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LIMIT OF TORSION SEMISTABLE GALOIS REPRESENTATIONS WITH UNBOUNDED WEIGHTS

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Abstract. Let K be a complete discrete valuation field of characteristic $(0,p)$ with perfect residue field, and let T be an integral $\mathbb{Z}_p$-representation of $\mathrm{Gal}(\overline{K}/K)$. A theorem of T. Liu says that if T/p^nT is torsion semistable (resp., crystalline) of *uniformly bounded* Hodge-Tate weights for all $n \geq 1$, then T is also semistable (resp., crystalline). In this paper, we show that we can relax the condition of "uniformly bounded Hodge-Tate weights" to an *unbounded* (log-)growth condition.

1. Introduction

We first introduce some notation. Let p be a prime, let k be a perfect field of characteristic p, let $W(k)$ be the ring of Witt vectors, let $K_0 = W(k)[\frac{1}{p}]$ be the fraction field, let K be a finite totally ramified extension of K_0, let $e = e(K/K_0)$ be the ramification index, and let $G_K = \mathrm{Gal}(\overline{K}/K)$ be the absolute Galois group for a fixed algebraic closure $\overline{K}$ of K.

We use $\mathrm{Rep}^{\mathrm{tor}}_{\mathbb{Z}_p}(G_K)$ (resp., $\mathrm{Rep}^{\mathrm{fr}}_{\mathbb{Z}_p}(G_K)$) to denote the category of finite p-power torsion (resp., $\mathbb{Z}_p$-finite free) representations of G_K. Let r be an integer in the range $[0,\infty]$ (including infinity). We use $\mathrm{Rep}^{\mathrm{fr,st},[-r,0]}_{\mathbb{Z}_p}(G_K)$ (resp., $\mathrm{Rep}^{\mathrm{fr,cris},[-r,0]}_{\mathbb{Z}_p}(G_K)$) to denote the category of finite free $\mathbb{Z}_p$-lattices in semistable (resp., crystalline) representations of G_K with Hodge-Tate weights in the range $[-r,0]$.

Definition 1.1. Let r be an integer in the range $[0,\infty]$ (including infinity). $T_\infty \in \mathrm{Rep}^{\mathrm{tor}}_{\mathbb{Z}_p}(G_K)$ is called torsion semistable (resp., crystalline) of weight r if there exist two objects L and L' in $\mathrm{Rep}^{\mathrm{fr,st},[-r,0]}_{\mathbb{Z}_p}(G_K)$ (resp., $\mathrm{Rep}^{\mathrm{fr,cris},[-r,0]}_{\mathbb{Z}_p}(G_K)$) such that $T_\infty = L/L'$.

The following result was first conjectured by Fontaine ([Fon97]) and was fully proved in [Liu07] (some partial results were known by work of Ramakrishna, Berger and Breuil; see [Liu07, §1] for a historical account).

Theorem 1.2 ([Liu07]). *Let $T \in \mathrm{Rep}^{\mathrm{fr}}_{\mathbb{Z}_p}(G_K)$. Suppose that there exists an $r \in [0,\infty)$ such that T/p^nT is torsion semistable (resp., crystalline) of weight r for all $n \geq 1$. Then $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semistable (resp., crystalline) with Hodge-Tate weights in $[-r,0]$.*

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It is *necessary* to have $r < \infty$ in the above theorem because of the following result.

Theorem 1.3 ([GL, Thm. 3.3.2]). *Suppose K is a finite extension of $\mathbb{Q}_p$. For any $T_\infty \in \mathrm{Rep}^{\mathrm{tor}}_{\mathbb{Z}_p}(G_K)$, it is torsion semistable (in fact, torsion crystalline).*

In fact, suppose T is in $\mathrm{Rep}^{\mathrm{fr}}_{\mathbb{Z}_p}(G_K)$ of rank d (with $K/\mathbb{Q}_p$ finite extension). Then it is shown in [GL, Rem. 3.3.5] that T/p^nT is torsion crystalline of weight $h(n) \le n(p^{fd}+p-2)$ (where f is the inertia degree of K). Namely, the growth of the (crystalline) weight of T/p^nT is *linear.*

During a conversation with Ruochuan Liu, he proposed the following question.

Question 1.4. Let $T \in \mathrm{Rep}^{\mathrm{fr}}_{\mathbb{Z}_p}(G_K)$. For each $n \ge 1$, suppose T/p^nT is torsion semistable (resp., crystalline) of weight $h(n)$. Is it still possible to show that $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semistable (resp., crystalline) if we allow $h(n)$ to go to infinity?

By the paragraph above the question, it is necessary that $h(n)$ cannot grow as fast as $n(p^{fd}+p-2)$ (in the case when $K/\mathbb{Q}_p$ is a finite extension). So, one would expect that $h(n)$ has to grow more slowly than linear-growth. The first natural guess is the log-growth, and this is precisely what we obtained.

Theorem 1.5. *Let $T \in \mathrm{Rep}^{\mathrm{fr}}_{\mathbb{Z}_p}(G_K)$ of rank d. For each $n \ge 1$, suppose T/p^nT is torsion semistable (resp., crystalline) of weight $h(n)$. If*

$$h(n) < \frac{1}{2d}\log_{16} n, \forall n >> 0,$$

then $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semistable (resp., crystalline).

One of the motivations of our work is the study of local-global compatibility problems in the construction of Galois representations (associated to automorphic representations). Indeed, many such Galois representations are constructed via congruence methods. A good motivational explanation of the situation can be found in the introduction in Jorza's thesis [Jor10]. Namely, certain p^n-torsion semistable (or crystalline) representations will be constructed via congruence methods. However, the weights of these p^n-torsion representations grow (quite rapidly) to infinity, and so Theorem 1.2 is no longer applicable. Unfortunately, our Theorem 1.5 also seems useless in this respect. To name one example, in the case [Jor12, Thm. 2.1, Thm. 3.1], the weights of these torsion representations grow exponentially. We do hope some of the techniques in our paper can be useful for future studies in local-global compatibility problems, perhaps combined with methods from analytic continuation of semistable periods.

Notation. Let $\mathcal{O}_{\overline{K}}$ be the ring of integers of $\overline{K}$. Let $R := \varprojlim_{x \to x^p} \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$, and let $W(R)$ be the ring of Witt vectors of R. Let A_{cris} be the usual period ring.

We fix a uniformizer $\pi \in \mathcal{O}_K$ and let $E(u) \in W(k)[u]$ be the Eisenstein polynomial of π. Define $\pi_n \in \overline{K}$ inductively such that $\pi_0 = \pi$ and $(\pi_{n+1})^p = \pi_n$. Then $\{\pi_n\}_{n\ge 0}$ defines an element $\underline{\pi} \in R$, and let $[\underline{\pi}] \in W(R)$ be the Techmüller representative of $\underline{\pi}$.

Define $\mu_n \in \overline{K}$ inductively such that μ_1 is a primitive pth root of unity and $(\mu_{n+1})^p = \mu_n$. Set $K_\infty := \bigcup_{n=1}^\infty K(\pi_n)$, $K_{p^\infty} = \bigcup_{n=1}^\infty K(\mu_n)$, and $\hat{K} := \bigcup_{n=1}^\infty K(\pi_n, \mu_n)$. Let $G_\infty := \mathrm{Gal}(\overline{K}/K_\infty)$, $H_\infty := \mathrm{Gal}(\overline{K}/\hat{K})$, $H_K := \mathrm{Gal}(\hat{K}/K_\infty)$, and $\hat{G} := \mathrm{Gal}(\hat{K}/K)$.

When V is a semistable representation of G_K, we let $D_{\mathrm{st}}(V) := (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V^\vee)^{G_K}$, where $V^\vee$ is the dual representation of V (and B_{st} is the usual period ring). The Hodge-Tate weights of V are defined to be $i \in \mathbb{Z}$ such that $\mathrm{gr}^i(K \otimes_{K_0} D_{\mathrm{st}}(V)) \neq 0$. For example, for the cyclotomic character ε_p, its Hodge-Tate weight is $\{1\}$.

2. Integral and torsion p-adic Hodge theory

In this section, we recall some tools in integral and torsion p-adic Hodge theory.

2.1. **Étale φ-modules and étale (φ, τ)-modules.** Recall that $\mathfrak{S} = W(k)[\![u]\!]$ with the Frobenius endomorphism $\varphi_{\mathfrak{S}} \colon \mathfrak{S} \to \mathfrak{S}$ which acts on $W(k)$ via arithmetic Frobenius and sends u to u^p. Via the map $u \mapsto [\underline{\pi}]$, there is an embedding $\mathfrak{S} \hookrightarrow W(R)$ which is compatible with Frobenious endomorphisms. Denote $\mathfrak{S}_n := \mathfrak{S}/p^n\mathfrak{S}$.

Recall that $\mathcal{O}_{\mathcal{E}}$ is the p-adic completion of $\mathfrak{S}[1/u]$. Our fixed embedding $\mathfrak{S} \hookrightarrow W(R)$ determined by $\underline{\pi}$ uniquely extends to a φ-equivariant embedding $\iota \colon \mathcal{O}_{\mathcal{E}} \hookrightarrow W(\mathrm{Fr}R)$ (here $\mathrm{Fr}R$ denotes the fractional field of R), and we identify $\mathcal{O}_{\mathcal{E}}$ with its image in $W(\mathrm{Fr}R)$. Denote $\mathcal{O}_{\mathcal{E},n} := \mathcal{O}_{\mathcal{E}}/p^n\mathcal{O}_{\mathcal{E}}$. We note that $\mathcal{O}_{\mathcal{E}}$ is a complete discrete valuation ring with uniformizer p and residue field $k(\!(\underline{\pi})\!)$ as a subfield of $\mathrm{Fr}\, R$. Let $\mathcal{E}$ denote the fractional field of $\mathcal{O}_{\mathcal{E}}$, $\mathcal{E}^{\mathrm{ur}}$ the maximal unramified extension of $\mathcal{E}$ inside $W(\mathrm{Fr}R)[\frac{1}{p}]$, and $\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$ the ring of integers of $\mathcal{E}^{\mathrm{ur}}$. Set $\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}$ the p-adic completion of $\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$.

Definition 2.1. Let $'\mathrm{Mod}^{\varphi}_{\mathcal{O}_{\mathcal{E}}}$ denote the category of finite type $\mathcal{O}_{\mathcal{E}}$-modules M equipped with a $\varphi_{\mathcal{O}_{\mathcal{E}}}$-semilinear endomorphism $\varphi_M \colon M \to M$ such that $1 \otimes \varphi \colon \varphi^* M \to M$ is an isomorphism. Morphisms in this category are just $\mathcal{O}_{\mathcal{E}}$-linear maps compatible with φ's. We call objects in $'\mathrm{Mod}^{\varphi}_{\mathcal{O}_{\mathcal{E}}}$ *étale φ-modules.*

Let $'\mathrm{Rep}_{\mathbb{Z}_p}(G_\infty)$ (resp., $'\mathrm{Rep}_{\mathbb{Z}_p}(G_K)$) denote the category of finite type $\mathbb{Z}_p$-modules V with a continuous $\mathbb{Z}_p$-linear G_∞ (resp., G_K)-action. For M in $'\mathrm{Mod}^{\varphi}_{\mathcal{O}_{\mathcal{E}}}$, define

$$V(M) := (\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M)^{\varphi=1}.$$

For V in $'\mathrm{Rep}_{\mathbb{Z}_p}(G_\infty)$, define

$$\underline{M}(V) := (\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}} \otimes_{\mathbb{Z}_p} V)^{G_\infty}.$$

Theorem 2.2 ([Fon90, Prop. A 1.2.6])**.** *The functors V and $\underline{M}$ induce an exact tensor equivalence between the categories $'\mathrm{Mod}^{\varphi}_{\mathcal{O}_{\mathcal{E}}}$ and $'\mathrm{Rep}_{\mathbb{Z}_p}(G_\infty)$.*

Recall that $H_\infty = \mathrm{Gal}(\overline{K}/\hat{K})$. Let $F_\tau := (\mathrm{Fr}R)^{H_\infty}$. As a subring of $W(\mathrm{Fr}R)$, $W(F_\tau)$ is stable on G_K-action and the action factors through $\hat{G}$.

Definition 2.3. An étale (φ, τ)-module is a triple $(M, \varphi_M, \hat{G})$, where

- (M, φ_M) is an étale φ-module;
- $\hat{G}$ is a continuous $W(F_\tau)$-semilinear $\hat{G}$-action on $\hat{M} := W(F_\tau) \otimes_{\mathcal{O}_{\mathcal{E}}} M$, and $\hat{G}$ commutes with $\varphi_{\hat{M}}$ on $\hat{M}$, i.e., for any $g \in \hat{G}$, $g\varphi_{\hat{M}} = \varphi_{\hat{M}} g$;
- regarding M as an $\mathcal{O}_{\mathcal{E}}$-submodule in $\hat{M}$, then $M \subset \hat{M}^{H_K}$.

Given an étale (φ, τ)-module $\hat{M} = (M, \varphi_M, \hat{G})$, we define

$$\mathcal{T}^*(\hat{M}) := (W(\mathrm{Fr}\, R) \otimes_{\mathcal{O}_{\mathcal{E}}} M)^{\varphi=1} = \left(W(\mathrm{Fr}\, R) \otimes_{W(F_\tau)} \hat{M}\right)^{\varphi=1},$$

which is a representation of G_K.

Proposition 2.4 ([GL, Prop. 2.1.7]). *Notation as the above. Then*

(1) $\mathcal{T}^*(\hat{M})|_{G_\infty} \simeq V(M)$.
(2) *The functor $\mathcal{T}^*$ induces an equivalence between the category of étale (φ, τ)-modules and the category ${}'\mathrm{Rep}_{\mathbb{Z}_p}(G_K)$.*

2.2. Kisin modules and $(\varphi, \hat{G})$-modules.

Definition 2.5. For a nonnegative integer r, we write ${}'\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ for the category of finite-type $\mathfrak{S}$-modules $\mathfrak{M}$ equipped with a $\varphi_{\mathfrak{S}}$-semilinear endomorphism $\varphi_{\mathfrak{M}}\colon \mathfrak{M} \to \mathfrak{M}$ satisfying

- the cokernel of the linearization $1 \otimes \varphi\colon \varphi^*\mathfrak{M} \to \mathfrak{M}$ is killed by $E(u)^r$;
- the natural map $\mathfrak{M} \to \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$ is injective.

Morphisms in ${}'\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ are φ-compatible $\mathfrak{S}$-module homomorphisms.

We call objects in ${}'\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ *Kisin modules of $E(u)$-height r*. The category of *finite free Kisin modules of $E(u)$-height* r, denoted $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$, is the full subcategory of ${}'\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ consisting of those objects which are finite free over $\mathfrak{S}$. We call an object $\mathfrak{M} \in {}'\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ a *torsion Kisin module of $E(u)$-height* r if $\mathfrak{M}$ is killed by p^n for some n. Since $E(u)$ is always fixed in this paper, we often drop $E(u)$ from the above notions.

Let $\mathfrak{M} \in {}'\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ be a Kisin module of height r; we define

$$T^*_{\mathfrak{S}}(\mathfrak{M}) := (\mathfrak{M} \otimes_{\mathfrak{S}} W(\mathrm{Fr}R))^{\varphi=1} .$$

Since $\mathfrak{S} \subset W(R)^{G_\infty}$, we see that G_∞ acts on $T^*_{\mathfrak{S}}(\mathfrak{M})$. Note that this is the covariant version of the more usual (contra-variant) functor (see [GL, §2.3]).

Now let us review the theory of $(\varphi, \hat{G})$-modules. We denote by S the p-adic completion of the divided power envelope of $W(k)[u]$ with respect to the ideal generated by $E(u)$. There is a unique map (Frobenius) $\varphi_S\colon S \to S$ which extends the Frobenius on $\mathfrak{S}$. One can show that the embedding $W(k)[u] \to W(R)$ via $u \mapsto [\underline{\pi}]$ extends to the embedding $S \hookrightarrow A_{\mathrm{cris}}$. Inside $B^+_{\mathrm{cris}} = A_{\mathrm{cris}}[\frac{1}{p}]$, define a subring,

$$\mathcal{R}_{K_0} := \left\{ x = \sum_{i=0}^{\infty} f_i t^{\{i\}}, f_i \in S[\frac{1}{p}] \text{ and } f_i \to 0 \text{ as } i \to +\infty \right\},$$

where $t^{\{i\}} = \frac{t^i}{p^{\tilde{q}(i)}\tilde{q}(i)!}$ and $\tilde{q}(i)$ satisfies $i = \tilde{q}(i)(p-1) + r(i)$ with $0 \le r(i) < p-1$. Define $\widehat{\mathcal{R}} := W(R) \cap \mathcal{R}_{K_0}$. One can show that $\mathcal{R}_{K_0}$ and $\widehat{\mathcal{R}}$ are stable under the G_K-action and that the G_K-action factors through $\hat{G}$ (see [Liu10, §2.2]). Let I_+R be the maximal ideal of R and let $I_+\widehat{\mathcal{R}} = W(I_+R) \cap \widehat{\mathcal{R}}$. By [Liu10, Lem. 2.2.1], one has $\widehat{\mathcal{R}}/I_+\widehat{\mathcal{R}} \simeq \mathfrak{S}/u\mathfrak{S} = W(k)$.

Definition 2.6. Following [Liu10], a finite free (resp., torsion), $(\varphi, \hat{G})$-module of height r is a triple $(\mathfrak{M}, \varphi, \hat{G})$ where

(1) $(\mathfrak{M}, \varphi_{\mathfrak{M}}) \in {}'\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ is a finite free (resp., torsion) Kisin module of height r;
(2) $\hat{G}$ is a continuous $\widehat{\mathcal{R}}$-semilinear $\hat{G}$-action on $\hat{\mathfrak{M}} := \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$;
(3) $\hat{G}$ commutes with $\varphi_{\hat{\mathfrak{M}}}$ on $\hat{\mathfrak{M}}$, i.e., for any $g \in \hat{G}$, $g\varphi_{\hat{\mathfrak{M}}} = \varphi_{\hat{\mathfrak{M}}} g$;
(4) regard $\mathfrak{M}$ as a $\varphi(\mathfrak{S})$-submodule in $\hat{\mathfrak{M}}$; then $\mathfrak{M} \subset \hat{\mathfrak{M}}^{H_K}$;
(5) $\hat{G}$ acts on $W(k)$-module $M := \hat{\mathfrak{M}}/I_+\widehat{\mathcal{R}}\hat{\mathfrak{M}} \simeq \mathfrak{M}/u\mathfrak{M}$ trivially.

Morphisms between $(\varphi, \hat{G})$-modules are morphisms of Kisin modules that commute with the $\hat{G}$-action on $\hat{\mathfrak{M}}$'s.

Given $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G})$ a $(\varphi, \hat{G})$-module, either finite free or torsion, we define

$$\hat{T}^*(\hat{\mathfrak{M}}) := (W(\mathrm{Fr}R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M})^{\varphi=1},$$

and it is a $\mathbb{Z}_p[G_K]$-module.

Theorem 2.7 ([GL, Thm 2.3.2]).

(1) *$\hat{T}^*$ induces an equivalence between the category of finite free $(\varphi, \hat{G})$-modules of height r and the category of G_K-stable $\mathbb{Z}_p$-lattices in semistable representations of G_K with Hodge-Tate weights in $[-r, 0]$.*

(2) *For $\hat{\mathfrak{M}}$ a $(\varphi, \hat{G})$-module, either finite free or torsion, there exists a natural isomorphism $T^*_{\mathfrak{S}}(\mathfrak{M}) \xrightarrow{\sim} \hat{T}^*(\hat{\mathfrak{M}})$ of $\mathbb{Z}_p[G_\infty]$-modules.*

We record a useful lemma which can identify crystalline representations from $(\varphi, \hat{G})$-modules.

Lemma 2.8. *Suppose $K_\infty \cap K_{p^\infty} = K$ (which is always true when $p > 2$), and let $\hat{\mathfrak{M}}$ be a finite free $(\varphi, \hat{G})$-module. Then $\hat{T}^*(\hat{\mathfrak{M}})$ is a crystalline representation if and only if*

$$(\widetilde{\tau} - 1)(\mathfrak{M}) \in \hat{\mathfrak{M}} \cap (u^p \varphi(\mathfrak{t}) W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}).$$

Here $\widetilde{\tau}$ is a topological generator of G_{p^∞} such that $\mu_n = \frac{\widetilde{\tau}(\pi_n)}{\pi_n}$ for all n, and $\mathfrak{t} \in W(R) \setminus pW(R)$ such that $\varphi(\mathfrak{t}) = \frac{pE(u)}{E(0)}\mathfrak{t}$ (note that $\mathfrak{t}$ is unique up to units of $\mathbb{Z}_p$).

Proof. This is a combination of [GLS14, Prop. 5.9] and [Oze14, Thm. 21]. Note that the running assumption $p > 2$ in both papers is to guarantee $K_\infty \cap K_{p^\infty} = K$ (see the footnote in [GLS14, Prop. 4.7]). When $p = 2$ and $K_\infty \cap K_{p^\infty} = K$, all the proofs still work. $\square$

Definition 2.9.

(1) Given an étale φ-module M in $'\mathrm{Mod}^{\varphi}_{\mathcal{O}_\mathcal{E}}$, if $\mathfrak{M} \in {'\mathrm{Mod}}^{\varphi, r}_{\mathfrak{S}}$ is a Kisin module so that $M = \mathcal{O}_\mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{M}$, then $\mathfrak{M}$ is called a *Kisin model* of M, or simply a *model* of M.

(2) Given $\hat{M} := (M, \varphi_M, \hat{G}_M)$ a torsion (resp., finite free) (φ, τ)-module, a torsion (resp., finite free) $(\varphi, \hat{G})$-module $\hat{\mathfrak{M}} := (\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G})$ is called a *model* of $\hat{M}$ if $\mathfrak{M}$ is a model of M and the isomorphism

$$W(F_\tau) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}} \simeq W(F_\tau) \otimes_{\varphi, W(F_\tau)} \hat{M}$$

induced by $\mathcal{O}_\mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{M} \simeq M$ is compatible with $\hat{G}$-actions on both sides.

Suppose T is a G_K-stable $\mathbb{Z}_p$-lattice in a semistable representation of G_K with Hodge-Tate weights in $[-r, 0]$. Let $\hat{M}$ be the (φ, τ)-module associated to T via Proposition 2.4, and let $\hat{\mathfrak{M}}$ be the $(\varphi, \hat{G})$-module associated to T via Theorem 2.7. Then $\hat{\mathfrak{M}}$ is a model of $\hat{M}$ ([GL, Lem. 2.4.3]).

Now suppose that T_n is a p-power torsion representation of G_K and $\hat{M}_n$ is the associated étale (φ, τ)-module. Suppose there exists a surjective map of G_K-representations $f \colon L \twoheadrightarrow T_n$, where L is a semistable finite free $\mathbb{Z}_p$-representation with Hodge-Tate weights in $[-r, 0]$ (we call such f *a loose semistable lift*). The loose semistable lift induces a surjective map (which we still denote by f) $f\colon \hat{\mathcal{L}} \twoheadrightarrow \hat{M}_n$, where $\hat{\mathcal{L}}$ is the étale (φ,τ)-module associated to L. Suppose $\hat{\mathfrak{L}}$ is the $(\varphi,\hat{G})$-module associated to L. Then it is easy to see that $f(\hat{\mathfrak{L}})$ is a $(\varphi,\hat{G})$-model of $\hat{M}_n$.

2.3. **Torsion Kisin modules.** Let $\mathfrak{M} \in {}'\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ be a torsion Kisin module such that $M := \mathfrak{M}[\frac{1}{u}]$ is a finite free $\mathfrak{S}_n[\frac{1}{u}]$-module (i.e., the torsion G_∞-representation associated to M is finite free over $\mathbb{Z}/p^n\mathbb{Z}$). For each $0 \le i < j \le n$, we define

$$\mathfrak{M}^{i,j} := \mathrm{Ker}(p^i\mathfrak{M} \xrightarrow{p^{j-i}} p^j\mathfrak{M}).$$

Following the discussion above [Liu07, Lem. 4.2.4], we have $\mathfrak{M}^{i,j} \in {}'\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$. We also have $\mathfrak{M}^{i,j}[\frac{1}{u}] = p^{n-j+i}M$, and so it is finite free over $\mathcal{O}_{\mathcal{E},j-i}$.

Define the function $\mathfrak{c}(r) := 4\cdot 4^r e^2 r^3$. This is (bigger than) the $\mathfrak{c}$ in [Liu07, p. 653].

The following three lemmas are extracted from [Liu07] and played important roles there.

Lemma 2.10. *Let $\hat{\mathfrak{M}}$ be a torsion $(\varphi,\hat{G})$-module, and suppose it is torsion semistable in the sense that it is the quotient of two finite free $(\varphi,\hat{G})$-modules (with height r). Let $0 \le i < j$; then $\hat{\mathfrak{M}}^{i,j} := \mathrm{Ker}(p^i\hat{\mathfrak{M}} \xrightarrow{p^{j-i}} p^j\hat{\mathfrak{M}})$ is also torsion semistable. In fact, if $\hat{\mathfrak{M}} = \hat{\mathfrak{L}}/\hat{\mathfrak{L}}'$, then there exist finite free $\hat{\mathfrak{N}}$ and $\hat{\mathfrak{N}}'$ such that $\hat{\mathfrak{M}}^{i,j} = \hat{\mathfrak{N}}/\hat{\mathfrak{N}}'$, which furthermore satisfy:*

$$\hat{T}^*(\hat{\mathfrak{L}})[\frac{1}{p}] = \hat{T}^*(\hat{\mathfrak{L}}')[\frac{1}{p}] = \hat{T}^*(\hat{\mathfrak{N}})[\frac{1}{p}] = \hat{T}^*(\hat{\mathfrak{N}}')[\frac{1}{p}].$$

Proof. The lemma is extracted from the proof of [Liu07, Lem. 4.4.1].

Let $\hat{\mathfrak{N}} := \mathrm{Ker}(p^j\hat{\mathfrak{L}} \to p^j\hat{\mathfrak{M}})$ and let $\hat{\mathfrak{N}}' := \mathrm{Ker}(p^i\hat{\mathfrak{L}} \to p^i\hat{\mathfrak{M}})$. Both $\hat{\mathfrak{N}}$ and $\hat{\mathfrak{N}}'$ are finite free $(\varphi,\hat{G})$-modules by [Liu07, Cor. 2.3.8] (also note that the functor $\mathfrak{M} \mapsto \hat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ is exact, by [CL11, Lem. 3.1.2]). There is a commutative diagram of $(\varphi,\hat{G})$-modules:

$$\begin{tikzcd}
0 \arrow[r] & \hat{\mathfrak{N}}' \arrow[r] \arrow[d] & p^i\hat{\mathfrak{L}} \arrow[r] \arrow[d, "\simeq"] & p^i\hat{\mathfrak{M}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \hat{\mathfrak{N}} \arrow[r] & p^j\hat{\mathfrak{L}} \arrow[r] & p^j\hat{\mathfrak{M}} \arrow[r] & 0
\end{tikzcd}$$

where all the vertical arrows are the $\times p^{j-i}$ map. By the snake lemma, we have

$$0 \to \hat{\mathfrak{N}}' \to \hat{\mathfrak{N}} \to \hat{\mathfrak{M}}^{i,j} \to 0.$$

$\square$

Lemma 2.11. *Suppose $\mathfrak{M}, \mathfrak{N} \in {}'\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$, both finite free over $\mathfrak{S}_n$, and $\mathfrak{M}[1/u] = \mathfrak{N}[1/u]$. Suppose $n \ge \mathfrak{c}(r)$; then $p^{\mathfrak{c}(r)}\mathfrak{M} = p^{\mathfrak{c}(r)}\mathfrak{N}$.*

Proof. This is [Liu07, Cor. 4.2.5]. $\square$

Lemma 2.12. *Suppose $\mathfrak{M} \in {}'\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ such that $\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ is finite free over $\mathcal{O}_{\mathcal{E},n}$. Suppose $n > 2\mathfrak{c}(r)$; then $\mathfrak{M}^{\mathfrak{c}(r),n-\mathfrak{c}(r)}$ is finite free over $\mathfrak{S}_{n-2\mathfrak{c}(r)}$.*

Proof. This is extracted from [Liu07, Lem 4.3.1]. $\square$

3. Limit of torsion representations

In this section, we prove our main theorem.

Theorem 3.1. *Let $T \in \mathrm{Rep}^{\mathrm{fr}}_{\mathbb{Z}_p}(G_K)$ of rank d. For each $n \geq 1$, suppose T/p^nT is torsion semistable (resp., crystalline) of weight $h(n)$. If*

$$h(n) < \frac{1}{2d}\log_{16} n, \quad \forall n \gg 0,$$

then $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semistable (resp., crystalline).

Proof. Suppose $\hat{M}$ is the étale (φ, τ)-module associated to T. For each $n \geq 1$, since T/p^nT is torsion semistable (resp., crystalline), let $\hat{\mathfrak{M}}_n$ be a $(\varphi, \hat{G})$-model of $\hat{M}/p^n\hat{M}$ associated to a loose semistable (resp., crystalline) lift of T/p^nT.

Denote $y_n = n^2 - 2\mathfrak{c}(h(n^2))$. It is easy to check that there exists some n_0 such that when $n \geq n_0$, we have:

- $\mathfrak{c}(h(n)) < \sqrt{n}$, which implies that $y_n > 0$ and $y_{n+1} - y_n > 0$,
- and $y_n - n > \mathfrak{c}(\lceil \log_{16}(n+1) \rceil)$, where $\lceil \cdot \rceil$ is the ceiling function.

Now, for any $n \geq n_0$, let

$$A_n := \mathfrak{M}_{n^2}^{\mathfrak{c}(h(n^2)),\quad n^2 - \mathfrak{c}(h(n^2))}.$$

By Lemma 2.12 (note that A_n is a Kisin model of $M/p^{y_n}M$), A_n is a Kisin module finite free over $\mathfrak{S}_{y_n}$ (of rank d) of height bounded by $h(n^2)$. Let $\mathfrak{M}'_n := p^{y_n - n}A_n$; then it is finite free over $\mathfrak{S}_n$ of height bounded by $h(n^2)$. Now we claim that $p\mathfrak{M}'_{n+1} = \mathfrak{M}'_n$ $\forall n \geq n_0$.

To show the claim, consider A_n and $p^{y_{n+1} - y_n}A_{n+1}$, where both are finite free over $\mathfrak{S}_{y_n}$ (and both are models of $M/p^{y_n}M$), with heights bounded by $\log_{16}(n+1)$ (because $\max\{h(n^2), h((n+1)^2)\} < \log_{16}(n+1)$). So by Lemma 2.11, we have

$$p^{\mathfrak{c}(\lceil \log_{16}(n+1) \rceil)}A_n = p^{\mathfrak{c}(\lceil \log_{16}(n+1) \rceil)}p^{y_{n+1} - y_n}A_{n+1}.$$

Multiply both sides with $p^{y_n - n - \mathfrak{c}(\lceil \log_{16}(n+1) \rceil)}$; we get $p\mathfrak{M}'_{n+1} = \mathfrak{M}'_n$.

Now, define $\widetilde{\mathfrak{M}} := \varprojlim_{n \geq n_0} \mathfrak{M}'_n$. Then it is a finite free $\mathfrak{S}$-module of rank d and there is a natural φ-action on it. We claim that

- $\widetilde{\mathfrak{M}}$ is a Kisin module, i.e., it is of finite $E(u)$-height.

To prove the claim, pick any $\mathfrak{S}$-basis of $\widetilde{\mathfrak{M}}$ and consider the matrix A of φ with respect to the basis. It is sufficient to show that there exists $b \in \mathfrak{S}$ such that $(\det A) \cdot b = E(u)^s$ for some s. Note that for each $n \geq n_0$, there exists some $B_n \in \mathrm{Mat}_d(\mathfrak{S})$ such that $AB_n = E(u)^{h(n^2)}\mathrm{Id} \pmod{p^n}$ (where Id is the identity matrix), and so

$$\det A \cdot \det B_n = E(u)^{dh(n^2)} \pmod{p^n}.$$

We then conclude by Lemma 3.2 below.

Next we show that we can upgrade $\widetilde{\mathfrak{M}}$ to a $(\varphi, \hat{G})$-module. The strategy is quite similar to what is done in [Liu07, §§5, 6, 7, 8]. However, because of the work [Liu10] (which substantially used the results in [Liu07, §§5, 6, 7, 8]), it is much easier now.

As we have shown that $\widetilde{\mathfrak{M}}$ is a Kisin module, it is obvious that $T^*_{\mathfrak{S}}(\widetilde{\mathfrak{M}}) = T|_{G_\infty}$, and so $\widetilde{\mathfrak{M}}$ is a Kisin model of M. Consider the $\hat{G}$-action on

$$W(F_\tau) \otimes_{\varphi, W(F_\tau)} \hat{M} = W(F_\tau) \otimes_{\varphi, \mathfrak{S}} \widetilde{\mathfrak{M}}.$$

By Lemma 2.10, all the modules

$$\hat{\mathfrak{M}}'_n = \mathrm{Ker}(p^{\mathfrak{c}(h(n^2))+y_n-n}\hat{\mathfrak{M}}_{n^2} \xrightarrow{p^n} p^{n^2-\mathfrak{c}(h(n^2))}\hat{\mathfrak{M}}_{n^2})$$

are also torsion semistable, and so the $\hat{G}$-actions on $W(F_\tau)\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}'_n$ descend to $\hat{G}$-actions on $\hat{\mathcal{R}}\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}'_n$. By taking the inverse limit, the $\hat{G}$-action on $W(F_\tau)\otimes_{\varphi,\mathfrak{S}}\widetilde{\mathfrak{M}}$ descends to a $\hat{G}$-action on $\hat{\mathcal{R}}\otimes_{\varphi,\mathfrak{S}}\widetilde{\mathfrak{M}}$, and so $(\widetilde{\mathfrak{M}},\varphi,\hat{G})$ is a $(\varphi,\hat{G})$-module. Now it is obvious that $\hat{T}^*(\hat{\widetilde{\mathfrak{M}}}) = T$, and so T is semistable.

Now we only need to deal with the crystalline case. When the conditions in Lemma 2.8 are satisfied (that is, when $p>2$, or when $p=2$ and $K_\infty\cap K_{p^\infty}=K$), then the $\hat{G}$-actions on $\hat{\mathcal{R}}\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}'_n$ satisfy the (torsion version of the) conclusion in loc. cit., and so the $\hat{G}$-action on $\hat{\widetilde{\mathfrak{M}}}$ satisfies the conclusion in loc. cit. as well (note that $u^p\varphi(\mathfrak{t})W(R)$ is p-adically closed in $W(R)$), and so $\hat{\widetilde{\mathfrak{M}}}$ is crystalline.

When $p=2$ and $K_\infty\cap K_{p^\infty}\neq K$, then we can argue similarly as in the very final paragraph of [Liu10] (which is the errata for [Liu07]). Namely, we can show that T is crystalline over both $K(\pi_1)$ and $K(\mu_4)$, and so T is crystalline over $K(\pi_1)\cap K(\mu_4)=K$. $\square$

Lemma 3.2. *Let $\alpha\in\mathfrak{S}$ and suppose there exists some n_0 such that for any $n\geq n_0$, there exists $\beta_n\in\mathfrak{S}$ such that*

$$\alpha\beta_n = E(u)^{dh(n^2)} \pmod{p^n\mathfrak{S}},$$

where d and $h(n^2)$ are as in Theorem 3.1. Then there exist some $s\in\mathbb{Z}^{\geq 0}$ and $\gamma\in\mathfrak{S}$ such that $\alpha\gamma=E(u)^s$.

Before we prove the lemma, we recall a useful lemma. Note that $E(u)$ is not a zero divisor in $\mathfrak{S}_n$ $\forall n\geq 0$, so it is OK to do "division by $E(u)$" in $\mathfrak{S}_n$.

Lemma 3.3 ([Liu07, Lem. 4.2.2])**.** *Suppose $f,g\in\mathfrak{S}_n$ with $n\geq 2$ and suppose $E(u)|fg \pmod{p^n}$. Then we have*

$$E(u)|f \pmod{p)^{\lfloor \frac{n}{2}\rfloor}} \text{ or } E(u)|g \pmod{p^{\lfloor \frac{n}{2}\rfloor}},$$

where $\lfloor\cdot\rfloor$ is the floor function.

The following easy corollary is convenient for our use.

Corollary 3.4. *Suppose $f,g\in\mathfrak{S}_{2^n}$. Suppose $E(u)^k|fg \pmod{p^{2^n}}$ where $k<n$. Then we will have*

$$E(u)^a|f \pmod{p^{2^{n-k}}} \text{ and } E(u)^b|g \pmod{p^{2^{n-k}}}$$

for some $a,b\geq 0$ such that $a+b=k$.

Proof of Lemma 3.2. First we have $u\nmid\alpha$. This is because when n is big enough, $f_0(\alpha\beta_n)=f_0(E(u)^{dh(n^2)}+p^n\theta_n)$ for some $\theta_n\in\mathfrak{S}$, and the right-hand side is $\neq 0$, because $dh(n^2)<n$. Here f_0 is the $W(k)$-linear map $\mathfrak{S}\to W(k)$ with $f_0(u)=0$.

Next, suppose $E(u)^{x_n}|\alpha$ in $\mathfrak{S}_n$. Then we claim that there exists s such that $x_n\leq s$ $\forall n\geq n_0$. To prove the claim, write $\alpha=E(u)^{x_n}\theta_{1,n}+p^n\theta_{2,n}$ for some $\theta_{1,n},\theta_{2,n}\in\mathfrak{S}$ $\forall n\geq n_0$. Since $f_0(\alpha)\neq 0$ and $p|f_0(E(u))$, it is easy to see that the sequence $\{x_n\}$ has to be bounded.

Finally, we claim that for all $n\gg 0$, there exists $\gamma_n\in\mathfrak{S}_n$ such that $\alpha\gamma_n=E(u)^s \pmod{p^n}$ (note that such γ_n is unique). We only need to show the existence of such γ_n for a sequence $\{n_m\}$ going to infinity.

For all $m > \max\{s, n_0\}$, consider $n = 16^{dm}$, so $h(n^2) \le m$. We can and do assume that $\alpha\beta_n = E(u)^{dm} \pmod{p^{16^{dm}}}$ (when $h(n^2) < m$, we can simply multiply some $E(u)$-power to β_n, and it does not affect our result). We want to show that there exists $\gamma_n \in \mathfrak{S}_n$ such that $\alpha\gamma_n = E(u)^s \pmod{p^{16^{dm}}}$.

Take any $m' > 2m$ and let $n' = 16^{dm'}$, so we have $\alpha\beta_{n'} = E(u)^{dm'} \pmod{p^{16^{dm'}}}$. Apply Corollary 3.4; then we will have

$$E(u)^a | \alpha \pmod{p^{2^{4dm'-dm'}}} \text{ and } E(u)^b | \beta_{n'} \pmod{p^{2^{4dm'-dm'}}},$$

where $a + b = dm'$. However, we always have $a \le s$, and so $b \ge dm' - s$ (and $dm' - s > 0$ because $m > s$). That is, we now have (note that $2^{4dm'-dm'} > 16^{dm}$)

$$\alpha\beta_{n'} = E(u)^{dm'} \pmod{p^{16^{dm}}} \text{ and } E(u)^{dm'-s} \mid \beta_{n'} \pmod{p^{16^{dm}}}.$$

So we can simply let $\gamma_n = \frac{\beta_{n'}}{E(u)^{dm'-s}}$.

Now simply let $\gamma := \varprojlim_{n>>0} \gamma_n$, and we are done. $\square$

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