were recently found by Brasco and Santambrogio [4].

Fractional differentiability estimates come from the following elementary observation: if β is θ -Hölder continuous and $V \in W^{1,2}$, then the composition $\beta(V)$ lies in $\mathcal{N}^{\theta, 2/\theta}$ (see Lemma 5.1 for the precise statement).

In this context, we recall a well-known result due to Bojarski and Iwaniec [3]: if $u \in W^{1,p}$ is p -harmonic, then

$$V := |\nabla u|^{\frac{p-2}{2}} \nabla u \in W_{\text{loc}}^{1,2}.$$

One can recover ∇u from V as $\nabla u = \beta(V)$, where $\beta(w) = |w|^{\frac{2}{p}-1} w$ is $\frac{2}{p}$ -Hölder continuous, thus obtaining $\nabla u \in \mathcal{N}_{\text{loc}}^{2/p, p}$ as a corollary. This was shown for a quite general class of systems by Mingione [12]. Note that both proofs [3, 12] rely on testing the equation with the same test function.

Our aim is therefore to obtain $W^{1,2}$ estimates for some nonlinear expressions of the gradient – similar to V , only with smaller exponents. In this way we are able to improve $\mathcal{N}^{2/p, p}$ regularity of the gradient to almost $\mathcal{N}^{\frac{2}{p-1}, p-1}$.

Theorem 1.4. *Let $p \geq 3$ and assume that $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ solves the p -Laplace system (1) with $f \in W^{1,p'}(\Omega, \mathbb{R}^N)$. Then*

$$|\nabla u|^{s-1} \nabla u \in W_{\text{loc}}^{1,2}(\Omega, \mathbb{R}^{nN})$$

for each $\frac{p-1}{2} < s \leq \frac{p}{2}$. Moreover,

$$\| |\nabla u|^{s-1} \nabla u \|_{W^{1,2}(\Omega')} \leq C \left(\|u\|_{W^{1,p}(\Omega)}^s + \|f\|_{W^{1,p'}(\Omega)}^{\frac{s}{p-1}} \right) \quad \text{for } \Omega' \Subset \Omega.$$

Regularity of the same nonlinear expression for values $1 \leq s \leq p - 1$ was studied in a very recent paper by Avelin, Kuusi, and Mingione [2]. Their study concerns the limiting case of equation (1) when the source term f is a measure, so available regularity [2, Th. 1.3] is lower than in our case:

$$|\nabla u|^{s-1} \nabla u \in W_{\text{loc}}^{\sigma \frac{s-1}{p-1}, \frac{p-1}{s}}(\Omega, \mathbb{R}^n) \quad \text{for } 1 \leq s \leq p - 1, \sigma \in (0, 1).$$

The proof of Theorem 1.4 follows roughly by differentiating the p -Laplace system (1) and testing the obtained system with the function $\eta^2|\nabla u|^{2s-p}\nabla u$ (η being a cut-off function). Since this process involves the second order derivatives of u , it cannot be carried out directly. The problem lies in the fact that for $p > 2$ the p -Laplace system (1) is degenerate at points where $\nabla u = 0$. This difficulty is bypassed by approximating u with solutions of some uniformly elliptic systems. For fixed $\varepsilon > 0$ we consider the following approximation of the Dirichlet p -energy functional:

$$F_\varepsilon(u) = \frac{1}{p} \int_\Omega (\varepsilon^2 + |\nabla u|^2)^{p/2} + \int_\Omega \langle u, f_\varepsilon \rangle_{\mathbb{R}^N},$$

where f_ε is a smooth approximation of f . By standard theory, F_ε has a unique smooth minimizer $u_\varepsilon \in u + W_0^{1,p}(\Omega, \mathbb{R}^N)$. Since the elliptic constant vanishes as $\varepsilon \rightarrow 0$, regularity of u_ε might be lost in the limit, so our goal is to obtain estimates similar to those in Theorem 1.4 uniformly in ε (this is done in Lemma 3.2).

The method outlined above is an extension of the one employed by Cellina [5] in the case $2 \leq p < 3$. Indeed, the proof of Theorem 1.4 carries over to this case, leading to the following result.

Theorem 1.5. *Let $2 \leq p < 3$ and assume that $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ solves the p -Laplace system (1) with $f \in W^{1,p'}(\Omega, \mathbb{R}^N)$. Then*

$$|\nabla u|^{s-1}\nabla u \in W_{loc}^{1,2}(\Omega, \mathbb{R}^{nN})$$

for each $1 \leq s \leq \frac{p}{2}$. Moreover,

$$\| |\nabla u|^{s-1}\nabla u \|_{W^{1,2}(\Omega')} \leq C \left(\|u\|_{W^{1,p}(\Omega)}^s + \|f\|_{W^{1,p'}(\Omega)}^{\frac{s}{p-1}} \right) \quad \text{for } \Omega' \Subset \Omega.$$

We can take s equal to 1 in the above theorem, thus recovering the following result due to Cellina [5]. In this case one does not need to use the fractional differentiability lemma (Lemma 5.1).

Corollary 1.6 ([5, Th. 1]). *Let $2 \leq p < 3$ and assume that $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ solves the p -Laplace system (1) with $f \in W^{1,p'}(\Omega, \mathbb{R}^N)$. Then $\nabla u \in W_{loc}^{1,2}(\Omega, \mathbb{R}^{nN})$ with*

$$\|\nabla u\|_{W^{1,2}(\Omega')} \leq C \left(\|u\|_{W^{1,p}(\Omega)} + \|f\|_{W^{1,p'}(\Omega)}^{\frac{1}{p-1}} \right) \quad \text{for } \Omega' \Subset \Omega.$$

For the sake of clarity, the following exposition is restricted to the case $N = 1$, i.e., to the single p -Laplace equation. The general case follows exactly the same lines, but one has to keep track of the additional indices.

2. FRACTIONAL SOBOLEV SPACES

The main result is concerned with the estimates in Nikol'skiĭ spaces [13] (see also [1]), which we now define. Below $\Omega \subseteq \mathbb{R}^n$ is an open domain, and for each $\delta > 0$ we denote $\Omega_\delta = \{x \in \Omega : \mathbf{B}(x, \delta) \subseteq \Omega\}$.

Definition 2.1. Let $u \in L^q(\Omega)$, $\theta \in [0, 1]$. The Nikol'skiĭ seminorm $[u]_{\mathcal{N}^{\theta,q}(\Omega)}$ is defined as the smallest constant A such that

$$\left(\int_{\Omega_{|v|}} |u(x+v) - u(x)|^q \right)^{1/q} \leq A|v|^\theta$$

holds for all vectors $v \in \mathbb{R}^n$ of length $|v| \leq \delta$. The norm in $\mathcal{N}^{\theta,q}(\Omega)$ is

$$\|u\|_{\mathcal{N}^{\theta,q}(\Omega)} := \|u\|_{L^q(\Omega)} + [u]_{\mathcal{N}^{\theta,q}(\Omega)}.$$

Changing the value of $\delta > 0$ amounts to choosing an equivalent norm.

In the context of this paper, only local results are available due to the use of cut-off functions. Therefore we may fix a subdomain $\Omega' \Subset \Omega$, choose $\delta = \text{dist}(\Omega', \partial\Omega)$, and look for estimates of the form

$$\left(\int_{\Omega'} |u(x+v) - u(x)|^q \right)^{1/q} \leq A|v|^\theta \quad \text{for vectors of length } |v| \leq \delta.$$

Note that the seminorms $\mathcal{N}^{1,q}$ and $W^{1,q}$ are equivalent for $q > 1$ due to the difference quotient characterization of Sobolev spaces. This will be exploited in Lemma 5.1. Other basic examples are $\mathcal{N}^{0,q} = L^q$ and $\mathcal{N}^{\theta,\infty} = C^{0,\theta}$. For the sake of comparison, let us also mention the embeddings

$$\mathcal{N}^{\theta+\varepsilon,q}(\Omega) \hookrightarrow W^{\theta,q}(\Omega) \hookrightarrow \mathcal{N}^{\theta,q}(\Omega)$$

valid for any $\varepsilon > 0$ [1, 7.73]. Here $W^{\theta,q}$ stands for the fractional Slobodeckii-Sobolev space.

3. REGULARITY OF NONLINEAR EXPRESSIONS

Let us introduce a slight change of notation. The functions u, f solving the degenerate equation (1) shall be henceforth referred to as u_0, f_0 . For $\varepsilon > 0$ we introduce $u_\varepsilon, f_\varepsilon$ as smooth solutions to a non-degenerate approximate equation.

Since the claim is local, we can assume without loss of regularity that the domain $\Omega \subseteq \mathbb{R}^n$ is bounded. For fixed $\varepsilon > 0$ we consider the following approximations:

$$\begin{aligned} l_\varepsilon(w) &= (\varepsilon^2 + |w|^2)^{1/2} && \text{for } w \in \mathbb{R}^n, \\ L_\varepsilon(w) &= \frac{1}{p} l_\varepsilon(w)^p && \text{for } w \in \mathbb{R}^n, \\ F_\varepsilon(u) &= \int_\Omega L_\varepsilon(\nabla u) + u f_\varepsilon && \text{for } u \in u_0 + W_0^{1,p}(\Omega). \end{aligned}$$

We choose f_ε to be some family of smooth functions such that $f_\varepsilon \rightarrow f_0$ in $W^{1,p'}(\Omega)$. Taking the limit $\varepsilon \rightarrow 0$, one recovers the p -energy F_0 .

We begin by noting some basic properties needed in the sequel.

Lemma 3.1. *For $l_\varepsilon, L_\varepsilon$ defined as above,*

- (a) $\max(\varepsilon, |w|) \leq l_\varepsilon(w) \leq \varepsilon + |w|$,
- (b) $l_\varepsilon(w) \searrow |w|$ as $\varepsilon \searrow 0$,
- (c) L_ε is smooth and

$$\frac{\partial^2 L_\varepsilon}{\partial w_i \partial w_j}(w) = l_\varepsilon(w)^{p-2} \delta_{ij} + (p-2) l_\varepsilon(w)^{p-4} w_i w_j,$$

hence it is uniformly elliptic:

$$(\varepsilon^2 + |w|^2)^{\frac{p-2}{2}} |v|^2 \leq \frac{\partial^2 L_\varepsilon}{\partial w_i \partial w_j}(w) v_i v_j \leq (p-1)(\varepsilon^2 + |w|^2)^{\frac{p-2}{2}} |v|^2$$

holds for any $v, w \in \mathbb{R}^n$.

The straightforward computations behind Lemma 3.1 are omitted; these and later computations can be simplified by noting that

$$\begin{aligned} \frac{\partial l_\varepsilon}{\partial w_i}(w) &= l(w)^{-1}w_i && \text{for } w \in \mathbb{R}^n, \\ \frac{\partial}{\partial x_i}(l_\varepsilon(\nabla u)^s) &= s \cdot l_\varepsilon(\nabla u)^{s-2} \langle \nabla u_{x_i}, \nabla u \rangle && \text{for } u: \Omega \rightarrow \mathbb{R}. \end{aligned}$$

A useful remark here is that the outcome of all computations depends on ε only via the function l_ε , allowing us to show estimates uniform in ε .

The regularity result in Theorem 1.4 shall be first shown for similar nonlinear expressions of the gradients of the approximate solutions. For fixed parameters $s, \varepsilon > 0$ let us introduce the smooth function

$$\alpha_\varepsilon^s: \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad \alpha_\varepsilon^s(w) = l_\varepsilon(w)^{s-1}w.$$

Notice that for $\varepsilon = 0$ we recover the familiar expression $\alpha_0^s(w) = |w|^{s-1}w$ together with its inverse $\alpha_0^{1/s}$.

Lemma 3.2. *Fix a solution $u_0 \in W^{1,p}(\Omega)$ of the equation (1) with $f_0 \in W^{1,p'}(\Omega)$. For each $\varepsilon \in (0, 1)$ the functional F_ε has a unique smooth minimizer $u_\varepsilon \in u_0 + W_0^{1,p}(\Omega)$. Moreover,*

- (a) *the functions u_ε are uniformly bounded in $W^{1,p}(\Omega)$,*
- (b) *for each $\frac{p-1}{2} < s \leq \frac{p}{2}$, the functions $\alpha_\varepsilon^s(\nabla u_\varepsilon)$ are uniformly bounded in $W_{loc}^{1,2}(\Omega)$ with respect to ε , i.e.,*

$$\|\alpha_\varepsilon^s(\nabla u_\varepsilon)\|_{W^{1,2}(\Omega')} \leq C(\Omega, \Omega', n, p, s, \|u_0\|_{W^{1,p}(\Omega)}, \|f_0\|_{W^{1,p'}(\Omega)}) \quad \text{for } \Omega' \Subset \Omega.$$

Proof of part (a). The existence of unique minimizer $u_\varepsilon \in W^{1,p}(\Omega)$ is a standard result, and $C^{1,\alpha}$ regularity was shown by Tolksdorf [15] (also in the case of systems of equations). Since the resulting elliptic equation is non-degenerate, u_ε is smooth by a bootstrap argument (although only C^2 regularity is needed in the sequel).

We turn our attention to the uniform $W^{1,p}$ estimates. First, $u_\varepsilon - u_0 \in W_0^{1,p}(\Omega)$, hence

$$\begin{aligned} \|u_\varepsilon\|_{L^p(\Omega)} &\leq \|u_\varepsilon - u_0\|_{L^p(\Omega)} + \|u_0\|_{L^p(\Omega)} \\ &\leq C\|\nabla u_\varepsilon - \nabla u_0\|_{L^p(\Omega)} + \|u_0\|_{L^p(\Omega)} \\ &\leq C\|\nabla u_\varepsilon\|_{L^p(\Omega)} + C\|u_0\|_{W^{1,p}(\Omega)} \end{aligned}$$

by Poincaré’s inequality; thus we only need to bound $\|\nabla u_\varepsilon\|_{L^p(\Omega)}$. Using the minimality of u_ε and the monotonicity from Lemma 3.1, we obtain the bound

$$\int_\Omega \frac{1}{p} |\nabla u_\varepsilon|^p + u_\varepsilon f_\varepsilon \leq F_\varepsilon(u_\varepsilon) \leq F_\varepsilon(u_0) \leq \int_\Omega \frac{1}{p} (1 + |\nabla u_0|^2)^{p/2} + u_0 f_\varepsilon,$$

which together with the previous one yields a uniform bound for $\|\nabla u_\varepsilon\|_{L^p(\Omega)}$. \square

Part (b) of Lemma 3.2 is the key part of this paper; it will be proved in Section 4. Taking it for granted, we can pass to the limit and prove Theorem 1.4.

Proof of Theorem 1.4. By Lemma 3.2(a) we can choose a sequence $\varepsilon \searrow 0$ such that u_ε converges weakly in $W^{1,p}(\Omega)$ to some \bar{u} ; in particular $\bar{u} = u_0$ on $\partial\Omega$. It also shows that the linear parts of the functionals F_0, F_ε converge:

$$\int_\Omega u_\varepsilon f_0, \int_\Omega u_\varepsilon f_\varepsilon \rightarrow \int_\Omega \bar{u} f_0.$$

As for the nonlinear part, we argue again by minimality and monotonicity:

$$\begin{aligned} F_0(\bar{u}) &\leq \liminf_{\varepsilon \rightarrow 0} F_0(u_\varepsilon) \\ &\leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \\ &\leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_0) \\ &= F_0(u_0). \end{aligned}$$

Recall that the solution u_0 of the p -Laplace system (1) is unique and easily seen to minimize the p -energy F_0 . Hence \bar{u} has to coincide with u_0 as another minimizer of F_0 .

After fixing $\Omega' \Subset \Omega$, we use Lemma 3.2(b) in a similar way, obtaining $\alpha_\varepsilon^s(\nabla u_\varepsilon) \rightarrow \bar{\alpha}$ weakly in $W^{1,2}(\Omega')$ and a.e. We can assume that $\bar{\alpha} = \alpha_0^s(v)$ for some vector field v , as α_0^s is invertible. An elementary pointwise reasoning shows that the convergence $\alpha_\varepsilon^s(\nabla u_\varepsilon) \rightarrow \alpha_0^s(v)$ leads to $\nabla u_\varepsilon \rightarrow v$ a.e. Combining this with weak convergence $\nabla u_\varepsilon \rightarrow \nabla u_0$, we infer that $v = \nabla u_0$ and $\bar{\alpha} = \alpha_0^s(\nabla u_0)$, in consequence

$$\|\alpha_0^s(\nabla u_0)\|_{W^{1,2}(\Omega')} \leq C(\Omega, \Omega', n, p, s, \|u_0\|_{W^{1,p}(\Omega)}, \|f_0\|_{W^{1,p'}(\Omega)}).$$

To show that the constant has the desired form, we note the scaling properties of the p -Laplace system (1). For each $\lambda > 0$, the functions λu_0 and $\lambda^{p-1} f_0$ also solve (1); let us choose λ small enough so that their norms do not exceed 1. Then by the above discussion $\|\alpha_0^s(\nabla(\lambda u_0))\|_{W^{1,2}(\Omega')} \leq C$, where C is independent of the functions involved. Since α_0^s is s -homogeneous, this yields $\|\alpha_0^s(\nabla u_0)\|_{W^{1,2}(\Omega')} \leq C\lambda^{-s}$, which is equivalent to our claim. \square

4. A PRIORI ESTIMATES

Throughout this section, the value of $\varepsilon > 0$ is fixed and the subscript ε is omitted in $u_\varepsilon, f_\varepsilon, l_\varepsilon, L_\varepsilon, F_\varepsilon, \alpha_\varepsilon^s$.

Proof of Lemma 3.2(b). Fix the subdomain $\Omega' \Subset \Omega$ and a cut-off function $\eta \in C_c^\infty(\Omega)$ such that $\eta \equiv 1$ on Ω' and $\eta \geq 0$. Choose the parameter $\frac{p-1}{2} < s \leq \frac{p}{2}$ and additionally denote $q = p - 2s + 2$, thus $2 \leq q < 3$.

Since u is a smooth minimizer of F , it satisfies the Euler-Lagrange equation $\operatorname{div}(\nabla L(\nabla u)) = f$ and also the differentiated system

$$\operatorname{div}(D^2 L(\nabla u) \nabla u_{x_j}) = f_{x_j} \quad \text{for } j = 1, 2, \dots, n.$$

This system can be tested with the vector-valued function $\gamma = l(\nabla u)^{2-q} \nabla u$ multiplied by the cut-off function η^2 , resulting in

$$\begin{aligned} \int_\Omega \sum_{j=1}^n \eta^2 \langle D^2 L(\nabla u) \nabla u_{x_j}, \nabla \gamma^j \rangle &= - \int_\Omega \sum_{j=1}^n \gamma^j \langle D^2 L(\nabla u) \nabla u_{x_j}, \nabla \eta^2 \rangle \\ &\quad - \int_\Omega \sum_{j=1}^n \eta^2 \gamma^j f_{x_j}. \end{aligned}$$

Let us denote the integrands above by **I**, **II**, **III**.

The estimate for the left-hand side is crucial. A straightforward calculation based on Lemma 3.1 leads to

$$\begin{aligned} (D^2L(\nabla u)\nabla u_{x_j})^i &= l(\nabla u)^{p-2}u_{x_ix_j} + (p-2)l(\nabla u)^{p-4}\langle \nabla u_{x_j}, \nabla u \rangle u_{x_i}, \\ \frac{\partial}{\partial x_i}\gamma^j &= l(\nabla u)^{2-q}u_{x_ix_j} + (2-q)l(\nabla u)^{-q}\langle \nabla u_{x_i}, \nabla u \rangle u_{x_j}, \end{aligned}$$

which gives us

$$\begin{aligned} \mathbf{I} &= \eta^2 l(\nabla u)^{p-q} \left(|D^2u|^2 + (p-q) \left| D^2u \cdot \frac{\nabla u}{l(\nabla u)} \right|^2 \right. \\ &\quad \left. - (p-2)(q-2) \left| \left\langle D^2u \cdot \frac{\nabla u}{l(\nabla u)}, \frac{\nabla u}{l(\nabla u)} \right\rangle \right|^2 \right) \\ &\geq \min(1, (p-1)(3-q)) \cdot \eta^2 l(\nabla u)^{p-q} |D^2u|^2. \end{aligned}$$

In the last line we used the inequality $|\nabla u| \leq l(\nabla u)$, the Cauchy-Schwarz inequality

$$|\langle D^2u \cdot v, v \rangle| \leq |D^2u \cdot v| \leq |D^2u| \quad \text{for any vector } |v| \leq 1,$$

and our choice of q :

$$1 + (p-q) - (p-2)(q-2) = (p-1)(3-q) > 0.$$

The right-hand side is estimated in the standard way using Young's inequality:

$$\begin{aligned} \mathbf{II} &\leq \sum_{j=1}^n |\gamma^j| \cdot |D^2L(\nabla u)\nabla u_{x_j}| \cdot |\nabla \eta^2| \\ &\leq C(\Omega, \Omega', n, p, q) \cdot \eta l(\nabla u)^{p-q+1} |D^2u| \\ &\leq \delta C \cdot \eta^2 l(\nabla u)^{p-q} |D^2u|^2 + \frac{1}{\delta} C \cdot l(\nabla u)^{p-q+2}. \end{aligned}$$

For small enough $\delta > 0$, the first term can be absorbed by the left-hand side, and the second is bounded using Hölder's inequality

$$(2) \quad \int_{\Omega} l(\nabla u)^{p-q+2} \leq |\Omega|^{\frac{q-2}{p}} \left(\int_{\Omega} l(\nabla u)^p \right)^{\frac{p-q+2}{p}} \leq C(\Omega, p, q, \|u_0\|_{W^{1,p}(\Omega)}),$$

where the second inequality above was shown in the proof of Lemma 3.2(a). The last term is similar:

$$(3) \quad \mathbf{III} \leq \left(\int_{\Omega} l(\nabla u)^{p(3-q)} \right)^{1/p} \|\nabla f\|_{L^{p'}(\Omega)} \leq C(\Omega, p, q, \|u_0\|_{W^{1,p}(\Omega)}, \|f_0\|_{W^{1,p'}(\Omega)}).$$

Note that one could apply Hölder's inequality with exponents $(\frac{p}{3-q}, \frac{p}{p+q-3})$ instead of (p, p') , thus using weaker estimates on f_0 .

Recalling $\eta \equiv 1$ on Ω' , we can summarize these estimates with

$$(4) \quad \int_{\Omega'} l(\nabla u)^{p-q} |D^2u|^2 \leq C(\Omega, \Omega', n, p, q, \|u_0\|_{W^{1,p}(\Omega)}, \|f_0\|_{W^{1,p'}(\Omega)}),$$

where the constant may depend on everything except ε .

The function $V := \alpha^s(\nabla u) = l(\nabla u)^{s-1}\nabla u$ is smooth as a composition of smooth functions. Since $|V| \leq l(\nabla u)^s$, the $L^2(\Omega)$ -norm of V has been estimated in (2). Similarly, $|\nabla V| \leq C(n, s)l(\nabla u)^{s-1}|D^2u|$; hence (4) gives a bound on $\|\nabla V\|_{L^2(\Omega')}$ and finishes the proof. \square

5. FRACTIONAL DIFFERENTIABILITY

Lemma 5.1 (Fractional differentiability lemma). *Assume that $\beta: \mathbb{R}^k \rightarrow \mathbb{R}^k$ is Hölder continuous with exponent $\theta \in (0, 1)$ and constant $M > 0$, i.e.,*

$$|\beta(w) - \beta(v)| \leq M|w - v|^\theta \quad \text{for } w, v \in \mathbb{R}^k.$$

If $V \in W^{1,2}(\Omega, \mathbb{R}^k)$ and $\Omega' \Subset \Omega$, then $\beta(V) \in \mathcal{N}^{\theta, 2/\theta}(\Omega', \mathbb{R}^k)$ with

$$[\beta(V)]_{\mathcal{N}^{\theta, 2/\theta}(\Omega')} \leq C(n)M[V]_{W^{1,2}(\Omega)}^\theta.$$

Proof. Choose $\delta = \text{dist}(\Omega', \partial\Omega)$ and fix some vector $v \in \mathbb{R}^n$ of length $|v| \leq \delta$. For any $x \in \Omega'$,

$$\begin{aligned} |\beta(V(x+v)) - \beta(V(x))|^{2/\theta} &\leq (M|V(x+v) - V(x)|^\theta)^{2/\theta} \\ &= M^{2/\theta}|V(x+v) - V(x)|^2. \end{aligned}$$

Integrating the above over Ω' yields

$$\begin{aligned} \int_{\Omega'} |\beta(V(x+v)) - \beta(V(x))|^{2/\theta} &\leq M^{2/\theta} \int_{\Omega'} |V(x+v) - V(x)|^2 \\ &\leq C(n)M^{2/\theta}[V]_{W^{1,2}(\Omega)}^2|v|^2. \end{aligned}$$

□

Proof of Theorem 1.1. Choose $\theta \in [\frac{2}{p}, \frac{2}{p-1})$ and $s = 1/\theta$. Then by Theorem 1.4 $V = |\nabla u_0|^{s-1} \nabla u_0 \in W_{\text{loc}}^{1,2}(\Omega)$. We introduce the function

$$\beta: \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad \beta(w) = |w|^{\theta-1}w,$$

so that $u_0 = \beta(V)$. To see that it is Hölder continuous, consider its inverse – the elementary inequality (see e.g. [10, Ch. 10])

$$\langle |w|^{s-1}w - |v|^{s-1}v, w - v \rangle \geq \frac{1}{2} (|w|^{s-1} + |v|^{s-1}) |w - v|^2$$

implies that $\||w|^{s-1}w - |v|^{s-1}v\| \geq 2^{-s}|w - v|^s$ and in consequence β is θ -Hölder continuous with constant 2. By Lemma 5.1, this implies $\nabla u_0 \in \mathcal{N}_{\text{loc}}^{\theta, 2/\theta}(\Omega)$ together with the desired estimates. □

6. SHARPNESS OF THE ESTIMATES

Let $p \geq 3$ and $u(x_1, \dots, x_n) = \frac{1}{p'}|x_1|^{p'}$. It is easily seen that u solves the inhomogeneous p -Laplace equation $\text{div}(|\nabla u|^{p-2}\nabla u) = 1$ in \mathbb{R}^n . For fixed $q \in [1, \infty]$ we can find the largest $\theta > 0$ for which $u \in \mathcal{N}^{\theta, q}(\mathbf{B}(0, 1))$, arriving at

$$\nabla u \in \begin{cases} C^{0, \frac{1}{p-1}} & \text{for } q = \infty, \\ \mathcal{N}^{\theta, q}, \theta = \frac{1}{p-1} + \frac{1}{q} & \text{for } \frac{p-1}{p-2} < q < \infty, \\ W^{1, q} & \text{for } 1 \leq q < \frac{p-1}{p-2}. \end{cases}$$

As a special case, we note that $\nabla u \in \mathcal{N}^{\frac{2}{p-1}, p-1}$ for $p > 3$ (but not for $p = 3$). Moreover, this is optimal in the sense that $\nabla u \notin \mathcal{N}^{\frac{2}{p-1}, q}$ for $q > p - 1$ and $\nabla u \notin \mathcal{N}^{\theta, p-1}$ for $\theta > \frac{2}{p-1}$.

It is natural to ask whether the claim of Theorem 1.1 can be strengthened to cover the endpoint case $\theta = \frac{2}{p-1}$ for $p > 3$. However, in view of this example one cannot hope for more regularity.

ACKNOWLEDGMENTS

The author would like to thank Anna Zatorska-Goldstein and Paweł Strzelecki for their support and helpful suggestions, and Michał Łasica for bringing [5] and [11] to the author's attention.

REFERENCES

- [1] Robert A. Adams, *Sobolev spaces*, Pure and Applied Mathematics, Vol. 65, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. MR0450957
- [2] Benny Avelin, Tuomo Kuusi, and Giuseppe Mingione, *Nonlinear Calderón-Zygmund theory in the limiting case*, Arch. Ration. Mech. Anal. **227** (2018), no. 2, 663–714. MR3740385
- [3] B. Bojarski and T. Iwaniec, *p -harmonic equation and quasiregular mappings*, Partial differential equations (Warsaw, 1984), Banach Center Publ., vol. 19, PWN, Warsaw, 1987, pp. 25–38. MR1055157
- [4] L. Brasco and F. Santambrogio, *A sharp estimate à la Calderón-Zygmund for the p -Laplacian*, Commun. Contemp. Math. (2017).
- [5] Arrigo Cellina, *The regularity of solutions to some variational problems, including the p -Laplace equation for $2 \leq p < 3$* , ESAIM Control Optim. Calc. Var. **23** (2017), no. 4, 1543–1553. MR3716932
- [6] E. DiBenedetto, *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal. **7** (1983), no. 8, 827–850. MR709038
- [7] Lawrence C. Evans, *A new proof of local $C^{1,\alpha}$ regularity for solutions of certain degenerate elliptic p.d.e.*, J. Differential Equations **45** (1982), no. 3, 356–373. MR672713
- [8] Olga A. Ladyzhenskaya and Nina N. Ural'tseva, *Linear and quasilinear elliptic equations*, Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis, Academic Press, New York-London, 1968. MR0244627
- [9] John L. Lewis, *Regularity of the derivatives of solutions to certain degenerate elliptic equations*, Indiana Univ. Math. J. **32** (1983), no. 6, 849–858. MR721568
- [10] Peter Lindqvist, *Notes on the p -Laplace equation*, Report. University of Jyväskylä Department of Mathematics and Statistics, vol. 102, University of Jyväskylä, Jyväskylä, 2006. MR2242021
- [11] Juan J. Manfredi and Allen Weitsman, *On the Fatou theorem for p -harmonic functions*, Comm. Partial Differential Equations **13** (1988), no. 6, 651–668. MR934377
- [12] Giuseppe Mingione, *The singular set of solutions to non-differentiable elliptic systems*, Arch. Ration. Mech. Anal. **166** (2003), no. 4, 287–301. MR1961442
- [13] S. M. Nikol'skiĭ, *Approximation of functions of several variables and imbedding theorems*, translated by Jr.; Die Grundlehren der Mathematischen Wissenschaften translated from the Russian by John M. Danskin Band 205, Springer-Verlag, New York-Heidelberg., 1975. MR0374877
- [14] Berardino Sciunzi, *Regularity and comparison principles for p -Laplace equations with vanishing source term*, Commun. Contemp. Math. **16** (2014), no. 6, 1450013, 20. MR3277953
- [15] Peter Tolksdorf, *Everywhere-regularity for some quasilinear systems with a lack of ellipticity*, Ann. Mat. Pura Appl. (4) **134** (1983), 241–266. MR736742
- [16] Peter Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations **51** (1984), no. 1, 126–150. MR727034
- [17] K. Uhlenbeck, *Regularity for a class of non-linear elliptic systems*, Acta Math. **138** (1977), no. 3-4, 219–240. MR0474389
- [18] N. N. Ural'ceva, *Degenerate quasilinear elliptic systems* (Russian), Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **7** (1968), 184–222. MR0244628

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