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SOME SUFFICIENT CONDITIONS FOR NOVIKOV'S CRITERION

NGUYEN TIEN DUNG

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ABSTRACT. In this note, we employ the techniques of Malliavin calculus to provide some sufficient conditions for a stochastic process to satisfy Novikov's criterion. In particular, we obtain an improvement for Buckdahn's results established in Probab. Theory Related Fields 89 (1991), 211-238 and a generalization of Borell-TIS inequality.

1. Introduction

It is known that Girsanov's theorem is one of the most fundamental tools in stochastic analysis. There are several useful conditions for a stochastic process to satisfy Girsanov's theorem (we refer the reader to the monograph [5] for more details and for applications in finance), but the most popular one was suggested and proved by Novikov [7]. We recall that a stochastic $\{u_t, t \in [0,T]\}$ is said to satisfy Novikov's criterion if

$$E\exp\left(\frac{1}{2}\int_0^T u_s^2 ds\right) < \infty \tag{1.1}$$

(the constant $1/2$ is the best possible). Obviously, (1.1) is fulfilled whenever u is bounded. Otherwise, it should be noted that Novikov's criterion is often difficult to verify directly. Motivated by this note, the aim of the present paper is to introduce some sufficient conditions which provide us a common method to verify Novikov's criterion. Our idea is pretty simple and based on the explicit expression

$$E\exp\left(\frac{1}{2}\int_0^T u_s^2 ds\right) = 1 + \int_0^\infty x e^{\frac{1}{2}x^2} P(X > x)dx,$$

where $X := \left(\int_0^T u_s^2 ds\right)^{\frac{1}{2}}$. In order to be able to check (1.1) we use the techniques of Malliavin calculus to establish an upper bound for the tail probability, $P(X > x)$, of the random variable X. Our main results are formulated in Theorem 2.1.

On the other hand, in recent years, stochastic differential equations involving the maximum processes have been used in various fields. For example, the following equation has been investigated for applications in finance (see, e.g., [4] and

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references therein):

$$dX_t = (r - \delta(S_t, Y_t))X_t dt + \sigma(S_t, Y_t)X_t dB_t,\ X_0 = x,$$

where B_t is a standard Brownian motion, $S_t = s \vee \max_{0\le u\le t} X_u$, and $Y_t = y \vee \max_{0\le u\le t}(S_u - X_u)$. It is known from Theorem 12.1.8 in [8] that the normalized market $\{X_t\}$ is arbitrage free if

$$E\exp\left(\frac{1}{2}\int_0^T \frac{(r-\delta(S_t,Y_t))^2}{\sigma^2(S_t,Y_t)}dt\right) < \infty.$$

Hence, it is necessary to check Novikov's criterion for maximum functionals. In Theorem 2.2, we prove an inequality for the Malliavin derivative of maximum processes and apply Theorem 2.1 to obtain Novikov's criterion of the form

$$E\exp\left(\frac{1}{2}\int_0^T h^2(\max_{0\le s\le t} u_s, v_t)\,dt\right) < \infty, \tag{1.2}$$

where h is a Lipschitz function and u, v are Malliavin differentiable stochastic processes.

2. The main results

For the reader's convenience, let us recall the definition of Malliavin derivatives (for more details see [9]). We suppose that $(W_t)_{t\in[0,T]}$ is defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is a natural filtration generated by the Brownian motion W. For $h \in L^2[0,T]$, we denote by $W(h)$ the Wiener integral

$$W(h) = \int_0^T h(t)dW_t.$$

Let $\mathcal{S}$ denote the dense subset of $L^2(\Omega, \mathcal{F}, P)$ consisting of smooth random variables of the form

$$F = f(W(h_1), \ldots, W(h_n)), \tag{2.1}$$

where $n \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^n), h_1, \ldots, h_n \in L^2[0,T]$. If F has the form (2.1), we define its Malliavin derivative as the process $DF := \{D_t F, t \in [0,T]\}$ given by

$$D_t F = \sum_{k=1}^n \frac{\partial f}{\partial x_k}(W(h_1), \ldots, W(h_n))h_k(t).$$

For any $1 \le p < \infty$, we shall denote by $\mathbb{D}^{1,p}$ the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p}^p := E|F|^p + E\left[\int_0^T |D_u F|^p du\right].$$

A random variable F is said to be Malliavin differentiable if it belongs to $\mathbb{D}^{1,2}$.

The main results of this paper are stated in the following theorem, where we obtain two different conditions for Novikov's criterion.

Theorem 2.1. *Let* $\{u_t, 0 \le t \le T\}$ *be a stochastic process in* $\mathbf{L}^{1,2} := L^2([0,T], \mathbb{D}^{1,2})$. *Assume that one of the following two conditions holds:*

$$\int_0^T \int_0^T |D_r u_s|^2 dr ds \le c^2 < 1 \ a.s., \tag{2.2}$$

$$\int_0^T \int_0^T E[|D_r u_s|^2 | \mathcal{F}_r] dr ds \le c^2 < 1 \ a.s., \tag{2.3}$$

where c is a constant. Then, Novikov's criterion (1.1) *is satisfied.*

Proof. We separate the proof into two parts.
Part 1. In this part, we show that the condition (2.2) implies Novikov's criterion (1.1). We put

$$X := \left(\int_0^T u_s^2 ds \right)^{\frac{1}{2}}.$$

Since $u \in \mathbf{L}^{1,2}$, this implies that $X \in \mathbb{D}^{1,2}$, and its Malliavin derivative is given by

$$D_r X = \frac{\int_0^T u_s D_r u_s ds}{\left(\int_0^T u_s^2 ds \right)^{\frac{1}{2}}}.$$

By the Hölder inequality, we deduce that

$$|D_r X| \le \left(\int_0^T |D_r u_s|^2 ds \right)^{\frac{1}{2}} \quad \text{a.s.} \tag{2.4}$$

As a consequence,

$$\int_0^T |D_r X|^2 dr \le \int_0^T \int_0^T |D_r u_s|^2 dr ds \le c^2 \ \text{a.s.} \tag{2.5}$$

This, combined with Theorem 9.1.1 in [11], yields

$$P(X > x) \le 2e^{-\frac{(x-\mu)^2}{2c^2}}, \ \forall\, x > \mu, \tag{2.6}$$

where $\mu = EX$.

Obviously, we have

$$\begin{aligned} E \exp\left(\frac{1}{2} \int_0^T u_s^2 ds \right) &= Ee^{\frac{1}{2}X^2} \\ &= 1 + \int_0^\infty x e^{\frac{1}{2}x^2} P(X > x) dx \\ &= 1 + \int_0^\mu x e^{\frac{1}{2}x^2} P(X > x) dx + \int_\mu^\infty x e^{\frac{1}{2}x^2} P(X > x) dx \\ &\le 1 + \mu^2 e^{\frac{1}{2}\mu^2} + 2 \int_\mu^\infty x e^{\frac{1}{2}x^2} e^{-\frac{(x-\mu)^2}{2c^2}} dx. \end{aligned} \tag{2.7}$$

The last addend in the right hand side of (2.7) is finite because $c^2 < 1$. So we finish the proof.

Part 2. Let us now point out that (2.3) implies (1.1). It follows from (2.4) and Hölder inequality that

$$E[|D_rX||\mathcal{F}_r] \le E\bigg[\bigg(\int_0^T |D_r u_s|^2 ds\bigg)^{\frac{1}{2}} |\mathcal{F}_r\bigg]$$
$$\le \bigg(\int_0^T E[|D_r u_s|^2|\mathcal{F}_r] ds\bigg)^{\frac{1}{2}} \text{ a.s.}$$

Hence,

$$\int_0^T (E[|D_rX||\mathcal{F}_r])^2 dr \le \int_0^T \int_0^T E[|D_r u_s|^2|\mathcal{F}_r] dr ds \le c^2 \text{ a.s.}$$

By the Clark-Ocone formula we have

$$X = EX + \int_0^T E[D_rX|\mathcal{F}_r] dW_r.$$

Set $M_t = E[X - EX|\mathcal{F}_t]$. Then M_t is a martingale with $M_0 = 0, M_T = X - EX$, and

$$M_t = \int_0^t E[D_rX|\mathcal{F}_r] dW_r, \;\; 0 \le t \le T.$$

The quadratic variation of M_t satisfies

$$\langle M\rangle_t = \int_0^t (E[D_rX|\mathcal{F}_r])^2 dr \le \int_0^T (E[|D_rX||\mathcal{F}_r])^2 dr \le c^2 \text{ a.s.}$$

for all $0 \le t \le T$. Consequently, by the exponential martingale inequality (see, e.g., the inequality (A.5) in [9])

$$P(|X - EX| > x) = P(|M_T| > x) \le 2e^{-\frac{x^2}{2c^2}}, \;\; \forall\, x > 0.$$

The above estimate implies (2.6) because

$$P(X > x) = P(X - EX > x - \mu) \le P(|X - EX| > x - \mu), \; \forall\, x > \mu.$$

So we can finish the proof of Part 2. □

Remark 2.1. An interesting feature of Theorem 2.1 is that we do not require the adaptness of the stochastic process $\{u_t, 0 \le t \le T\}$. Hence, it can be applied to both the classical Girsanov's theorem and anticipative ones. In [2], Buckdahn introduced an anticipative Girsanov theorem which is a fundamental and important result. He provided the following set of sufficient conditions (see Theorem 4.9 in [2]):

(i) The condition (2.2) holds.

(ii) There exists a constant $q > 1$ such that

$$E \exp\bigg(\frac{q}{2}\int_0^T u_s^2 ds\bigg) < \infty. \tag{2.8}$$

We notice that (i) and (ii) are also sufficient conditions for the anticipative Girsanov theorem given by Enchev [3]. We observe from the proof of Theorem 2.1 that (i) implies that (2.8) holds for any $q < \frac{1}{c^2}$. Thus one can remove the condition (ii) required by Buckdahn. In other words, if a stochastic process u satisfies the condition (2.2), then it satisfies the anticipative Girsanov theorems proved by Buckdahn and Enchev.

Remark 2.2. When u is a functional of diffusion processes, the condition (2.3) seems to be easier to verify than (2.2). Let us give here an example. Consider the stochastic process $u_t = h(X_t)$, where h is a Lipschitz function with Lipschitz constant L_h and X_t is the solution to the delay stochastic differential equation

$$X_t = \varphi(0) + \int_0^t b(X_s)ds + \int_0^t \sigma(X_{s-\tau})dW_s, \quad t \in [0,T], \tag{2.9}$$

$X_t = \varphi(t), t \in [-\tau, 0]$, where $\tau > 0$ and φ is a continuous function on $[-\tau, 0]$. For the existence, uniqueness, and Malliavin differentiability of solutions, we assume that the coefficients b, σ are Lipschitz functions with Lipschitz constant L_b, L_σ, respectively. In addition, we assume that

$$\|\sigma\|_\infty = \sup_{x\in\mathbb{R}} |\sigma(x)| < \infty.$$

When $0 \le r \le t - \tau$, the Malliavin derivative of the solution satisfies

$$D_r X_t = \sigma(X_{r-\tau}) + \int_r^t \bar{b}(s) D_r X_s ds + \int_r^t \bar{\sigma}(s) D_r X_{s-\tau} \mathbb{1}_{[r+\tau,t]}(s) dW_s,$$

where $\bar{b}(s), \bar{\sigma}(s)$ are adapted stochastic processes bounded by L_b, L_σ, respectively. Hence, we can get

$$\begin{aligned}(D_r X_t)^2 = \sigma^2(X_{r-\tau}) + \int_r^t [2\bar{b}(s)(D_r X_s)^2 + \bar{\sigma}^2(s)(D_r X_{s-\tau})^2 \mathbb{1}_{[r+\tau,t]}(s)]ds \\ + \int_r^t 2\bar{\sigma}(s) D_r X_{s-\tau} D_r X_s \mathbb{1}_{[r+\tau,t]}(s) dW_s\end{aligned}$$

and

$$\begin{aligned}&E[(D_r X_t)^2|\mathcal{F}_r] \\ &\quad = \sigma^2(X_{r-\tau}) + E\left[\int_r^t [2\bar{b}(s)(D_r X_s)^2 + \bar{\sigma}^2(s)(D_r X_{s-\tau})^2 \mathbb{1}_{[r+\tau,t]}(s)]ds \Big| \mathcal{F}_r\right] \\ &\quad \le \sigma^2(X_{r-\tau}) + (2L_b + L_\sigma^2) \int_r^t E[(D_r X_s)^2|\mathcal{F}_r]ds.\end{aligned}$$

By Gronwall's lemma

$$E[(D_r X_t)^2|\mathcal{F}_r] \le \|\sigma\|_\infty^2 e^{(2L_b + L_\sigma^2)(t-r)} \text{ a.s.}$$

When $t - \tau < r \le t$, we have

$$D_r X_t = \sigma(X_{r-\tau}) + \int_r^t \bar{b}(s) D_r X_s ds,$$

which gives us

$$D_r X_t = \sigma(X_{r-\tau}) e^{\int_r^t \bar{b}(s)ds}$$

and

$$E[(D_r X_t)^2|\mathcal{F}_r] \le \|\sigma\|_\infty^2 e^{2L_b(t-r)}.$$

Hence, we can obtain the estimates

$$\begin{aligned}\int_0^T E[|D_r u_t|^2|\mathcal{F}_r]dr &= \int_0^t E[|D_r u_t|^2|\mathcal{F}_r]dr \leq L_h^2 \int_0^t E[|D_r X_t|^2|\mathcal{F}_r]dr \\ &\leq L_h^2 \int_0^{t-\tau} \|\sigma\|_\infty^2 e^{(2L_b+L_\sigma^2)(t-r)}dr + L_h^2 \int_{t-\tau}^t \|\sigma\|_\infty^2 e^{2L_b(t-r)}dr \\ &= L_h^2\|\sigma\|_\infty^2 \frac{e^{(2L_b+L_\sigma^2)t} - e^{(2L_b+L_\sigma^2)\tau}}{2L_b+L_\sigma^2} + L_h^2\|\sigma\|_\infty^2 \frac{e^{2L_b\tau}-1}{2L_b}\end{aligned}$$

and

$$\begin{aligned}&\int_0^T \int_0^T E[|D_r u_t|^2|\mathcal{F}_r]drdt \\ &\quad \leq L_h^2\|\sigma\|_\infty^2 \left[\left(\frac{e^{2L_b\tau}-1}{2L_b} - \frac{e^{(2L_b+L_\sigma^2)\tau}}{2L_b+L_\sigma^2}\right)T + \frac{e^{(2L_b+L_\sigma^2)T}-1}{(2L_b+L_\sigma^2)^2}\right].\end{aligned}$$

Thus the stochastic process $u_t = h(X_t)$ satisfies Novikov's criterion if

$$L_h^2\|\sigma\|_\infty^2 \left[\left(\frac{e^{2L_b\tau}-1}{2L_b} - \frac{e^{(2L_b+L_\sigma^2)\tau}}{2L_b+L_\sigma^2}\right)T + \frac{e^{(2L_b+L_\sigma^2)T}-1}{(2L_b+L_\sigma^2)^2}\right] < 1.$$

Theorem 2.2. *Let $\{u_t, 0 \leq t \leq T\}$ and $\{v_t, 0 \leq t \leq T\}$ be stochastic processes in $\mathbf{L}^{1,2}$. Suppose that u is of continuous paths and $\{\max_{0\leq s\leq t} u_s, 0 \leq t \leq T\} \in \mathbf{L}^{1,2}$. Let $h : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ be a Lipschitz function, i.e.,*

$$|h(x_1, y_1) - h(x_2, y_2)| \leq h_1|x_1 - y_1| + h_2|x_2 - y_2|, \ \forall\, x_1, y_1, x_2, y_2 \in \mathbb{R},$$

where h_1, h_2 are positive constants. If u, v, and h satisfy the relation

$$2h_1^2 \int_0^T \left(\max_{0\leq s\leq t} \int_0^T |D_r u_s|^2 dr\right) dt + 2h_2^2 \int_0^T \int_0^T |D_r v_t|^2 drdt \leq c^2 < 1 \text{ a.s.}, \tag{2.10}$$

then it holds that

$$E \exp\left(\frac{1}{2}\int_0^T h^2(\max_{0\leq s\leq t} u_s, v_t)\, dt\right) < \infty. \tag{2.11}$$

Proof. We first establish an estimate for the Malliavin derivative of maximum processes. We claim that, for $0 \leq t \leq T$,

$$\int_0^T (D_r \max_{0\leq s\leq t} u_s)^2 dr \leq \max_{0\leq s\leq t} \int_0^T (D_r u_s)^2 dr \text{ a.s.} \tag{2.12}$$

When $t = 0$, the claim is clear. We now fix $t \in (0, T]$ and define the sequence

$$u_t^{(\lambda)} := \frac{1}{\lambda}\log\left(\int_0^t e^{\lambda u_s} ds\right),\ \lambda > 0.$$

For each $\lambda > 0$, $u_t^{(\lambda)}$ belongs to $\mathbb{D}^{1,2}$, and its derivative is given by

$$D_r u_t^{(\lambda)} = \frac{\int_0^t e^{\lambda X_s} D_r u_s ds}{\int_0^t e^{\lambda u_s} ds},\ 0 \leq r \leq T.$$

Moreover, by the continuity of u and Laplace's principle (see, e.g., [10]), we have

$$\lim_{\lambda\to\infty} u_t^{(\lambda)} = \max_{0\leq s\leq t} u_s \text{ a.s.}$$

By considering a subsequence if necessary, the convergence also holds in $\mathbb{D}^{1,2}$ because of the closability of Malliavin derivatives.

By the Hölder inequality

$$|D_r u_t^{(\lambda)}|^2 \leq \frac{\int_0^t e^{\lambda X_s} |D_r u_s|^2 ds}{\int_0^t e^{\lambda u_s} ds} \text{ a.s.}$$

Hence,

$$\begin{aligned} \int_0^T |D_r u_t^{(\lambda)}|^2 dr &\leq \frac{\int_0^T \int_0^t e^{\lambda u_s} |D_r u_s|^2 ds dr}{\int_0^t e^{\lambda u_s} ds} \\ &= \frac{\int_0^t \left(\int_0^T |D_r u_s|^2 dr\right) e^{\lambda u_s} ds}{\int_0^t e^{\lambda u_s} ds} \\ &\leq \max_{0\leq s\leq t} \int_0^T |D_r u_s|^2 dr \text{ a.s.,} \end{aligned}$$

which implies the claim (2.12).

We now consider the stochastic process $w_t := h(\max_{0\leq s\leq t} u_s, v_t)$. By Proposition 1.2.4 in [9] we have $w_t \in \mathbb{D}^{1,2}$ and, moreover,

$$|D_r w_t| \leq h_1 |D_r \max_{0\leq s\leq t} u_s| + h_2 |D_r v_t| \text{ a.s.}$$

Hence,

$$|D_r w_t|^2 \leq 2h_1^2 |D_r \max_{0\leq s\leq t} u_s|^2 + 2h_2 |D_r v_t|^2 \text{ a.s.}$$

This, together with (2.10) and (2.12), yields

$$\int_0^T \int_0^T |D_r w_s|^2 dr ds \leq c^2 < 1 \text{ a.s.}$$

The above estimate points out that the process w_t satisfies the condition (2.2) of Theorem 2.1. So the proof is complete. □

Remark 2.3. For the Malliavin differentiability of maximum processes, we refer the reader to Proposition 2.1.10 in [9]. The estimate (2.12) itself is of independent interest. For example, we can obtain a generalization of Borell-TIS inequality as in the following proposition, which follows directly from Theorem 9.1.1 in [11].

Proposition 2.1. *Let $\{u_t, 0 \leq t \leq T\}$ be a centered stochastic process in $\mathbb{D}^{1,2}$. Suppose that $\max_{0\leq s\leq t} u_s \in \mathbb{D}^{1,2}$ for each $t \in [0,T]$ and that there exists a deterministic function $\sigma^2(t)$ such that almost surely*

$$\int_0^T |D_r u_t|^2 dr \leq \sigma^2(t).$$

Then, for $\sigma^2 := \sup_{t\in[0,T]} \sigma^2(t)$ and $x > 0$, it holds that

$$P\left(|\sup_{t\in[0,T]} u_t - E\sup_{t\in[0,T]} u_t| > x\right) \leq 2\exp\left(-\frac{x^2}{2\sigma^2}\right).$$

When u is a Gaussian process, we have $\int_0^T |D_r u_t|^2 dr = E|u_t|^2 := \sigma^2(t)$, and hence Proposition 2.1 reduces to the classical Borell-TIS inequality (see, e.g., Chapter 2 in [1]).

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Department of Mathematics, FPT University, Hoa Lac High Tech Park, Hanoi, Vietnam

Email address: dung_nguyentien10@yahoo.com, dungnt@fpt.edu.vn