PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 8, August 2018, Pages 3295–3305
http://dx.doi.org/10.1090/proc/14030
Article electronically published on April 18, 2018

INVERSE VALUES OF THE MODULAR j-INVARIANT AND HOMOTOPY LIE THEORY

KWANG HYUN KIM, YESULE KIM, AND JEEHOON PARK

(Communicated by Matthew A. Papanikolas)

Abstract. The goal of this article is to give a simple arithmetic application of the enhanced homotopy (Lie) theory for algebraic varieties developed by the second and third authors. Namely, we compute an inverse value of the modular j-invariant by using a deformation theory for period matrices of elliptic curves based on homotopy Lie theory. Another key ingredient in our approach is J. Carlson and P. Griffiths' explicit computation regarding infinitesimal variations of Hodge structures.

1. Introduction

It is known that the inverse function of the modular j-invariant can be expressed in terms of the hypergeometric functions (for example, see [1]). In fact, for a given number $N \in \mathbb{C}$, solving the equation $j(\tau) = N$ for τ (unique to up $\mathrm{SL}_2(\mathbb{Z})$-action) can be done in several ways.

In this paper, we give another method to compute the inverse value of the modular j-invariant using a modern deformation theory based on (shifted) differential graded Lie algebras (homotopy Lie algebras) and the Maurer–Cartan equations, which was developed in [5] and [4]. We briefly explain our method of computation in the introduction.

The starting point is to know a particular value of the modular j-function. For example, we know $j(e^{2\pi i/3}) = 0$ and $j(i) = 1728$. Since the elliptic curves over $\mathbb{C}$ are classified by the j-invariant, there is a unique (up to isomorphism) elliptic curve E whose j-invariant is 1728. Let us choose an affine Weierstrass equation of E, namely,

$$E : y^2 - 4x^3 + x = 0. \tag{1}$$

Since $j(i) = 1728$, the complex uniformization theorem for elliptic curves says that there is a symplectic integral homology basis $a, b \in H_1(E(\mathbb{C}), \mathbb{Z})$ such that the 1×2 period matrix of E has the form

$$\left(\int_a \omega \quad \int_b \omega\right) = \left(\int_a \omega \quad i\int_a \omega\right), \tag{2}$$

where ω is the algebraic invariant holomorphic differential 1-form on $E(\mathbb{C})$, which is given by $\omega = \frac{dx}{y} = \frac{dx}{\sqrt{4x^3 - x}}$ in terms of affine coordinates. We fix such an integral

Received by the editors March 1, 2017, and, in revised form, November 10, 2017.
2010 *Mathematics Subject Classification.* Primary 11F03, 11Y99, 13D10; Secondary 32G20.
Key words and phrases. Modular j-invariant, homotopy Lie theory, period integrals of elliptic curves, differential graded Lie algebra.
Jeehoon Park was supported by Samsung Science & Technology Foundation (SSTF-BA1502).

basis a, b throughout the article. Let $\varepsilon := \int_a \omega$. Then the explicit value of ε is given in [9, p. 444],

$$\varepsilon := 2\int_{\frac{1}{2}}^{\infty} \frac{dx}{\sqrt{4x^3 - x}} = \frac{\Gamma(\frac{1}{4})^2}{2\sqrt{\pi}} = 3.7081\ldots. \tag{3}$$

Now, for a given nonzero complex number N, we like to compute τ such that $j(\tau) = N$. For such a purpose, observe that the elliptic curve E_B defined by the affine Weierstrass equation

$$E_B : y^2 - 4x^3 + x + B = 0, \quad B \in \mathbb{C}, \tag{4}$$

has j-invariant

$$1728\frac{1}{1 - 27B^2}.$$

One can find a complex number B such that $N = 1728\frac{1}{1-27B^2}$. Note that $E_B(\mathbb{C})$ is topologically isomorphic to $E(\mathbb{C})$ and we fix an (orientation preserving) isomorphism $H_1(E(\mathbb{C}), \mathbb{Z}) \simeq H_1(E_B(\mathbb{C}), \mathbb{Z})$ once and for all. This gives a symplectic integral basis a_B, b_B of $H_1(E_B(\mathbb{C}), \mathbb{Z})$ such that

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} \rho_1 & \rho_2 \\ \rho_3 & \rho_4 \end{pmatrix} = \begin{pmatrix} a_B & b_B \end{pmatrix}, \tag{5}$$

where $\left(\begin{smallmatrix} \rho_1 & \rho_2 \\ \rho_3 & \rho_4 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Let ω_B be the algebraic invariant holomorphic differential 1-form on $E_B(\mathbb{C})$, which is again given by $\omega_B = \frac{dx}{y} = \frac{dx}{\sqrt{4x^3-x-B}}$. Then $j(\tau) = N$ implies that the period matrix of the elliptic curve E_B,

$$\begin{pmatrix} \tau_1 & \tau_1' \end{pmatrix} := \left(\int_{a_B} \omega_B \quad \int_{b_B} \omega_B \right), \tag{6}$$

satisfies that $\tau = \frac{\tau_1'}{\tau_1}$ (or $\frac{\tau_1}{\tau_1'}$). The main theorem of [4], based on a deformation theory of DGLA (differential graded Lie algebra), says that there is an explicit algorithm to compute $\begin{pmatrix} \tau_1 & \tau_1' \end{pmatrix}$ from $\begin{pmatrix} \varepsilon & i\varepsilon \end{pmatrix} = \left(\int_a \omega \quad \int_b \omega \right)$, if we know an explicit algebraic recipe to express the antiholomorphic differential 1-form $\overline{\omega}$ on $E(\mathbb{C})$, where $\bar{\cdot}$ is the complex conjugation. We will provide such an explicit recipe (see 22) using the computations of Carlson and Griffiths, [2], regarding infinitesimal variations of Hodge structures. More precisely, there is an algorithm to compute a 2×2 matrix $\mathcal{M}$ such that[1]

$$\begin{pmatrix} \tau_1 & \tau_1' \\ \tau_2 & \tau_2' \end{pmatrix} = \mathcal{M} \cdot \begin{pmatrix} \int_a \omega & \int_b \omega \\ \int_a \overline{\omega} & \int_b \overline{\omega} \end{pmatrix} \cdot \begin{pmatrix} \rho_1 & \rho_2 \\ \rho_3 & \rho_4 \end{pmatrix} = \mathcal{M} \cdot \begin{pmatrix} \varepsilon & i\varepsilon \\ \overline{\varepsilon} & \overline{i\varepsilon} \end{pmatrix} \cdot \begin{pmatrix} \rho_1 & \rho_2 \\ \rho_3 & \rho_4 \end{pmatrix}. \tag{7}$$

Therefore this gives an explicit algorithm to compute an inverse value $\tau = \frac{\tau_1'}{\tau_1}$ (or $\frac{\tau_1}{\tau_1'}$, depending on which expression belongs to the upper half-plane) such that $j(\tau) = N$ from the period matrix $\begin{pmatrix} \varepsilon & i\varepsilon \end{pmatrix}$ of the simple elliptic curve E.

Now we briefly indicate the contents of the paper. In Section 2, we review basic facts on j-invariants and elliptic curves. In Section 3, we explain in detail the algorithm of how to find the matrix $\mathcal{M}$ (in the case of elliptic curves) in (7), which was developed in [5] and [4] in the case of smooth projective complete intersection varieties. In Section 4, we briefly recall Carlson and Griffiths' computation in [2]

[1]Note that $\tau_2 = \int_{a_B} \eta, \tau_2' = \int_{b_B} \eta$ for some antiholomorphic 1-form η. But the algorithm in [4] does not guarantee that this 1-form η is equal to $\overline{\omega_B}$. In fact, η depends on the deformation data $B \in \mathbb{C}$. Also note that τ_1 and τ_1' are more important if we are interested in the inverse value of the j-function.

and how we use it here. In the final Section 5, we summarise all the previous results to give an algorithm to solve $j(\tau) = N$.

2. Elliptic curves and the j-invariant

Let $\mathcal{H}$ be an upper half-plane on which the full modular group $\mathrm{SL}_2(\mathbb{Z})$ acts by the linear fractional transformation:

$$\gamma \cdot \tau = \frac{a\tau + b}{c\tau + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

The modular j-function

$$\begin{aligned} &j(\tau) := 1728 \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27 g_3(\tau)^2}, \\ &g_2(\tau) := 60 \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m+n\tau)^4}, \quad g_3(\tau) := 140 \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m+n\tau)^6} \end{aligned} \tag{8}$$

is a holomorphic function from $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ to $\mathbb{C}$. It is well known that the j-function classifies the complex structures of the compact smooth surface of genus 1. We briefly review how it classifies, since it is relevant for our later discussion; we refer to chapter 1 of [8] for details.

Let $\mathrm{ELL}_{\mathbb{C}}$ be the set of $\mathbb{C}$-isomorphism classes of elliptic curves over $\mathbb{C}$. Let $\{E\} \in \mathrm{ELL}_{\mathbb{C}}$ be an isomorphism class of elliptic curves, and choose a Weierstrass equation (for an affine part), which exists by the uniformization theorem for elliptic curves over $\mathbb{C}$ ([8, Corollary 4.3, Ch. 1]),

$$E : y^2 = 4x^3 + Ax + B$$

for some curve E in this class. Then $\frac{dx}{y}$ becomes an invariant holomorphic differential 1-form on $E(\mathbb{C})$. Take a symplectic integral basis a_E, b_E for the homology group $H_1(E(\mathbb{C}), \mathbb{Z})$ and compute the periods

$$p_1 = \int_{a_E} \frac{dx}{y}, \quad p_2 = \int_{b_E} \frac{dx}{y}. \tag{9}$$

Switching p_1 and p_2 if necessary, we may assume that

$$\tau_E = \frac{p_1}{p_2} \in \mathcal{H}.$$

Then evaluate $j(\tau)$ at $\tau = \tau_E$. The [8, Proposition 4.4, Ch. 1], implies that the above map $\{E\} \mapsto j(\tau_E)$ is bijective and factors through $j : \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H} \to \mathbb{C}$.

Let L_{τ_E} be the lattice of $\mathbb{C}$ generated by τ_E and 1. The uniformization theorem says that the map

$$\begin{aligned} \Phi : \mathbb{C}/L_{\tau_E} &\to E(\mathbb{C}), \\ z &\mapsto [\mathcal{P}(z, L_{\tau_E}), \mathcal{P}'(z, L_{\tau_E}), 1], \quad \text{for } z \neq 0, \\ 0 &\mapsto [0, 1, 0], \end{aligned} \tag{10}$$

where $\mathcal{P}(z, L_{\tau_E})$ is the Weierstrass $\mathcal{P}$-function, induces an isomorphism of complex Lie groups. Note that the holomorphic differential $\frac{dx}{y}$ pulls back to dz on $\mathbb{C}/L_{\tau_E}$, i.e., $\Phi^*(\frac{dx}{y}) = dz$.

3. Deformation of period matrices

In this section, let $G(\underline{x})$ be an arbitrary homogeneous polynomial of degree 3, which defines an elliptic curve E_G inside the 2-projective space $\mathbb{P}^2$. Let $X = E_G(\mathbb{C})$ and let $\mathbf{P} = \mathbb{P}^2(\mathbb{C})$. Let $\mathbb{H} = H^1_{dR}(X/\mathbb{C})$. In [5], the authors constructed a DGBV (Differential Gerstenhaber Batalin-Vilkovisky) algebra $(\mathcal{A}_X, Q_X, K_X)$ which computes $H^2_{dR}(\mathbf{P}\setminus X)$, and hence $\mathbb{H}$ via the residue isomorphism res. See [6] for details on the residue isomorphism,

$$\mathrm{res}_X : H^2_{dR}(\mathbf{P}\setminus X) \simeq \mathbb{H}.$$

More precisely, it was shown that the 0th cohomology $H^0(\mathcal{A}_X, K_X)$ of $(\mathcal{A}_X, K_X)$ is isomorphic to $\mathbb{H}$.

Let us briefly recall the construction of $(\mathcal{A}_X, Q_X, K_X)$; see [5] for details. We denote the homogeneous coordinate ring of $\mathbf{P}$ by $\mathbb{C}[\underline{x}] = \mathbb{C}[x_0, x_1, x_2]$. Introduce a formal variable y and consider a commutative ring $A = \mathbb{C}[y, x_0, x_1, x_2]$. For notational convenience, let $y_{-1} = y, y_0 = x_0, y_1 = x_1, y_2 = x_2$, and $A = \mathbb{C}[\underline{y}]$. Then the $\mathbb{Z}$-graded supercommutative algebra $\mathcal{A}_X$ and differentials K_X and Q_X are given explicitly as follows:

$$\begin{aligned}
\mathcal{A}_X &:= \mathbb{C}[\underline{y}][\underline{\eta}] = \mathbb{C}[y_{-1}, y_0, y_1, y_2][\eta_{-1}, \eta_0, \eta_1, \eta_2],\\
K_X &:= \sum_{i=-1}^{2} \left(\frac{\partial (yG(\underline{x}))}{\partial y_i} + \frac{\partial}{\partial y_i} \right) \frac{\partial}{\partial \eta_i} : \mathcal{A}_X \to \mathcal{A}_X,\\
Q_X &:= \sum_{i=-1}^{2} \frac{\partial (yG(\underline{x}))}{\partial y_i} \frac{\partial}{\partial \eta_i} : \mathcal{A}_X \to \mathcal{A}_X.
\end{aligned}$$

The $\mathbb{Z}$-grading of $\mathcal{A}_X$ is given by the rules

$$|y_i| = 0, |\eta_i| = -1, \quad i = -1, \ldots, 2, \tag{11}$$

and we thus have the cochain complex

$$0 \to \mathcal{A}_X^{-4} \xrightarrow{K_X} \mathcal{A}_X^{-3} \xrightarrow{K_X} \cdots \xrightarrow{K_X} \mathcal{A}_X^{0} = A \to 0.$$

The supercommutativity means that $a \cdot b = (-1)^{|a||b|} b \cdot a$ for homogeneous elements a, b. Hence we see that $\eta_i \cdot \eta_j = -\eta_j \cdot \eta_i$, which implies that $\eta_i^2 = 0$ and $\mathcal{A}_X^{-5} = \mathcal{A}_X^{-6} = \cdots 0$. Since $\frac{\partial (yG(\underline{x}))}{\partial y_i} \frac{\partial}{\partial \eta_i}$ is a differential operator of order 1, the differential Q_X is a derivation of the product of $\mathcal{A}$. Thus $(\mathcal{A}_X, \cdot, Q_X)$ is a CDGA (commutative differential graded algebra). But K_X is *not* a derivation of the product, because the differential operator $\frac{\partial}{\partial y_i} \frac{\partial}{\partial \eta_i}$ has order 2. We also introduce the $\mathbb{C}$-linear map

$$\Delta := K_X - Q_X = \sum_{i=-1}^{2} \frac{\partial}{\partial y_i} \frac{\partial}{\partial \eta_i} : \mathcal{A}_X \to \mathcal{A}_X.$$

Note that Δ is also a differential of degree 1 (Q_X and K_X also have degree 1), i.e., $\Delta^2 = 0$. Therefore we have

$$\Delta Q_X + Q_X \Delta = 0.$$

The DGBV algebra provides us a DGLA (differential graded Lie algebra): the triple $(\mathcal{A}_X, K_X, \ell_2^K)$, where $\ell_2^K(a, b) := K(a \cdot b) - K(a) \cdot b - (-1)^{|a|} a \cdot K(b)$, is a

DGLA, i.e.,

$$\begin{aligned} \ell_2^K(a,b) - (-1)^{|a||b|}\ell_2^K(b,a) &= 0, \\ \ell_2^K(a,\ell_2^K(b,c)) + (-1)^{|a|}\ell_2^K(\ell_2^K(a,b),c) + (-1)^{(|a|+1)|b|}\ell_2^K(b,\ell_2^K(a,c)) &= 0, \\ K\ell_2^K(a,b) + \ell_2^K(Ka,b) + (-1)^{|a|}\ell_2^K(a,Kb) &= 0, \end{aligned}$$

for any homogeneous elements $a, b \in \mathcal{A}$. This DGLA $(\mathcal{A}_X, K_X, \ell_2^K)$ provides us with a deformation functor which is given by solving the Maurer–Cartan equations. In [4], the authors used this deformation functor to find an explicit formula for the period matrices of deformations of X from the period matrix of X. In summary, we have the following theorem.

Theorem 3.1. *The triple $(\mathcal{A}_X, K_X, Q_X)$ is a DGBV algebra. If we define $J(y^k h(\underline{x}))$ $:= (-1)^k k! \frac{h(\underline{x})}{G(\underline{x})^{k+1}}\Omega_x$ for a nonnegative integer k, then J induces an isomorphism*

$$H^0(\mathcal{A}_X, K_X) \xrightarrow{J} H^2_{dR}(\mathbf{P} \setminus X),$$

and consequently $\mathrm{res}_X \circ J$ *gives an isomorphism* $H^0(\mathcal{A}_X, K_X) \simeq \mathbb{H}$.

Let $\{e_\beta\}_{\beta \in I}$ be a basis of $\mathbb{H}$. Let $\{\gamma_\alpha\}_{\alpha \in I}$ be a basis of $H_1(X, \mathbb{C})$ by noting that

$$\dim_{\mathbb{C}} H_1(X, \mathbb{C}) = \dim_{\mathbb{C}} \mathbb{H}.$$

Let $\Omega(X) = \left(\Omega_\beta^\alpha(X)\right)$ be the matrix of X whose (β, α)-entry is given by

$$\Omega_\beta^\alpha(X) := \int_{\gamma_\alpha} e_\beta, \quad \alpha, \beta \in I,$$

where I is an index set for the dimension of $\mathbb{H}$. Let $E_U \subset \mathbb{P}^2$ be another elliptic curve which is deformed from E_G by a nonzero homogeneous polynomial $H(\underline{x})$ of degree 3, i.e., $U(\underline{x}) = G(\underline{x}) + H(\underline{x})$ is the defining equation for E_U. Let $X_U = E_U(\mathbb{C})$. Note that $\mathbb{H}$ has Hodge decomposition $\mathbb{H} = \mathbb{H}^{1,0} \oplus \mathbb{H}^{0,1}$ and the size of I is 2. Let $I = \{\alpha_1, \alpha_2\}$. Thus both $\Omega(X_U)$ and $\Omega(X)$ are 2×2 matrices. We introduce a set of formal variables $\{s, t\} = \{t^{\alpha_1}, t^{\alpha_2}\} = \{\underline{t}\}$ indexed by I. The main theorem of [4] applied to elliptic curves gives the following theorem.

Theorem 3.2. *There is an algorithmic construction of a power series $T^\rho(s,t) \in \mathbb{C}[[s,t]]$ such that*

$$\Omega(X_U) = \mathcal{M} \cdot \Omega(X), \tag{12}$$

where $\mathcal{M}$ is the 2×2 matrix whose (β, ρ)-entry is given by

$$\mathcal{M}_\beta^\rho = \left(\frac{\partial}{\partial t^\beta} T^\rho(\underline{t}) \right) \Big|_{\substack{t=1 \\ s=0}}$$

for each $\beta, \rho \in I$.

This theorem says that $\Omega(X) = \Omega(X_G)$ and $\Omega(X_U)$ are "transcendental" invariants but their relationship is "algorithmically computable up to desired precision": if we know the period matrix $\Omega(X)$ and the polynomials $G(\underline{x}), H(\underline{x})$, then there is an algebraic algorithm to compute the period matrix $\Omega(X_U)$.

We need to explain more details about Theorem 3.2 in the case of elliptic curves (1) and (4) considered in the introduction, in order to get the algorithm of the inverse value of the j-function. From now on, we use the following homogeneous equations:

$$G(\underline{x}) := x_2x_1^2 - 4x_0^3 + x_2^2x_0, \quad U(\underline{x}) := x_2x_1^2 - 4x_0^3 + x_2^2x_0 + Bx_2^3,$$
$$\text{and } H(\underline{x}) = U(\underline{x}) - G(\underline{x})$$

so that $X = E_G(\mathbb{C}) = E(\mathbb{C})$ and $X_U = E_U(\mathbb{C}) = E_B(\mathbb{C})$.

The key idea in the proof of Theorem 3.2 is to interpret a period integral of $X = E_G(\mathbb{C})$ as a $\mathbb{C}$-linear map $C_\gamma^X : \mathbb{C}[y, x_0, x_1, x_2] \to \mathbb{C}$ for each $\gamma \in H_1(X, \mathbb{Z})$ such that $C_\gamma^X \circ K_X = 0$. In fact, for $y^{k-1}F(\underline{x})$ where $F(\underline{x})$ is a homogeneous polynomial of degree $3k-3$ and $k \geq 1$, the $\mathbb{C}$-linear map C_γ was defined as follows:

$$\begin{aligned} C_\gamma^X(y^{k-1}F(\underline{x})) &= -\frac{1}{2\pi i}\int_{\tau(\gamma)}\left(\int_0^\infty y^{k-1}F(\underline{x})\cdot e^{yG(\underline{x})}dy\right)\Omega_x \\ &= \frac{(-1)^{k-1}(k-1)!}{2\pi i}\int_{\tau(\gamma)}\frac{F(\underline{x})}{G(\underline{x})^k}\Omega_x = \int_\gamma (\mathrm{res}_X \circ J)(y^{k-1}F(\underline{x})), \end{aligned} \tag{13}$$

where $\tau : H_1(X, \mathbb{Z}) \to H_1(\mathbf{P} - X, \mathbb{Z})$ is the tubular neighborhood map and

$$\Omega_x = \sum_{i=0}^{2}(-1)^i x_i(dx_0\wedge\cdots\wedge\hat{dx_i}\wedge\cdots\wedge dx_2) = x_0dx_1\wedge dx_2 - x_1dx_0\wedge dx_2 + x_2dx_0\wedge dx_1.$$

For a fixed symplectic integral basis $a, b \in H_1(X, \mathbb{Z})$ and $a_B, b_B \in H_1(X_U, \mathbb{Z})$, we get

$$\int_a \omega = C_a^X(2), \qquad \int_b \omega = C_b^X(2),$$
$$\int_{a_B} \omega_B = C_{a_B}^{X_U}(2), \qquad \int_{b_B} \omega_B = C_{b_B}^{X_U}(2)$$

because the algebraic holomorphic differential 1-forms ω and ω_B are given by

$$\omega = (\mathrm{res}_X \circ J)(2) := \mathrm{res}_X\left(\frac{2\Omega_x}{G(\underline{x})}\right), \quad \omega_B = \mathrm{res}_{X_U}\left(\frac{2\Omega_x}{U(\underline{x})}\right).$$

The computation (13) implies that

$$C_a^X(v\cdot e^{yH(\underline{x})}) = C_a^{X_U}(v), \quad C_b^X(v\cdot e^{yH(\underline{x})}) = C_b^{X_U}(v), \quad v \in A = \mathbb{C}[\underline{y}],$$

on which the algorithm for the matrix $\mathcal{M}$ relies. Also note that

$$\begin{pmatrix} C_a^{X_U}(v) & C_b^{X_U}(v)\end{pmatrix}\begin{pmatrix}\rho_1 & \rho_2 \\ \rho_3 & \rho_4\end{pmatrix} = \begin{pmatrix} C_{a_B}^{X_U}(v) & C_{b_B}^{X_U}(v)\end{pmatrix},$$

since $\begin{pmatrix} a & b\end{pmatrix}\left(\begin{smallmatrix}\rho_1 & \rho_2 \\ \rho_3 & \rho_4\end{smallmatrix}\right) = \begin{pmatrix} a_B & b_B\end{pmatrix}$ for $\left(\begin{smallmatrix}\rho_1 & \rho_2 \\ \rho_3 & \rho_4\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

The power series $T^\rho(s,t)$ depends on choices of $\mathbb{C}$-basis of $H^0(\mathcal{A}_{X_U}, K_{X_U})$ and $H^0(\mathcal{A}_X, K_X)$ (note that $\mathcal{A}_X = \mathcal{A}_{X_U}$); we explain how it depends on such choices below. If we let $u_{\alpha_1} = 2$ and $u_{\alpha_2} = yH(\underline{x})$, then one can check (see [4, Lemma 2.6]) that

$$\{u_\alpha \bmod K_{X_U}(\mathcal{A}_{X_U}^{-1})\} \text{ is a } \mathbb{C}\text{-basis of } H^0(\mathcal{A}_{X_U}, K_{X_U}).$$

Let $g_{\alpha_1} = 2$ and $g_{\alpha_2} = yg(\underline{x})$, where $yg(\underline{x}) \in \mathbb{C}[y, x_0, x_1, x_2]$ is the polynomial which satisfies that $\overline{\omega} = \mathrm{res}\circ J(yg(\underline{x}))$ where $\overline{\omega}$ is the complex conjugation of ω. We will find $yg(\underline{x})$ explicitly in the next section. Then one can also check that

$$\{g_\alpha \bmod K_X(\mathcal{A}_X^{-1})\} \text{ is a } \mathbb{C}\text{-basis of } H^0(\mathcal{A}_X, K_X).$$

For each $\rho \in I$, we define a (unique) power series $T^\rho(\underline{t}) \in \mathbb{C}[[\underline{t}]]$ by the formula

$$\sum_{\rho \in I} T^\rho(\underline{t}) \cdot g_\rho + K_X(\Lambda(\underline{t})) = e^{\sum_{\alpha \in I} t^\alpha u_\alpha} - 1 \tag{14}$$

for some (not unique) $\Lambda(\underline{t}) \in \mathcal{A}_X^{-1}[[\underline{t}]]$. Note that (14) uniquely determines $T^\rho(\underline{t})$. In the case of elliptic curves of X and X_U, Theorem 3.2 says that

$$\begin{pmatrix} C_a^{X_U}(2) & C_b^{X_U}(2) \\ C_a^{X_U}(yH(\underline{x})) & C_b^{X_U}(yH(\underline{x})) \end{pmatrix} = \mathcal{M} \begin{pmatrix} C_a^X(2) & C_b^X(2) \\ C_a^X(yg(\underline{x})) & C_b^X(yg(\underline{x})) \end{pmatrix} = \mathcal{M} \begin{pmatrix} \varepsilon & i\varepsilon \\ \overline{\varepsilon} & -i\overline{\varepsilon} \end{pmatrix}, \tag{15}$$

where $\mathcal{M}_\beta^\rho = \left(\frac{\partial}{\partial t^\beta} T^\rho(\underline{t})\right)\Big|_{\substack{t=1 \\ s=0}}$.

Remark 3.3. We remark that $C_\gamma^X(v \cdot e^{s+tyH(\underline{x})})$, for $v \in A, \gamma = a, b$, is a formal family of periods on the parameters s and t. This formal family is a solution of a system of partial differential equations with respect to $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$. This leads to the solution of the linear order differential equations, called the Picard–Fuchs equations (put $s = 0$), by elimination method. We refer to [5, Theorem 3.23] for details.

4. Choice of a basis and complex conjugation

For our application of the deformation theory to inverse values of the modular j-function, we need to choose an appropriate $\mathbb{C}$-basis $\{g_\alpha : \alpha \in I\} = \{g_{\alpha_1}, g_{\alpha_2}\}$ in (14). The goal here is to find a polynomial $g_{\alpha_2} = y \cdot g(\underline{x}) \in \mathbb{C}[y, x_0, x_1, x_2]$ such that $\int_a \overline{\omega} = C_a^X(g_{\alpha_2})$, where $\overline{\omega}$ is the complex conjugation of ω.

Let $\eta : X \to \mathbf{P}$ be a given closed embedding. Let us consider the following exact sequence of complexes of sheaves on $\mathbf{P}$:

$$0 \to \Omega_{\mathbf{P}}^\bullet \to \Omega_{\mathbf{P}}^\bullet(\log X) \overset{\text{res}}{\to} \eta_* \Omega_X^{\bullet - 1} \to 0, \tag{16}$$

where Ω_Y^p is a sheaf of holomorphic p-forms on Y and $\Omega_{\mathbf{P}}^p(\log X)$ is a sheaf of meromorphic p-forms ω on $\mathbf{P}$ such that ω and $d\omega$ are regular on $\mathbf{P} \setminus X$ and have at most a pole of order one along X. See [7, page 444] for more details. The main result of [2] is to give an explicit formula of the coboundary δ map in the Poincaré residue sequence induced from (16),

$$\delta : H^1(X, \Omega_X^1) \to H^2(\mathbf{P}, \Omega_{\mathbf{P}}^2). \tag{17}$$

We recall [2, theorem 3] in the case of elliptic curves.

Theorem 4.1. *Let a and b be nonnegative integers such that $a+b = 1$. For homogeneous polynomials $A(\underline{x}), B(\underline{x}) \in A = \mathcal{A}_X^0$ such that* $\deg(A) = 3a$ *and* $\deg(B) = 3b$,

$$\delta\Big(\text{res}_X\big(\frac{A(\underline{x})\Omega_x}{G(\underline{x})^{a+1}}\big) \cdot \text{res}_X\big(\frac{B(\underline{x})\Omega_x}{G(\underline{x})^{b+1}}\big)\Big) = c_{ab} \frac{A(\underline{x})B(\underline{x})\Omega_x}{\frac{\partial G}{\partial x_0}\frac{\partial G}{\partial x_1}\frac{\partial G}{\partial x_2}}, \tag{18}$$

where $\cdot$ is the cup product of the singular cocycles and

$$c_{ab} = 3\frac{(-1)^{\frac{a(a+1)}{2} + \frac{b(b+1)}{2} + 1 + b^2}}{a!b!}.$$

Note the right-hand side of (18) is viewed as a Cech cocycle on the covering of $\mathbf{P}$:

$$U_j = \{\underline{x} \in \mathbf{P} : \frac{\partial G(\underline{x})}{\partial x_j} \neq 0\}, \quad j = 0, 1, 2.$$

We define

$$\mathbb{D} = \mathbb{D}(\underline{x}) = \det\big(\frac{\partial^2 G(\underline{x})}{\partial x_i \partial x_j}\big), \quad i,j = 0,1,2.$$

Then $\mathbb{D} \in \mathbb{C}[\underline{x}]$ is a homogeneous polynomial of degree 3. For any $C \in \mathbb{C}$, we have the following formula:

$$\int_{\mathbf{P}} [\frac{C\mathbb{D}\Omega_x}{\frac{\partial G}{\partial x_0}\frac{\partial G}{\partial x_1}\frac{\partial G}{\partial x_2}}] = \frac{(2\pi i)^2}{2}\operatorname{Res}_0\{\frac{C\mathbb{D}}{\frac{\partial G}{\partial x_0}\frac{\partial G}{\partial x_1}\frac{\partial G}{\partial x_2}}\} = -2^4\pi^2 \cdot C. \tag{19}$$

where Res_0 is the Grothendieck residue given in [7, page 449]. We refer to [7, page 452], for the proof of (19): apply the remark [7, (12.10)] to the case $d_0 = 2$, $m = 2$, and $F_i = \frac{\partial G}{\partial x_i}$ in their notation.

Recall that $J(y^k h(\underline{x})) := (-1)^k k! \frac{h(\underline{x})}{G(\underline{x})^{k+1}}\Omega_x$ and J induces an isomorphism

$$H^0(\mathcal{A}_X, K_X) \xrightarrow{J} H^2_{dR}(\mathbf{P} \setminus X),$$

and $\operatorname{res}_X \circ J$ gives an isomorphism $H^0(\mathcal{A}_X, K_X) \simeq \mathbb{H}$. We know that $\operatorname{res}_X(J(1))$ is a basis of the holomorphic component $\mathbb{H}^{1,0}$. We like to find a homogeneous polynomial $g(\underline{x}) \in \mathbb{C}[\underline{x}]$ of degree 3 such that

$$\operatorname{res}_X(J(yg(\underline{x}))) = \operatorname{res}_X(-\frac{g(\underline{x})}{G(\underline{x})^2}\Omega_x) \in \mathbb{H}^{0,1}$$

represents the cohomology class of the complex conjugation $\overline{\operatorname{res}_X(J(2))}$ of $\operatorname{res}_X(J(2))$.

The relationship between the long exact sequence of hypercohomology in (17) and the Gysin long sequence is known as follows (see [7, Proposition (11.4)]):

$$\begin{tikzcd}
\cdots \arrow[r] & H^{k-2}(\Omega^\bullet_X) \arrow[r, "\delta"] & H^k(\Omega^\bullet_{\mathbf{P}}) \arrow[r] & H^k(\Omega^\bullet_{\mathbf{P}}(\log X)) \arrow[r, "\operatorname{res}"] & H^{k-1}(\Omega^\bullet_X) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^{k-2}(X,\mathbb{C}) \arrow[u] \arrow[r, "2\pi i\cdot\eta_!"] & H^k(\mathbf{P},\mathbb{C}) \arrow[u] \arrow[r] & H^k(\mathbf{P}-X,\mathbb{C}) \arrow[u] \arrow[r] & H^{k-1}(X,\mathbb{C}) \arrow[u] \arrow[r] & \cdots
\end{tikzcd}$$

where vertical maps are isomorphisms and all the diagrams commute. Recall that the lower shriek of η, $\eta_!$, can be constructed as the composition of three maps:

$$H^2(X,\mathbb{C}) \xrightarrow{\simeq} H_0(X,\mathbb{C}) \xrightarrow{\eta_*} H_0(\mathbf{P},\mathbb{C}) \xrightarrow{\simeq} H^4(\mathbf{P},\mathbb{C}), \tag{20}$$

where the first and third maps are isomorphisms given by the Poincare duality. Thus $\eta_! = \frac{1}{2\pi i}\delta$, where δ was defined in (17), satisfies that

$$\int_X \mu = \int_{\mathbf{P}} \eta_!(\mu) = \frac{1}{2\pi i}\int_{\mathbf{P}} \delta(\mu), \tag{21}$$

for $\mu \in H^2(X,\mathbb{C})$. Note that $\eta_!$ is an isomorphism since X is an odd-dimensional hypersurface (in particular, X is an elliptic curve).

The formulaes (18) and (19) imply that

$$\begin{aligned}
\int_X \operatorname{res}_X(\frac{2\Omega_x}{G(\underline{x})^1})\cdot\operatorname{res}_X(\frac{C\mathbb{D}\Omega_x}{G(\underline{x})^2}) &= \frac{1}{2\pi i}\int_{\mathbf{P}} \delta\Big(\operatorname{res}_X(\frac{2\Omega_x}{G(\underline{x})^1})\cdot\operatorname{res}_X(\frac{C\mathbb{D}\Omega_x}{G(\underline{x})^2})\Big) \\
&= \frac{1}{2\pi i}\int_{\mathbf{P}} c_{01}[\frac{2C\mathbb{D}\Omega_x}{\frac{\partial G}{\partial x_0}\frac{\partial G}{\partial x_1}\frac{\partial G}{\partial x_2}}] = -48\pi i \cdot C,
\end{aligned}$$

since $c_{01} = -3$. Thus we get

$$C = \frac{1}{-48\pi i} \int_X \operatorname{res}_X\left(\frac{2\Omega_x}{G(\underline{x})^1}\right) \cdot \operatorname{res}_X\left(\frac{C\mathbb{D}\Omega_x}{G(\underline{x})^2}\right).$$

Therefore, for $g(\underline{x}) := C\mathbb{D} := \left(\frac{1}{-48\pi i}\int_X \omega\wedge\bar{\omega}\right)\cdot\det\left(\frac{\partial^2 G(\underline{x})}{\partial x_i \partial x_j}\right)$, the anti-holomorphic 1-form

$$\operatorname{res}_X(J(yg(\underline{x}))) = \operatorname{res}_X\left(-\frac{g(\underline{x})}{G(\underline{x})^2}\Omega_x\right) \in \mathbb{H}^{0,1} \tag{22}$$

represents the cohomology class of the complex conjugation $\overline{\operatorname{res}_X(J(2))}$ of $\operatorname{res}_X(J(2))$ $\in \mathbb{H}^{1,0}$. The following lemma is well known (a nice exercise).

Lemma 4.2. *Let* $\omega = \operatorname{res}_X\left(\frac{2\Omega_x}{G(\underline{x})}\right)$. *For any* $\mathbb{Z}$*-basis* a, b *of* $H_1(X,\mathbb{Z})$ *such that* $\int_b \omega / \int_a \omega$ *is in the upper half-plane, we have*

$$\int_X \omega\wedge\bar{\omega} = \int_b \omega \cdot \overline{\int_a \omega} - \int_a \omega \cdot \overline{\int_b \omega}. \tag{23}$$

Note that $\frac{i}{2}\omega\wedge\bar{\omega}$ is the Kahler 2-form on the elliptic curve X.

5. Computation of the inverse value of the j-function

We summarise all the previous results to give an algorithmic strategy to compute the inverse value of the modular j-invariant, i.e., to solve $j(\tau) = N$ for a given nonzero $N \in \mathbb{C}$. Let $X = E_G(\mathbb{C})$, where E_G is an elliptic curve defined by $G(\underline{x}) = x_2x_1^2 - 4x_0^3 + x_2^2x_0$. Choose a positive quadratic number B such that $N = 1728\frac{1}{1-27B^2}$. Define

$$H(\underline{x}) = Bx_0^3, \quad g(\underline{x}) := \frac{-1}{3\pi}\varepsilon\overline{\varepsilon}\cdot(12x_0x_1^2 - 12x_0^2x_2 - x_2^3),$$

where $\varepsilon = \int_a \omega$ is given in (3).

Theorem 5.1. *If we define two variable power series* $T(s,t)$ *and* $S(s,t)$ *by using the equation*

$$S(s,t) + T(s,t)\cdot yg(\underline{x}) + K_X(\Lambda(s,t)) = e^{s+t\cdot yH(\underline{x})} - 1 \quad \textit{in } \mathcal{A}_X^0[[s,t]],$$

then an inverse value τ*, which is determined unique up to* $\mathrm{SL}_2(\mathbb{Z})$*-action, satisfying* $j(\tau) = N$ *is given by*[2]

$$\tau = \frac{S_t(0,1)i\varepsilon - iT_t(0,1)\overline{\varepsilon}}{S_s(0,1)\varepsilon + T_s(0,1)\overline{\varepsilon}} \quad (\textit{up to } \mathrm{SL}_2(\mathbb{Z})\textit{-action})$$

or its inverse, depending on which belongs to the upper half-plane.

Proof. The j-invariant of E_G is 1728 and the period matrix of X is $\begin{pmatrix}\varepsilon & i\varepsilon\end{pmatrix}$. In order to solve $j(\tau) = N \neq 0$, we observe that the Weierstrass equation

$$U(\underline{x}) = x_2x_1^2 - 4x_0^3 + x_2^2x_0 + Bx_2^3 = 0, \quad B\in\mathbb{C},$$

has j-invariant $1728\frac{1}{1-27B^2}$ and choose a quadratic number B such that $N = 1728\frac{1}{1-27B^2}$. Then the previous discussion (see (15)) implies that

$$\begin{pmatrix}\tau_1 & \tau_1' \\ \tau_2 & \tau_2'\end{pmatrix} := \begin{pmatrix} C_{a_B}^{X_U}(2) & C_{b_B}^{X_U}(2) \\ C_{a_B}^{X_U}(yH(\underline{x})) & C_{b_B}^{X_U}(yH(\underline{x}))\end{pmatrix} = \mathcal{M}\cdot\begin{pmatrix}\varepsilon & i\varepsilon \\ \overline{\varepsilon} & \overline{i\varepsilon}\end{pmatrix}\cdot\begin{pmatrix}\rho_1 & \rho_2 \\ \rho_3 & \rho_4\end{pmatrix}, \tag{24}$$

[2]We use the notation $S_s(s,t) = \frac{\partial}{\partial s}S(s,t), T_s(s,t) = \frac{\partial}{\partial s}T(s,t)$, etc.

where $\left(\begin{smallmatrix}\rho_1 & \rho_2\\ \rho_3 & \rho_4\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ is given in (5) and $\mathcal{M}$ is the 2×2 matrix whose (β,ρ)-entry is given by

$$\mathcal{M}_\beta^\rho = \left(\frac{\partial}{\partial t^\beta} T^\rho(\underline{t})\right)\Big|_{\substack{t=1\\ s=0}},$$

and the 2-variable power series $S(s,t) = T^{\alpha_1}(\underline{t})$ and $T(s,t) = T^{\alpha_2}(\underline{t})$ (where $I = \{\alpha_1,\alpha_2\}$, $s = t^{\alpha_1}, t = t^{\alpha_2}$) are uniquely determined by (14):

$$S(s,t) + T(s,t)\cdot yg(\underline{x}) + K_X(\Lambda(s,t)) = e^{s+t\cdot yH(\underline{x})} - 1,$$

for some $\Lambda(t,s) \in \mathcal{A}_X^{-1}[[\underline{t}]]$. Note that

$$\begin{aligned} g(\underline{x}) &= \Big(\frac{1}{-48\pi i}\int_X \omega\wedge\bar{\omega}\Big)\cdot \det\Big(\frac{\partial^2 G(\underline{x})}{\partial x_i \partial x_j}\Big) \\ &= \frac{1}{-24\pi}\varepsilon\bar{\varepsilon}\cdot \det\Big(\frac{\partial^2 G(\underline{x})}{\partial x_i \partial x_j}\Big) = \frac{-1}{3\pi}\varepsilon\bar{\varepsilon}\cdot(12x_0x_1^2 - 12x_0^2x_2 - x_2^3), \end{aligned}$$

by Lemma 4.2 and our formula of $g(\underline{x})$ above (22), which guarantees the correctness of our choice of $g(\underline{x})$. Also note that $K_X = Q_X + \Delta$ and the cochain complex $(\mathcal{A}_X, \cdot, Q_X)$ becomes a commutative differential graded algebra. Then this fact is utilised to compute the power series $S(s,t)$ and $T(s,t)$ by using the Gröbner basis algorithm: we refer to the [5, section 4.13] for details. Finally, τ such that $j(\tau) = N \neq 0$ is given by $\frac{\tau_1'}{\tau_1}$ (or $\frac{\tau_1}{\tau_1'}$), depending on which belongs to the upper half-plane. $\square$

Remark 5.2. In fact, the computation of periods of any elliptic curve E in terms of the AGM (arithmetic-geometric mean) of roots of the Weierstrass equation of E was given in [3]. Thus their work gives another way of solving the equation $j(\tau) = N$ directly in terms of the AGM. On the other hand, our Lie homotopical method based on [4] is a computation of the explicit relationship between periods of two elliptic curves.

Acknowledgment

The authors thank A. Caldararu for helping them figure out the first equality in (21).

References

[1] Bruce C. Berndt and Heng Huat Chan, *Ramanujan and the modular j-invariant*, Canad. Math. Bull. **42** (1999), no. 4, 427–440. MR1727340

[2] James A. Carlson and Phillip A. Griffiths, *Infinitesimal variations of Hodge structure and the global Torelli problem*, Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979, Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md., 1980, pp. 51–76. MR605336

[3] John E. Cremona and Thotsaphon Thongjunthug, *The complex AGM, periods of elliptic curves over $\mathbb{C}$ and complex elliptic logarithms*, J. Number Theory **133** (2013), no. 8, 2813–2841. MR3045217

[4] Yesule Kim and Jeehoon Park, *Deformations for period matrices of smooth projective complete intersections,* available at http://math.postech.ac.kr/ jeehoonpark/papers.html, submitted.

[5] Jae-Suk Park and Jeehoon Park, *Enhanced homotopy theory for period integrals of smooth projective hypersurfaces*, Commun. Number Theory Phys. **10** (2016), no. 2, 235–337. MR3528835

[6] Phillip A. Griffiths, *On the periods of certain rational integrals. I, II*, Ann. of Math. (2) **90** (1969), 460-495; ibid. (2) **90** (1969), 496–541. MR0260733

[7] Chris Peters and Joseph Steenbrink, *Infinitesimal variations of Hodge structure and the generic Torelli problem for projective hypersurfaces (after Carlson, Donagi, Green, Griffiths, Harris)*, Classification of algebraic and analytic manifolds (Katata, 1982), Progr. Math., vol. 39, Birkhäuser Boston, Boston, MA, 1983, pp. 399–463. MR728615

[8] Joseph H. Silverman, *Advanced topics in the arithmetic of elliptic curves*, Graduate Texts in Mathematics, vol. 151, Springer-Verlag, New York, 1994. MR1312368

[9] E. T. Whittaker and G. N. Watson, *A course of modern analysis,* An introduction to the general theory of infinite processes and of analytic functions; with an account of the principal transcendental functions; Reprint of the fourth (1927) edition. Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1996. MR1424469

Department of mathematics and computer science, Queensborough Community College, 222-05, 56th Avenue, Bayside, New York 11364
Email address: harpum@gmail.com

Department of Mathematics, Pohang University of Science and Technology, 77 Cheongam-ro, Nam-gu, Pohang, Gyeongbuk, Republic of Korea 37673
Email address: yesule@postech.ac.kr

Department of Mathematics, Pohang University of Science and Technology, 77 Cheongam-ro, Nam-gu, Pohang, Gyeongbuk, Republic of Korea 37673
Email address: jpark.math@gmail.com