

## $q$ -BESSEL FUNCTIONS AND ROGERS-RAMANUJAN TYPE IDENTITIES

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ABSTRACT. We evaluate  $q$ -Bessel functions at an infinite sequence of points and introduce a generalization of the Ramanujan function and give an extension of the  $m$ -version of the Rogers-Ramanujan identities. We also prove several generating functions for Stieltjes-Wigert polynomials with argument depending on the degree. In addition we give several Rogers-Ramanujan type identities.

### 1. INTRODUCTION

The Rogers-Ramanujan identities are

$$(1.1) \quad \begin{aligned} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} &= \frac{1}{(q, q^4; q^5)_{\infty}}, \\ \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} &= \frac{1}{(q^2, q^3; q^5)_{\infty}}, \end{aligned}$$

where the notation for the  $q$ -shifted factorials is the standard notation in [10], [12]. References for the Rogers-Ramanujan identities, their origins, and many of their applications are in [1], [2], and [4]. In particular we recall the partition theoretic interpretation of the first Rogers-Ramanujan identity as the partitions of an integer  $n$  into parts  $\equiv 1$  or  $4 \pmod{5}$  are equinumerous with the partitions of  $n$  into parts where any two parts differ by at least 2.

Garrett, Ismail, and Stanton [9] proved the  $m$ -version of the Rogers-Ramanujan identities

$$(1.2) \quad \sum_{n=0}^{\infty} \frac{q^{n^2+mn}}{(q; q)_n} = \frac{(-1)^m q^{-\binom{m}{2}} a_m(q)}{(q, q^4; q^5)_{\infty}} - \frac{(-1)^m q^{-\binom{m}{2}} b_m(q)}{(q^2, q^3; q^5)_{\infty}},$$

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where

$$\begin{aligned}
 (1.3) \quad a_m(q) &= \sum_j q^{j^2+j} \begin{bmatrix} m-j-2 \\ j \end{bmatrix}_q, \\
 b_m(q) &= \sum_j q^{j^2} \begin{bmatrix} m-j-1 \\ j \end{bmatrix}_q.
 \end{aligned}$$

The polynomials  $a_m(q)$  and  $b_m(q)$  were considered by Schur in conjunction with his proof of the Rogers-Ramanujan identities; see [1] and [9] for details. We shall refer to  $a_m(q)$  and  $b_m(q)$  as the Schur polynomials. The closed form expressions for  $a_m$  and  $b_m$  in (1.3) were given by Andrews in [3], where he also gave a polynomial generalization of the Rogers-Ramanujan identities. In this paper we give a family of Rogers-Ramanujan type identities involving the evaluation of  $q$ -Bessel and allied functions at special points. We also give the partition theoretic interpretation of these identities. In Section 2 we define the functions and polynomials used in our analysis. In Section 3 we present our Rogers-Ramanujan type identities. They resemble the  $m$  form in (1.2).

In a series of papers from 1903 till 1905 F. H. Jackson introduced  $q$ -analogues of Bessel functions. The modern notation for the modified  $q$ -Bessel functions, that is,  $q$ -Bessel functions with imaginary argument, is [11]

$$(1.4) \quad I_\nu^{(1)}(z; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{n=0}^\infty \frac{(z/2)^{\nu+2n}}{(q, q^{\nu+1}; q)_n}, \quad |z| < 2,$$

$$(1.5) \quad I_\nu^{(2)}(z; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{n=0}^\infty \frac{q^{n(n+\nu)}}{(q, q^{\nu+1}; q)_n} (z/2)^{\nu+2n},$$

$$(1.6) \quad I_\nu^{(3)}(z; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{n=0}^\infty \frac{q^{\binom{n}{2}}}{(q, q^{\nu+1}; q)_n} (z/2)^{\nu+2n}.$$

The functions  $I_\nu^{(1)}$  and  $I_\nu^{(2)}$  are related via

$$(1.7) \quad I_\nu^{(1)}(z; q) = \frac{I_\nu^{(2)}(z; q)}{(z^2/4; q)_\infty},$$

[12, Theorem 14.1.3].

$$(1.8) \quad S_n(x; q) = \frac{1}{(q; q)_n} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2} (-x)^k = \frac{1}{(q; q)_n} \sum_{k=0}^n \frac{(q^{-n}; q)_k}{(q; q)_k} q^{\binom{k+1}{2}} (xq^n)^k,$$

respectively. Ismail and C. Zhang [14] proved the following symmetry relation for the Stieltjes-Wigert polynomials:

$$(1.9) \quad q^{n^2} (-t)^n S_n(q^{-2n}/t; q) = S_n(t; q).$$

Section 2 contains the evaluation of  $I_\nu^{(2)}$  at an infinite number of special points. These new sums seem to be new. In Section 3 we introduce a generalization of the Ramanujan function

$$(1.10) \quad A_q(z) := \sum_{n=0}^\infty \frac{(-z)^n}{(q; q)_n} q^{n^2},$$

which Ramanujan introduced and studied many of its properties in the lost notebook [18]. It was later realized that this is an analogue of the Airy function. In Section 4 we introduce a function  $B_q^\alpha$  and prove some identities it satisfies, and then use them to derive several Rogers-Ramanujan type identities. The function  $B_q^\alpha$  is also a generalization of the Ramanujan function and is expected to lead to numerous new Rogers-Ramanujan type identities. The Stieltjes-Wigert polynomials satisfy a second order  $q$ -difference equation of polynomial coefficients for

$$f(x)y(qx) + g(x)y(x) + h(x)y(x/q) = 0.$$

In Section 5 we evaluate  $y(q^n x)$  in terms of  $y(x)$  and  $y(x/q)$  with explicit coefficients. Section 6 contains miscellaneous properties of the Stieltjes-Wigert polynomials.

### 2. $q$ -BESSEL SUMS

Our first result is the following theorem.

**Theorem 2.1.** *The function  $I_\nu^{(2)}$  has the representation*

$$(2.1) \quad I_\nu^{(2)}(2z; q) = \frac{z^\nu}{(q; q)_\infty} {}_1\phi_1(z^2; 0; q, q^{\nu+1}).$$

In particular  $I_\nu^{(2)}$  takes the special values

$$(2.2) \quad I_\nu^{(2)}(2q^{-n/2}; q) = \frac{q^{\nu n/2} S_n(-q^{-\nu-n}; q)}{(q^{n+1}; q)_\infty}$$

and

$$(2.3) \quad I_\nu^{(2)}(2q^{-n/2}; q) = \frac{q^{-\nu n/2} S_n(-q^{\nu-n}; q)}{(q^{n+1}; q)_\infty}.$$

*Proof.* Recall the Heine transformation [10, (III.2)]

$$(2.4) \quad {}_2\phi_1\left(\begin{matrix} A, B \\ C \end{matrix} \middle| q, Z\right) = \frac{(C/B, BZ; q)_\infty}{(C, Z; q)_\infty} {}_2\phi_1\left(\begin{matrix} ABZ/C, B \\ BZ \end{matrix} \middle| q, \frac{C}{B}\right).$$

The left-hand side of (2.1) is

$$\begin{aligned} & \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} z^\nu \sum_{k=0}^\infty \frac{q^{k^2+k\nu} z^{2k}}{(q^{\nu+1}, q; q)_k} = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} z^\nu \lim_{a, b \rightarrow \infty} {}_2\phi_1\left(\begin{matrix} a, b \\ q^{\nu+1} \end{matrix} \middle| q, \frac{q^{\nu+1} z^2}{ab}\right) \\ & = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} z^\nu \frac{1}{(q^{\nu+1}; q)_\infty} \lim_{a, b \rightarrow \infty} {}_2\phi_1\left(\begin{matrix} z^2, b \\ z^2 q^{\nu+1}/a \end{matrix} \middle| q, \frac{q^{\nu+1}}{b}\right), \end{aligned}$$

which implies (2.1). When  $z = q^{-n/2}$  and in view of (1.8), the left-hand side of (2.2) equals its right-hand side. Formula (2.3) follows from the symmetry relation (1.9). □

The results (2.2)–(2.3) of Theorem 2.1 when written as a series become

$$(2.5) \quad \begin{aligned} \sum_{k=0}^\infty \frac{q^{k(k+\nu-n)}}{(q, q^{\nu+1}; q)_k} &= \frac{q^{n\nu}}{(q^{\nu+1}; q)_\infty} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2} q^{-k(\nu+n)} \\ &= \frac{1}{(q^{\nu+1}; q)_\infty} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2} q^{k(\nu-n)}. \end{aligned}$$

Another way to prove (2.2) for integer  $\nu$  is to use the generating function

$$(2.6) \quad \sum_{-\infty}^{\infty} q^{\binom{m}{2}} I_m^{(2)}(z; q) t^m = (-tz/2, -qz/2t; q)_{\infty}.$$

Carlitz [6] did this for  $n = 0, 1$  and used this to give another proof of the Rogers-Ramanujan identities.

**Theorem 2.2** ([1]). *The  $q$ -binomial coefficient  $\begin{bmatrix} n \\ k \end{bmatrix}_q$  is the generating function for integer partitions whose Ferrers diagrams fit inside a  $k \times (n - k)$  rectangle.*

Recall that

$$(2.7) \quad I_{\nu}^{(j)}(z; q) = e^{-i\nu\pi/2} J_{\nu}^{(j)}(e^{i\pi/2}z; q), \quad j = 1, 2.$$

Chen, Ismail, and Muttalib [8] established an asymptotic series for  $J_{\nu}^{(2)}(z; q)$ . Their main term for  $r > 0$  is

$$(2.8) \quad I_{\nu}^{(2)}(r; q) = (r/2)^{\nu} \frac{(q^{1/2}; q)_{\infty}}{2(q; q)_{\infty}} \times \left[ (rq^{(\nu+1/2)/2}/2; q^{1/2})_{\infty} + (-rq^{(\nu+1/2)/2}/2; q^{1/2})_{\infty} \right]$$

as  $r \rightarrow +\infty$ . This determines the large  $r$  behavior of the maximum modulus of  $I_{\nu}^{(2)}$ .

We next derive a Mittag-Leffler expansion for  $I_{\nu}^{(1)}$ .

**Theorem 2.3.** *We have the expansion*

$$(2.9) \quad I_{\nu}^{(1)}(z; q) = \frac{(z/2)^{\nu}}{(q; q)_{\infty}^2} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}} S_n(-q^{\nu-n}; q)}{(1 - z^2q^n/4)}$$

Using residues it is easy to see that the difference between  $I_{\nu}^{(1)}(z; q)/(z^2; q)_{\infty}$  and the right-hand side of (2.9) is entire. We give a direct proof that this difference is zero.

*Proof of Theorem 2.3.* Use (1.8) to see that the sum on the right-hand side of (2.9) is

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}}}{(1 - z^2q^n/4)} \sum_{k=0}^n \frac{q^{k^2+k(\nu-n)}}{(q; q)_k (q; q)_{n-k}} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(\nu+(k+1)/2)}}{(q; q)_k (1 - z^2q^k/4)} {}_1\phi_1(z^2q^k/4; z^2q^{k+1}/4; q, q). \end{aligned}$$

Now apply (III.4) of [10] with  $a = z^2q^k/4, b = 1, c = 0, z = q$  to see that the above sum is  $(q; q)_{\infty}/(z^2q^{k+1}/4; q)_{\infty}$ . This shows that the right-hand side of (2.9) is given by

$$\frac{(z/2)^{\nu}}{(q, z^2/4; q)_{\infty}} {}_1\phi_1(z^2/4; 0; q, q^{\nu+1}),$$

and the result follows from (2.1) and (1.7). □

3. A GENERALIZATION OF THE RAMANUJAN FUNCTION

The Rogers-Ramanujan identities evaluate  $A_q$  at  $z = -1, -q$ . The result (1.2) evaluates  $A_q(-q^m)$ . This motivated us to consider the function

$$(3.1) \quad u_m(a, q) := \sum_{n=-\infty}^{\infty} \frac{q^{n^2+mn}}{(aq; q)_n}$$

as a function of  $q^m$ . When  $a = 1$  we get the Rogers-Ramanujan function. It is clear that

$$q^{m+1}u_{m+2}(a, q) = \sum_{n=-\infty}^{\infty} \frac{(1 - aq^n)}{(aq; q)_n} q^{n^2+mn}.$$

Therefore

$$(3.2) \quad q^{m+1}u_{m+2}(a, q) = u_m(a, q) - au_{m+1}(a, q).$$

Let  $u_m(a, q) = q^{-\binom{m}{2}}(-1)^m \tilde{u}_m(a, q)$ . Then  $\{\tilde{u}_m(a, q)\}$  satisfies the difference equation

$$(3.3) \quad y_{m+1} = q^{m-1}y_{m-1} + ay_m, \quad m = 0, \pm 1, \dots$$

We now solve (3.3) for  $m \geq 0$  using generating functions. The generating function  $Y(t) := \sum_{n=0}^{\infty} y_n t^n$  satisfies

$$Y(t) = \frac{y_0 + t(y_1 - ay_0)}{1 - at} + \frac{t^2}{1 - at} Y(qt),$$

whose solution is

$$Y(t) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)}t^{2n}}{(at; q)_{n+1}} [y_0 + tq^n(y_1 - ay_0)].$$

We now need two initial conditions, so choose two solutions  $\{c_m(a, q)\}$  and  $\{d_m(a, q)\}$ :

$$(3.4) \quad c_0(a, q) = 1, \quad c_1(a, q) = 0, \quad d_0(a, q) = 0, \quad d_1(a, q) = 1.$$

**Theorem 3.1.** *The polynomials  $\{c_m(a, q)\}$  and  $\{d_m(a, q)\}$  have the generating functions*

$$(3.5) \quad \sum_{n=0}^{\infty} c_n(a, q)t^n = \sum_{n=0}^{\infty} \frac{q^{n(n-1)}}{(at; q)_n} t^{2n},$$

$$(3.6) \quad \sum_{n=0}^{\infty} d_n(a, q)t^n = \sum_{n=0}^{\infty} \frac{q^{n^2}t^{2n+1}}{(at; q)_{n+1}}.$$

It is clear from the initial conditions (3.4) and the recurrence relation (3.3) that both  $\{c_n(a, q)\}$  and  $\{d_n(a, q)\}$  are polynomials in  $a$  and in  $q$ .

**Theorem 3.2.** *The polynomials  $\{c_n(a, q)\}$  and  $\{d_n(a, q)\}$  have the explicit form*

$$(3.7) \quad c_n(a, q) = \sum_{j=0}^{\lfloor (n-2)/2 \rfloor} q^{j(j+1)} \begin{bmatrix} n-j-2 \\ j \end{bmatrix}_q a^{n-2j-2},$$

$$(3.8) \quad d_n(a, q) = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} q^{j^2} \begin{bmatrix} n-j-1 \\ j \end{bmatrix}_q a^{n-2j-1}.$$

The proof follows from equations (3.5) and (3.6) and the  $q$ -binomial theorem.

**Theorem 3.3.** *We have*

$$\sum_{n=-\infty}^{\infty} \frac{q^{n^2+mn}}{(aq; q)_n} = (-1)^m q^{-\binom{m}{2}} \times \left[ c_m(a, q) \sum_{n=-\infty}^{\infty} \frac{q^{n^2}}{(aq; q)_n} + d_m(a, q) \sum_{n=-\infty}^{\infty} \frac{q^{n^2+n}}{(aq; q)_n} \right].$$

The case  $a = 1$  is the  $m$ -version of the Rogers-Ramanujan identities in (1.2) first proved by Garret, Ismail, and Stanton [9].

We now solve (3.3) for  $m \leq 0$ . From the initial conditions (3.4) it is clear that

$$(3.9) \quad c_{-1}(a, q) = -aq, \quad d_{-1}(a, q) = q.$$

We follow the same generating function technique and conclude that

$$(3.10) \quad \sum_{n=0}^{\infty} c_{-n}(a, q)t^n = \sum_{n=0}^{\infty} (-1)^n q^{\binom{n+1}{2}} a^n t^n (qt/a; q)_n,$$

$$(3.11) \quad \sum_{n=0}^{\infty} d_{-n}(a, q)t^n = qt \sum_{n=0}^{\infty} (-1)^n a^n t^n (qt/a; q)_n q^{\binom{n+2}{2}}.$$

This establishes the following theorem.

**Theorem 3.4.** *For  $n \geq 0$  we have*

$$(3.12) \quad c_{-n}(a, q) = (-1)^n q^{\binom{n+1}{2}} \sum_{k=0}^{\lfloor n/2 \rfloor} q^{k(k-n)} \begin{bmatrix} n-k \\ k \end{bmatrix}_q a^{n-2k},$$

$$(3.13) \quad d_{-n-1}(a, q) = (-1)^n q^{\binom{n+2}{2}} \sum_{k=0}^{\lfloor n/2 \rfloor} q^{k(k-n)} \begin{bmatrix} n-k \\ k \end{bmatrix}_q a^{n-2k}.$$

The polynomials  $\{c_n(a, q)\}$  and  $\{d_n(a, q)\}$  first appeared in Carlitz’s paper [7], where he introduced them as Fibonacci polynomials. Štampach [20] observed that they are orthogonal polynomials and studied their moment problem, including computing the corresponding Nevanlinna matrix. In Štampach’s notation

$$(3.14) \quad d_n(x, a) = \phi_n(x; q), \quad T_n(x; q) = q^{-n} c_{n+2}(xq, q), \quad n = 0, 1, \dots$$

He did not consider the case  $n < 0$ .

#### 4. ROGERS-RAMANUJAN TYPE IDENTITIES

In this section we prove several identities of Rogers-Ramanujan type. One of the proofs uses the Ramanujan  ${}_1\psi_1$  sum [10, (II.29)]

$$(4.1) \quad \sum_{-\infty}^{\infty} \frac{(a; q)_n}{(b; q)_n} z^n = \frac{(q, b/a, az, q/az; q)_{\infty}}{(b, q/a, z, b/az; q)_{\infty}}, \quad \left| \frac{b}{a} \right| < |z| < 1.$$

Throughout this section we define  $\rho$  by

$$(4.2) \quad \rho = e^{2\pi i/3}.$$

**Lemma 4.1.** *For nonnegative integers  $j, k, \ell, m, n$ , and  $\rho = e^{2\pi i/3}$  we have*

$$(4.3) \quad \sum_{k=0}^n \frac{(a; q)_k (a; q)_{n-k} (-1)^k}{(q; q)_k (q; q)_{n-k}} = \begin{cases} 0 & n = 2m + 1, \\ \frac{(a^2; q^2)_m}{(q^2; q^2)_m} & n = 2m \end{cases}$$

and

$$(4.4) \quad \sum_{\substack{j+k+\ell=n \\ j,k,\ell \geq 0}} \frac{(a; q)_j (a; q)_k (a; q)_\ell}{(q; q)_j (q; q)_k (q; q)_\ell} \rho^{k+2\ell} = \begin{cases} 0 & 3 \nmid n, \\ \frac{(a^3; q^3)_m}{(q^3; q^3)_m} & n = 3m. \end{cases}$$

For  $j, k, m, \ell, n \in \mathbb{Z}$ , we have

$$(4.5) \quad \sum_{j+k=n} \frac{(a; q)_j (a; q)_k (-1)^k}{(b; q)_j (b; q)_k} = \begin{cases} 0 & n = 2m + 1, \\ \frac{(q, b/a, -b, -q/a; q)_\infty}{(-q, -b/a, b, q/a; q)_\infty} \frac{(a^2; q^2)_m}{(b^2; q^2)_m} & n = 2m \end{cases}$$

and

$$(4.6) \quad \sum_{j+k+\ell=n} \frac{(a; q)_j (a; q)_k (a; q)_\ell \rho^{k+2\ell}}{(b; q)_j (b; q)_k (b; q)_\ell} = 0$$

and for  $3 \nmid n$ , we have

$$(4.7) \quad \begin{aligned} & \sum_{j+k+\ell=3m} \frac{(a; q)_j (a; q)_k (a; q)_\ell \rho^{k+2\ell}}{(b; q)_j (b; q)_k (b; q)_\ell} \\ &= \frac{(q, b/a; q)_\infty^3}{(b, q/a; q)_\infty^3} \frac{(b^3, q^3 a^{-3}; q^3)_\infty}{(q^3, b^3 a^{-3}; q^3)} \frac{(a^3; q^3)_m}{(b^3; q^3)_m}. \end{aligned}$$

*Proof.* Formula (4.3) follows from

$$\frac{(at; q)_\infty (-at; q)_\infty}{(t; q)_\infty (-t; q)_\infty} = \frac{(a^2 t^2; q^2)_\infty}{(t^2; q^2)_\infty}, \quad |t| < 1,$$

while (4.4) follows from

$$\frac{(at; q)_\infty (apt; q)_\infty (a\rho^2 t; q)_\infty}{(t; q)_\infty (\rho t; q)_\infty (\rho^2 t; q)_\infty} = \frac{(a^3 t^3; q^3)_\infty}{(t^3; q^3)_\infty}, \quad |t| < 1.$$

For  $|ba^{-1}| < |x| < 1$ , apply the Ramanujan  ${}_1\psi_1$  sum (4.1) to the identity

$$\frac{(ax, q/(ax); q)_\infty (-ax, -q/(ax); q)_\infty}{(x, b/(ax); q)_\infty (-x, -b/(ax); q)_\infty} = \frac{(a^2 x^2, q^2/(a^2 x^2); q^2)_\infty}{(x^2, b^2/(a^2 x^2); q^2)_\infty}$$

to derive (4.5). Similarly we apply (4.1) to

$$\begin{aligned} & \frac{(a^3 x^3, q^3/(a^3 x^3); q^3)_\infty}{(x^3, b^3/(a^3 x^3); q^3)_\infty} \\ &= \frac{(ax\rho^2, q/(ax\rho^2); q)_\infty (ax\rho, -q/(ax\rho); q)_\infty}{(x\rho^2, b/(ax\rho^2); q)_\infty (x\rho, -b/(ax\rho); q)_\infty} \frac{(ax, q/(ax); q)_\infty}{(x, b/(ax); q)_\infty} \end{aligned}$$

and establish (4.6)-(4.7). □

It must be noted that (4.3) is essentially the evaluation of a continuous  $q$ -ultraspherical polynomial at  $x = 0$  [12, (12.2.19)].

For  $\alpha > 0$ , let

$$(4.8) \quad A_q^{(\alpha)}(a; t) = \sum_{n=0}^{\infty} \frac{(a; q)_n q^{\alpha n^2} t^n}{(q; q)_n},$$

in particular,

$$A_q^{(1)}(q; t) = \omega(t; q), \quad A_{q^2}^{(2)}(q^2; t^2) = \omega(t^2; q^4), \quad A_q^{(1)}(0; t) = A_q(-t),$$

where

$$\omega(v; q) = \sum_{n=0}^{\infty} q^{n^2} v^n.$$

**Theorem 4.2.** *Let  $\alpha \geq 0$ . Then*

$$(4.9) \quad A_{q^2}^{(2\alpha)}(a^2; t^2) = \sum_{j=0}^{\infty} \frac{(a; q)_j q^{\alpha j^2} (-t)^j}{(q; q)_j} A_q^{(\alpha)}(a; tq^{2\alpha j}).$$

For  $\rho = e^{2\pi i/3}$  we have

$$(4.10) \quad A_{q^3}^{(3\alpha)}(a^3; t^3) = \sum_{j,k=0}^{\infty} \frac{(a; q)_j (a; q)_k \rho^k q^{\alpha(j+k)^2} t^{j+k}}{(q; q)_j (q; q)_k} A_q^{(\alpha)}(a; \rho^2 q^{2\alpha(j+k)} t).$$

*Proof.* These two identities can be proved by applying (4.3) and (4.4) and straight-forward series manipulation. □

**Corollary 4.3.** *For any  $m = 0, 1, \dots$  we have*

$$(4.11) \quad \begin{aligned} & \frac{q^{-\binom{m}{2}} a_m(q^2)}{(q^2, q^8; q^{10})_{\infty}} - \frac{q^{-\binom{m}{2}} b_m(q^2)}{(q^4, q^6; q^{10})_{\infty}} \\ &= \sum_{j=0}^{\infty} \frac{(-q^{1-m})^j}{(q; q)_j q^{j^2}} \left\{ \frac{a_{m+2j}(q)}{(q, q^4; q^5)_{\infty}} - \frac{b_{m+2j}(q)}{(q^2, q^3; q^5)_{\infty}} \right\}. \end{aligned}$$

In particular for  $m = 0, 1$  we have

$$(4.12) \quad \frac{1}{(q^2, q^8; q^{10})_{\infty}} = \sum_{j=0}^{\infty} \frac{(-q)^j}{(q; q)_j q^{j^2}} \left\{ \frac{a_{2j}(q)}{(q, q^4; q^5)_{\infty}} - \frac{b_{2j}(q)}{(q^2, q^3; q^5)_{\infty}} \right\}$$

and

$$(4.13) \quad \frac{1}{(q^4, q^6; q^{10})_{\infty}} = \sum_{j=0}^{\infty} \frac{(-1)^j}{(q; q)_j q^{j^2}} \left\{ \frac{b_{2j+1}(q)}{(q^2, q^3; q^5)_{\infty}} - \frac{a_{2j+1}(q)}{(q^1, q^4; q^5)_{\infty}} \right\}.$$

*Proof.* Observe that

$$A_q^{(1)}(0; z) = A_q(-z), \quad A_{q^2}^{(2)}(0, z^2) = A_{q^2}(-z^2)$$

and

$$A_{q^2}(-t^2) = A_{q^2}^{(2)}(0; t^2) = \sum_{j=0}^{\infty} \frac{q^{j^2} (-t)^j}{(q; q)_j} A_q^{(1)}(0; tq^{2j}) = \sum_{j=0}^{\infty} \frac{q^{j^2} (-t)^j}{(q; q)_j} A_q(-tq^{2j}).$$

We rewrite (1.2) to get

$$\frac{a_m(q)}{(q, q^4; q^5)_{\infty}} - \frac{b_m(q)}{(q^2, q^3; q^5)_{\infty}} = (-1)^m q^{\binom{m}{2}} \sum_{n=0}^{\infty} \frac{q^{n^2+mn}}{(q; q)_n}$$

and

$$(4.14) \quad \frac{a_{m+2j}(q)}{(q, q^4; q^5)_{\infty}} - \frac{b_{m+2j}(q)}{(q^2, q^3; q^5)_{\infty}} = \mathcal{O}\left(q^{\binom{m+2j}{2}}\right)$$



as  $j \rightarrow \infty$ . Then for any nonnegative integer  $m$  we get

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{q^{2n^2+2mn}}{(q^2; q^2)_n} &= \frac{(-1)^m q^{-2\binom{m}{2}} a_m(q^2)}{(q^2, q^8; q^{10})_{\infty}} - \frac{(-1)^m q^{-2\binom{m}{2}} b_m(q^2)}{(q^4, q^6; q^{10})_{\infty}} \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j q^{j^2+mj}}{(q; q)_j} A_q(-q^{m+2j}) \\ &= \sum_{j=0}^{\infty} \frac{(-1)^{j+m} q^{j^2+mj}}{(q; q)_j q^{\binom{m+2j}{2}}} \left\{ \frac{a_{m+2j}(q)}{(q, q^4; q^5)_{\infty}} - \frac{b_{m+2j}(q)}{(q^2, q^3; q^5)_{\infty}} \right\}, \end{aligned}$$

which simplifies to (4.11). From (4.14) it is clear that the above series actually converges very fast. □

We now consider the following generalization of the  ${}_1\psi_1$  function. For  $\alpha \geq 0$ , define  $B_q^{(\alpha)}$  by

$$(4.15) \quad B_q^{(\alpha)}(a, b; x) = \sum_{n=-\infty}^{\infty} \frac{(a; q)_n}{(b; q)_n} q^{\alpha n^2} x^n.$$

**Theorem 4.4.** *We have*

$$(4.16) \quad \frac{(-b, -q/a, q, b/a; q)_{\infty}}{(-q, -b/a, b, q/a; q)_{\infty}} B_{q^2}^{(2\alpha)}(a^2, b^2; x^2) = \sum_{j=-\infty}^{\infty} \frac{(a; q)_j q^{\alpha j^2} (-x)^j}{(b; q)_j} B_q^{(\alpha)}(a, b; xq^{2\alpha j})$$

and

$$(4.17) \quad \begin{aligned} B_{q^3}^{(3\alpha)}(a^3, b^3; x^3) &= \frac{(b, q/a; q)_{\infty}^3 (q^3, b^3 a^{-3}; q^3)_{\infty}}{(q, b/a; q)_{\infty}^3 (b^3, q^3 a^{-3}; q^3)_{\infty}} \\ &\times \sum_{j,k=-\infty}^{\infty} \frac{(a; q)_j (a; q)_k \rho^k q^{\alpha(j+k)^2} x^{j+k}}{(b; q)_j (b; q)_k} B_q^{(\alpha)}(a, b; xq^{2\alpha(j+k)}). \end{aligned}$$

The proof follows from (4.5), (4.6), and (4.7) and straightforward series manipulation.

**Corollary 4.5.** *The following Rogers-Ramanujan type identities hold:*

$$(4.18) \quad \frac{(-a, -q/a, q, q; q)_{\infty}}{(a, q/a, -q, -q; q)_{\infty}} \sum_{n=-\infty}^{\infty} \frac{q^{4n^2} x^{2n}}{1 - a^2 q^{2n}} = \sum_{j,k=-\infty}^{\infty} \frac{q^{(j+k)^2} (-1)^j x^{j+k}}{(1 - aq^j)(1 - aq^k)},$$

$$(4.19) \quad \frac{(q, q; q)_{\infty}}{(-q, -q; q)_{\infty}} \sum_{n=-\infty}^{\infty} \frac{q^{4n^2} x^{2n}}{1 + q^{2n+1}} = \sum_{j,k=-\infty}^{\infty} \frac{q^{(j+k)^2} (-1)^j x^{j+k}}{(1 + iq^{j+1/2})(1 + iq^{k+1/2})}.$$

*Proof.* Formula (4.18) is the special case  $\alpha = 1$  and  $b = aq$  of (4.16), while (4.19) is the special case  $a = -q^{1/2}i$  of (4.18). □

The special choice  $\alpha = 1$  and  $b = aq$  in (4.17) establishes

$$(4.20) \quad \sum_{n=-\infty}^{\infty} \frac{q^{9n^2} x^{3n}}{1 - a^3 q^{3n}} = \frac{(q^3; q^3)_{\infty}^2}{(q; q)_{\infty}^6} \frac{(a, q/a; q)_{\infty}^3}{(a^3, q^3 a^{-3}; q^3)_{\infty}} \times \sum_{j,k,\ell=-\infty}^{\infty} \frac{\rho^{k+2\ell} q^{(j+k+\ell)^2} x^{j+k+\ell}}{(1 - aq^j)(1 - aq^k)(1 - aq^{\ell})}.$$

Two special cases of (4.20) are worth noting. First when  $a = q^{1/3}$  we find that

$$(4.21) \quad \frac{(q; q)_{\infty}^7}{(q^3; q^3)_{\infty}^3 (q^{1/3}, q^{2/3}; q)_{\infty}^3} \sum_{n=-\infty}^{\infty} \frac{q^{9n^2} x^{3n}}{1 - q^{3n+1}} = \sum_{j,k,\ell=-\infty}^{\infty} \frac{\rho^{k+2\ell} q^{(j+k+\ell)^2} x^{j+k+\ell}}{(1 - q^{j+1/3})(1 - q^{k+1/3})(1 - q^{\ell+1/3})}.$$

With  $a = -q^{1/3}$  in (4.20) we conclude that

$$(4.22) \quad \sum_{n=-\infty}^{\infty} \frac{q^{9n^2} x^{3n}}{1 + q^{3n+1}} = \frac{(q^3; q^3)_{\infty}^2}{(q; q)_{\infty}^6} \frac{(-q^{1/3}, -q^{2/3}; q)_{\infty}^3}{(-q^2, -q; q^3)_{\infty}} \times \sum_{j,k,\ell=-\infty}^{\infty} \frac{\rho^{k+2\ell} q^{(j+k+\ell)^2} x^{j+k+\ell}}{(1 + q^{j+1/3})(1 + q^{k+1/3})(1 + q^{\ell+1/3})}.$$

It is clear that one can generate other identities by specializing the parameters in the master formulas.

### 5. $q$ -LOMMEL POLYNOMIALS

Iterating the three term recurrence relation of the  $q$ -Bessel function leads to (5.1)

$$q^{n\nu+n(n-1)/2} J_{\nu+n}^{(2)}(x; q) = h_{n,\nu} \left( \frac{1}{x}; q \right) J_{\nu}^{(2)}(x; q) - h_{n-1,\nu+1} \left( \frac{1}{x}; q \right) J_{\nu-1}^{(2)}(x; q),$$

where  $h_{n,\nu}(x; q)$  are the  $q$ -Lommel polynomials introduced in [11], [12, §14.4]. It is more convenient to use the polynomials

$$(5.2) \quad p_{n,\nu}(x; q) := e^{-i\pi n/2} h_{n,\nu}(ix) = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(q^{\nu}, q; q)_{n-j}}{(q, q^{\nu}; q)_j (q; q)_{n-2j}} (2x)^{n-2j} q^{j(j+\nu-1)}.$$

The identity (5.1) expressed in terms of  $I_{\nu}$ 's is

$$(5.3) \quad \begin{aligned} & (-1)^n q^{n\nu+n(n-1)/2} I_{\nu+n}^{(2)}(x; q) \\ &= p_{n,\nu}(1/x; q) I_{\nu}^{(2)}(x; q) - p_{n-1,\nu+1}(1/x; q) I_{\nu-1}^{(2)}(x; q). \end{aligned}$$

When  $x = 2q^{-k/2}$  we obtain, after replacing  $\nu$  by  $\nu + k$ ,

$$\begin{aligned} (-1)^n q^{n(n+2\nu+k-1)/2} S_k(-q^{\nu+n}; q) &= p_{n,\nu+k}(q^{k/2}/2; q) S_k(-q^{\nu}; q) \\ &\quad - q^{k/2} p_{n-1,\nu+k+1}(q^{k/2}/2; q) S_k(-q^{\nu-1}; q). \end{aligned}$$

We now rewrite this as a functional equation in the form

$$(5.4) \quad \begin{aligned} y^n q^{n(n+k-1)/2} S_k(yq^n; q) &= u_n(q^{k/2}, -yq^k; q) S_k(y; q) \\ &\quad - q^{k/2} u_{n-1}(q^{k/2}, -yq^{k+1}; q) S_k(y/q; q) \end{aligned}$$

with

$$(5.5) \quad u_n(x, y) = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(y, q; q)_{n-j}}{(q, y; q)_j (q; q)_{n-2j}} x^{n-2j}.$$

Therefore

$$(5.6) \quad S_k(y; q) = \frac{y^n q^{n(n+k-1)/2} u_n(q^{k/2}, -yq^{k+1}; q)}{\Delta_n} S_k(yq^n; q) - \frac{y^{n+1} q^{(n+1)(n+k)/2} u_{n+1}(q^{k/2}, -yq^{k+1}; q)}{\Delta_n} S_k(-q^{\nu+n+1}; q),$$

where

$$(5.7) \quad \Delta_n = u_n(q^{k/2}, -yq^{k+1}; q) u_n(q^{k/2}, -yq^k; q) - u_{n+1}(q^{k/2}, -yq^k; q) u_{n-1}(q^{k/2}, -yq^{k+1}; q).$$

### 6. IDENTITIES INVOLVING STIELTJES-WIGERT POLYNOMIALS

In this section we state several identities involving Stieltjes-Wigert polynomials and the Ramanujan function:

$$(6.1) \quad (xt, -t; q)_\infty = \sum_{n=0}^\infty q^{\binom{n}{2}} t^n S_n(xq^{-n}; q),$$

$$(6.2) \quad \frac{q^{\binom{n}{2}} x^n}{(q; q)_n} = \sum_{k=0}^n \frac{(-1)^k q^{\binom{k}{2}}}{(q; q)_{n-k}} S_k(xq^{-k}; q),$$

$$(6.3) \quad S_n(x) = \sum_{k=0}^\infty \frac{q^{\binom{k+1}{2}} (xq^n)^k A_q(xq^k)}{(q; q)_n (q; q)_k},$$

$$(6.4) \quad S_n(ab; q) = b^n \sum_{k=0}^n \frac{(b^{-1}; q)_k (-q^{1-n})^k q^{\binom{k}{2}}}{(q; q)_k} S_{n-k}(aq^k; q),$$

$$(6.5) \quad S_n(a; q) = \frac{(-aq; q)_\infty}{(q, -aq; q)_n} \sum_{k=0}^\infty \frac{q^{k^2} (-a)^k}{(q, -aq^{n+1}; q)_k},$$

$$(6.6) \quad S_{2n+1}(q^{-2n-1}; q) = 0, \quad S_{2n}(q^{-2n}; q) = \frac{(-1)^n q^{-n^2}}{(q^2; q^2)_n},$$

$$(6.7) \quad S_n\left(-q^{-n+1/2}; q\right) = \frac{q^{-(n^2-n)/4}}{(q^{1/2}; q^{1/2})_n},$$

$$(6.8) \quad S_n\left(-q^{-n-1/2}; q\right) = \frac{q^{-(n^2+n)/4}}{(q^{1/2}; q^{1/2})_n},$$

$$(6.9) \quad A_q(wz) = (wq; q)_\infty \sum_{n=0}^\infty \frac{q^{n^2} w^n}{(wq; q)_n} S_n(zq^{-n}; q),$$

$$(6.10) \quad A_q(z) = (q; q)_m \sum_{n=0}^\infty \frac{q^{n^2+mn} (-z)^n}{(q; q)_n} S_m(zq^n; q).$$

*Proofs.* Formula (6.1) follows from the definition (1.8) and Euler’s identities. Dividing both sides of (6.1) by  $(-t; q)_\infty$  then expanding  $1/(-t; q)_\infty$  on the right-hand

side implies (6.2). The expansion (6.3) follows from (1.10) and the  $q$ -binomial theorem in the form

$$(6.11) \quad (x; q)_n = \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q (-x)^j q^{\binom{j}{2}}.$$

To prove (6.4) start with (6.1) as

$$\sum_{n=0}^{\infty} q^{\binom{n}{2}} t^n S_n(abq^{-n}; q) = (abt, -t; q)_{\infty} = (abt, -bt; q)_{\infty} \frac{(-t; q)_{\infty}}{(-bt; q)_{\infty}},$$

then expand the first product in  $S_k(aq^{-k}; q)$  and the second term using the  $q$ -binomial theorem. The proof of (6.5) consists of writing

$$(-aq; q)_{\infty} / (-aq; q)_n (-aq^{n+1}; q)_k$$

as  $-aq^{n+k+1}; q)_{\infty}$ , then expanding this infinite product and using (6.11). The special values in (6.6) follow from letting  $x = 1$  in (6.1) then equating like powers of  $t$ . Similarly the special values in (6.7) and (6.8) follow from putting  $x = -q^{\mp 1/2}$  in (6.1). Replace  $x$  by  $z$ , then multiply by  $(-w)^n q^{\binom{n+1}{2}}$ , and sum to prove (6.9). To prove (6.10) we expand the right-hand side in powers of  $z$  and realize that the coefficient of  $(-z)^n$  is

$$\frac{q^{n^2+mn}}{(q; q)_n} {}_2\phi_1(q^{-m}, q^{-n}; 0; q, q).$$

By the  $q$ -Chu-Vandermonde sum [10, (II.6)] the  ${}_2\phi_1$  equals  $q^{-mn}$ . □

We note that the polynomials  $\{S_n(xq^{-n}; q)\}$  are related to the  $q^{-1}$ -Hermite polynomials, [5], [13], which are defined by

$$(6.12) \quad h_n(\sinh \xi | q) = \sum_{k=0}^n \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} (-1)^k q^{k(k-n)} e^{(n-2k)\xi}.$$

Indeed

$$(6.13) \quad S_n(e^{-2\xi} q^{-n}; q) = \frac{1}{(q; q)_n} h_n(\sinh \xi | q).$$

In fact (6.1) is equivalent to the generating function for the  $q^{-1}$ -Hermite polynomials, [12]. Moreover (6.13) and the generating function [12, Theorem 21.3.1] lead to

$$(6.14) \quad \sum_{n=0}^{\infty} \frac{(q; q)_n q^{n^2/4}}{(\sqrt{q}; \sqrt{q})_n} t^n S_n(zq^{-n}; q) = \frac{(-tq^{1/4}, -tq^{1/4}z; \sqrt{q})_{\infty}}{(-t^2z; q)_{\infty}}.$$

The Poisson kernel of  $q^{-1}$ -Hermite polynomials, [12, Theorem 21.2.3], implies that

$$(6.15) \quad \sum_{n=0}^{\infty} (q; q)_n q^{\binom{n}{2}} t^n S_n(zq^{-n}; q) S_n(\zeta q^{-n}; q) = \frac{(-t, -tz\zeta, tz, t\zeta; q)_{\infty}}{(t^2z\zeta/q; q)_{\infty}}.$$

Similarly one can derive other generating relations.

It must be noted that (6.7) and (6.8) when written in terms of the  $q^{-1}$ -Hermite polynomials are the evaluation of  $h_n(0|q)$ ; see [12, Corollary 21.2.2]. It is easy to see that the evaluations (6.7) and (6.8) are equivalent to the identity in the following theorem.

**Theorem 6.1.** *We have*

$$(6.16) \quad A_{q^2}(-b^2) = (b\sqrt{q}; q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2/2} b^n}{(q, b\sqrt{q}; q)_n}.$$

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