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ON THE REGULARITY OF SOLUTIONS TO THE k-GENERALIZED KORTEWEG-DE VRIES EQUATION

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ABSTRACT. This work is concerned with special regularity properties of solutions to the k-generalized Korteweg-de Vries equation. In [Comm. Partial Differential Equations 40 (2015), 1336–1364] it was established that if the initial datum is $u_0 \in H^l((b,\infty))$ for some $l \in \mathbb{Z}^+$ and $b \in \mathbb{R}$, then the corresponding solution $u(\cdot,t)$ belongs to $H^l((\beta,\infty))$ for any $\beta \in \mathbb{R}$ and any $t \in (0,T)$. Our goal here is to extend this result to the case where $l > 3/4$.

1. INTRODUCTION

In this note we study the regularity of solutions to the initial value problem (IVP) associated to the k-generalized Korteweg-de Vries equation

$$\begin{cases} \partial_t u + \partial_x^3 u + u^k \partial_x u = 0, & x,t \in \mathbb{R},\ k \in \mathbb{Z}^+, \\ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

The starting point is a property found by Isaza, Linares, and Ponce [4] concerning the propagation of smoothness in solutions of the IVP (1.1). To state it we first recall the following well-posedness (WP) result for the IVP (1.1):

Theorem A1. *If $u_0 \in H^{3/4^+}(\mathbb{R})$, then there exist $T = T(\|u_0\|_{\frac{3}{4}^+,2};k) > 0$ and a unique solution $u = u(x,t)$ of the IVP (1.1) such that*

$$\begin{aligned}
&\text{(i)} && u \in C([-T,T] : H^{3/4^+}(\mathbb{R})),\\
&\text{(ii)} && \partial_x u \in L^4([-T,T] : L^\infty(\mathbb{R})) \quad (\textit{Strichartz}),\\
&\text{(iii)} && \sup_x \int_{-T}^{T} |J^r \partial_x u(x,t)|^2\,dt < \infty \quad \textit{for} \quad r \in [0, 3/4^+],\\
&\text{(iv)} && \int_{-\infty}^{\infty} \sup_{-T\le t\le T} |u(x,t)|^2\,dx < \infty,
\end{aligned} \tag{1.2}$$

with $J = (1-\partial_x^2)^{1/2}$. Moreover, the map data-solution $u_0 \to u(x,t)$ is locally continuous (smooth) from $H^{3/4+}(\mathbb{R})$ into the class $X_T^{3/4+}$ defined in (1.2).

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If $k \geq 2$, then the result holds in $H^{3/4}(\mathbb{R})$. If $k = 1, 2, 3$, then T can be taken arbitrarily large.

For the proof of Theorem A1 we refer to [6], [1], and [3]. The proof of our main result, Theorem 1.1, is based on an energy estimate argument for which the estimate (ii) in (1.2) (i.e., the time integrability of $\|\partial_x u(\cdot,t)\|_\infty$) is essential. However, we remark that from the WP point of view it is not optimal. For a detailed discussion on the WP of the IVP (1.1) we refer to [7, Chapters 7-8].

Now we enunciate the result obtained in [4] regarding propagation of regularities which motivates our study here:

Theorem A2 ([4]). *Let $u_0 \in H^{3/4^+}(\mathbb{R})$. If for some $l \in \mathbb{Z}^+$ and for some $x_0 \in \mathbb{R}$,*

$$\| \partial_x^l u_0\|^2_{L^2((x_0,\infty))} = \int_{x_0}^{\infty} |\partial_x^l u_0(x)|^2 dx < \infty, \tag{1.3}$$

then the solution $u = u(x,t)$ of the IVP (1.1) provided by Theorem A1 satisfies that for any $v > 0$ and $\epsilon > 0$,

$$\sup_{0\leq t\leq T} \int_{x_0+\epsilon-vt}^{\infty} (\partial_x^j u)^2(x,t)\, dx < c, \tag{1.4}$$

for $j = 0, 1, \dots, l$ with $c = c(l; \|u_0\|_{3/4^+,2}; \| \partial_x^l u_0\|_{L^2((x_0,\infty))}; v; \epsilon; T)$.

In particular, for all $t \in (0,T]$, the restriction of $u(\cdot,t)$ to any interval of the form (a,∞) belongs to $H^l((a,\infty))$.

Moreover, for any $v \geq 0$, $\epsilon > 0$, and $R > 0$,

$$\int_0^T \int_{x_0+\epsilon-vt}^{x_0+R-vt} (\partial_x^{l+1} u)^2(x,t)\, dxdt < c, \tag{1.5}$$

with $c = c(l; \|u_0\|_{3/4^+,2}; \| \partial_x^l u_0\|_{L^2((x_0,\infty))}; v; \epsilon; R; T)$.

Theorem A2 tells us that the H^l-regularity ($l \in \mathbb{Z}^+$) on the right hand side of the data travels forward in time with infinite speed. Notice that since the equation is reversible in time a gain of regularity in $H^s(\mathbb{R})$ cannot occur at $t > 0$, so $u(\cdot,t)$ fails to be in $H^l(\mathbb{R})$ due to its decay at $-\infty$. In this regard, it was also shown in [4] that for any $\delta > 0$ and $t \in (0,T)$ and $j = 1, \dots, l$,

$$\int_{-\infty}^{\infty} \frac{1}{\langle x_-\rangle^{j+\delta}} (\partial_x^j u)^2(x,t)\, dx \leq \frac{c}{t},$$

with $c = c(\|u_0\|_{3/4^+,2}; \|\partial_x^j u_0\|_{L^2((x_0,\infty))}; x_0; \delta)$, $x_- = \max\{0; -x\}$, and $\langle x\rangle = (1+x^2)^{1/2}$.

The aim of this note is to extend Theorem A2 to the case where the local regularity of the datum u_0 in (1.3) is measure with a fractional exponent. Thus, our main result is:

Theorem 1.1. *Let $u_0 \in H^{3/4^+}(\mathbb{R})$. If for some $s \in \mathbb{R}$, $s > 3/4$, and for some $x_0 \in \mathbb{R}$,*

$$\| J^s u_0\|^2_{L^2((x_0,\infty))} = \int_{x_0}^{\infty} |J^s u_0(x)|^2 dx < \infty, \tag{1.6}$$

then the solution $u = u(x,t)$ *of the IVP* (1.1) *provided by Theorem* A1 *satisfies that for any* $v > 0$*, and* $\epsilon > 0$*,*

$$\sup_{0\le t\le T}\int_{x_0+\epsilon-vt}^{\infty}(J^r u)^2(x,t)\,dx < c, \tag{1.7}$$

for $r \in (3/4, s]$ *with* $c = c(l; \|u_0\|_{3/4^+,2}; \|\,J^r u_0\|_{L^2((x_0,\infty))}; v; \epsilon; T)$.

Moreover, for any $v \ge 0$*,* $\epsilon > 0$*, and* $R > 0$*,*

$$\int_0^T\int_{x_0+\epsilon-vt}^{x_0+R-vt}(J^{s+1}u)^2(x,t)\,dxdt < c, \tag{1.8}$$

with $c = c(l; \|u_0\|_{{}_{3/4^+,2}}; \|\,J^s u_0\|_{L^2((x_0,\infty))}; v; \epsilon; R; T)$.

From the results in section 2 it will be clear that we need only consider the case $s \in (3/4,\infty) - \mathbb{Z}^+$.

The rest of this paper is organized as follows: section 2 contains some preliminary estimates required for Theorem 1.1, whose proof will be given in section 3.

2. Preliminary estimates

Let T_a be a pseudo-differential operator with the symbol

$$\sigma(T_a) = a(x,\xi) \in S^r, \quad r \in \mathbb{R}, \tag{2.1}$$

so that

$$T_a f(x) = \int_{\mathbb{R}^n} a(x,\xi)\widehat{f}(\xi)\,e^{2\pi i x\cdot\xi}d\xi. \tag{2.2}$$

The following result is the singular integral realization of a pseudo-differential operator, whose proof can be found in [8, Chapter 4].

Theorem A3. *Using the above notation* (2.1)-(2.2) *one has that*

$$T_a f(x) = \int_{\mathbb{R}^n} k(x,x-y)f(y)dy, \qquad \text{if} \quad x \notin \operatorname{supp}(f), \tag{2.3}$$

where $k \in C^\infty(\mathbb{R}^n \times R^n - \{0\})$ *satisfies :* $\forall\,\alpha,\beta \in (Z^+)^n\ \forall\, N \ge 0$,

$$|\partial_x^\alpha\partial_z^\beta k(x,z)| \le A_{\alpha,\beta,N,\delta}\,|z|^{-(n+m+|\beta|+N)}, \qquad |z| \ge \delta, \tag{2.4}$$

$$\text{if} \quad n+m+|\beta|+N > 0 \qquad \text{uniformly in} \quad x \in \mathbb{R}^n.$$

To simplify the exposition we restrict ourselves to the one-dimensional case $x \in \mathbb{R}$, where in the next section these results will be applied.

As a direct consequence of Theorem A3 one has

Corollary 2.1. *Let* $m \in \mathbb{Z}^+$ *and* $l \in \mathbb{R}$*. If* $g \in L^2(\mathbb{R})$ *and* $f \in L^p(\mathbb{R})$*,* $p \in [2,\infty]$*, with*

$$\textit{distance}(\operatorname{supp}(f); \operatorname{supp}(g)) \ge \delta > 0,$$

then

$$\|f\,\partial_x^m J^l g\|_2 \le c\|f\|_p\|g\|_2. \tag{2.5}$$

Next, let $\theta_j \in C^\infty(\mathbb{R}) - \{0\}$ with $\theta_j' \in C_0^\infty(\mathbb{R})$ for $j = 1,2$ and

$$\textit{distance}(\operatorname{supp}(1-\theta_1); \operatorname{supp}(\theta_2)) \ge \delta > 0. \tag{2.6}$$

Lemma 2.2. *Let $f \in H^s(\mathbb{R})$, $s < 0$, and T_a be a pseudo-differential operator of order zero ($a \in S^0$). If $\theta_1 f \in L^2(\mathbb{R})$, then*

$$\theta_2 T_a f \in L^2(\mathbb{R}). \tag{2.7}$$

Proof of Lemma 2.2. Since

$$\theta_2\, T_a f = \theta_2 T_a(\theta_1 f) + \theta_2 T_a((1-\theta_1)f),$$

combining the hypothesis and the continuity of T_a in $L^2(\mathbb{R})$ it follows that $\theta_2 T_a(\theta_1 f) \in L^2(\mathbb{R})$. Also

$$\begin{aligned}
&\theta_2(x)\, T_a((1-\theta_1)f)(x)\\
&= \int_{-\infty}^{\infty} \theta_2(x) a(x,\xi)(\widehat{(1-\theta_1)f})(\xi) e^{2\pi i x\xi} d\xi\\
&= \int \underbrace{\Big(\int \theta_2(x) a(x,\xi_1+\xi_2)(\widehat{1-\theta_1})(\xi_1) e^{2\pi i x\xi_1} d\xi_1\Big)}_{=:b(x,\xi)} \widehat{f}(\xi_2) e^{2\pi i x \xi_2} d\xi_2\\
&= T_b f(x) = \int \theta_2(x) k(x, x-z)(1-\theta_1(z)) f(z) dz\\
&= \int \theta_2(x) k(x,x-z)(1-\theta_1(z)) J^{2m} J^{-2m} f(z) dz
\end{aligned} \tag{2.8}$$

with $-2m < s$, $m \in \mathbb{Z}^+$, and $k(\cdot,\cdot)$ as in (2.4), so integration by parts and Theorem A3 yield the desired result.

Proposition 2.3. *Let $f \in L^2(\mathbb{R})$ and*

$$J^s f = (1-\partial_x^2)^{s/2} f \in L^2(\{x > a\}) \qquad s > 0.$$

Then for any $\epsilon > 0$ and any $r \in (0, s]$,

$$J^r f \in L^2(\{x > a + \epsilon\}). \tag{2.9}$$

Proof of Proposition 2.3. Define

$$g = J^s f \in L^2(\{x > a\});$$

thus $J^s f \in H^{-s}(\mathbb{R})$. Let $\theta_j \in C^\infty(\mathbb{R})$, $j = 1, 2$, with $\theta_1(x) = 1$ for $x \geq a + \epsilon/4$, supp $\theta_1 \subset \{x > a\}$, and $\theta_2(x) = 1$ for $x \geq a + \epsilon$ and supp $\theta_2 \subset \{x > a + \epsilon/2\}$; therefore $\theta_1 g \in L^2(\mathbb{R})$. Let $T = J^{i\beta}$, $\beta \in \mathbb{R}$. By Lemma 2.2

$$\theta_2 T g = \theta_2 J^{s+i\beta} f \in L^2(\mathbb{R}),$$

and since $f \in L^2(\mathbb{R})$,

$$\theta_2 J^{i\beta} f \in L^2(\mathbb{R}).$$

Hence, by the Three Lines Theorem it follows that

$$\theta_2 J^z f \in L^2(\mathbb{R}), \qquad z = r + i\beta, \quad r \in [0, s], \quad \beta \in \mathbb{R},$$

which completes the proof.

Remark 2.4. In a similar manner one has: for $\epsilon > 0$ let $\varphi_\epsilon \in C^\infty(\mathbb{R})$ with $\varphi_\epsilon(x) = 1$, $x \geq \epsilon$, supp $\varphi_\epsilon \subset \{x > \epsilon/2\}$, and $\varphi_\epsilon'(x) \geq 0$. Then:

(I) If $m \in \mathbb{Z}^+$ and $\varphi_\epsilon J^m f \in L^2(\mathbb{R})$, then $\forall\, \epsilon' > 2\epsilon$,

$$\varphi_{\epsilon'} \partial_x^j f \in L^2(\mathbb{R}), \qquad j = 0, 1, \dots, m.$$

(II) If $m \in \mathbb{Z}^+$ and $\varphi_\epsilon \partial_x^j f \in L^2(\mathbb{R})$, $j = 0, 1, \dots, m$, then $\forall \epsilon' > 2\epsilon$,
$$\varphi_{\epsilon'} J^m f \in L^2(\mathbb{R}).$$
(III) If $s > 0$ and $J^s(\varphi_\epsilon f)$, $f \in L^2(\mathbb{R})$, then $\forall \epsilon' > 2\epsilon$,
$$\varphi_{\epsilon'} J^s f \in L^2(\mathbb{R}).$$
(IV) If $s > 0$ and $\varphi_\epsilon J^s f$, $f \in L^2(\mathbb{R})$, then $\forall \epsilon' > 2\epsilon$,
$$J^s(\varphi_{\epsilon'} f) \in L^2(\mathbb{R}).$$

The same results hold with θ_1, θ_2, as in (2.6), instead of χ_ϵ, $\chi_{\epsilon'}$.

Next, we recall some inequalities obtained in [5].

Theorem A4 ([5])**.** *If $s > 0$ and $p \in (1, \infty)$, then*
$$\| J^s(fg)\|_p \le c(\|f\|_\infty \|J^s g\|_p + \|J^s f\|_p \|g\|_\infty) \tag{2.10}$$
and
$$\begin{aligned} \| [J^s; f]g\|_p &= \|J^s(fg) - fJ^s g\|_p \\ &\le c(\|\partial f\|_\infty \|J^{s-1} g\|_p + \|J^s f\|_p \|g\|_\infty). \end{aligned} \tag{2.11}$$

Also we shall use the following elementary estimate whose proof is similar to that found in [2, Chapter 6].

Lemma 2.5. *Let $\phi \in C^\infty(\mathbb{R})$ with $\phi' \in C_0^\infty(\mathbb{R})$. If $s \in \mathbb{R}$, then for any $l > |s-1| + 1/2$,*
$$\| [J^s; \phi]f\|_2 + \| [J^{s-1}; \phi]\partial_x f\|_2 \le c\, \|J^l \phi'\|_2\, \|J^{s-1} f\|_2. \tag{2.12}$$

3. Proof of Theorem 1.1

Without loss of generality $x_0 = 0$. For $\epsilon > 0$ and $b \ge 5\epsilon$ define the families of functions
$$\chi_{\epsilon,b},\ \phi_{\epsilon,b},\ \widetilde{\phi_{\epsilon,b}},\ \psi_\epsilon \in C^\infty(\mathbb{R}),$$
with $\chi'_{\epsilon,b} \ge 0$, $\chi_{\epsilon,b}(x) = 0$, $x \le \epsilon$, $\chi_{\epsilon,b}(x) = 1$, $x \ge b$:
$$\begin{aligned} &\chi'_{\epsilon,b}(x) \ge \frac{1}{10(b-\epsilon)}\, 1_{[2\epsilon, b-2\epsilon]}(x), \\ &\text{supp}(\phi_{\epsilon,b}),\ \text{supp}(\widetilde{\phi_{\epsilon,b}}) \subset [\epsilon/4, b], \\ &\phi_{\epsilon,b}(x) = \widetilde{\phi_{\epsilon,b}}(x) = 1,\ x \in [\epsilon/2, \epsilon], \\ &\text{supp}(\psi_\epsilon) \subset (-\infty, \epsilon/2], \\ &\chi_{\epsilon,b}(x) + \phi_{\epsilon,b}(x) + \psi_\epsilon(x) = 1, \qquad x \in \mathbb{R}, \\ &\chi^2_{\epsilon,b}(x) + \widetilde{\phi_{\epsilon,b}}^2(x) + \psi_\epsilon(x) = 1, \qquad x \in \mathbb{R}. \end{aligned} \tag{3.1}$$

Hence,
$$distance(\text{supp}(\chi_{\epsilon,b}); \text{supp}(\psi_\epsilon)) \ge \epsilon/2.$$

Formally, we apply the operator J^s to the equation in (1.1) and multiply by $J^s u\chi_\epsilon^2(x+vt)$ to obtain after integration by parts the identity

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int (J^s u)^2(x,t)\chi^2(x+vt)\,dx\\
&\underbrace{-\,v\int (J^s u)^2(x,t)\chi\,\chi'(x+vt)\,dx}_{A_1}\\
&+\frac{3}{2}\int (\partial_x J^s u)^2(x,t)\chi\,\chi'(x+vt)\,dx\\
&\underbrace{-\,\frac{1}{2}\int (J^s u)^2(x,t)\partial_x^3(\chi^2(x+vt))\,dx}_{A_2}\\
&\underbrace{+\int J^s(u\partial_x u)\,J^s u(x,t)\chi^2(x+vt)\,dx}_{A_3}=0,
\end{aligned}
\tag{3.2}
$$

where in χ the indexes ϵ, b were omitted. We shall do that from now on.

Case. $s\in(3/4,1)$.

First, we observe that combining (1.2) and the results in section 2 yields that for any $R>0$,

$$
\int_0^T\int_{-R}^{R}|J^r u(x,t)|^2\,dxdt<\infty \qquad \forall\, r\in[0,7/4^+]. \tag{3.3}
$$

Thus, after integration in time the terms A_1 and A_2 in (3.2) are bounded. So it only remains to handle A_3.

Thus,

$$
\begin{aligned}
J^s(u\partial_x u)\chi &= J^s(u\partial_x u\chi)-[J^s;\chi](u\partial_x u)\\
&= u\chi J^s\partial_x u+[J^s;u\chi]\partial_x u-[J^s;\chi](u\partial_x u)\\
&= u\chi J^s\partial_x u+[J^s;u\chi]\partial_x(u(\chi+\phi+\psi))-[J^s;\chi](u\partial_x u)\\
&= B_1+B_2+B_3+B_4+B_5.
\end{aligned}
\tag{3.4}
$$

Inserting B_1 in (3.2) one obtains a term which can be estimated by integration by parts, Gronwall's inequality, and (1.2). Using (2.11) it follows that

$$
\|B_2\|_2\le c\|\partial_x(u\chi)\|_\infty\|J^s(u\chi)\|_2 \tag{3.5}
$$

and

$$
\|B_3\|_2\le c(\|\partial_x(u\chi)\|_\infty\|J^s(u\phi)\|_2+\|\partial_x(u\phi)\|_\infty\|J^s(u\chi)\|_2). \tag{3.6}
$$

To bound B_4 and B_5 we apply Corollary 2.1 and (2.12), respectively, to get

$$
\|B_4\|_2=\|u\chi J^s\partial_x(u\psi)\|_2\le c\|u\|_\infty\|u\|_2 \tag{3.7}
$$

and

$$
\|B_5\|_2\le c\|\partial_x u\|_\infty\|u\|_2. \tag{3.8}
$$

Collecting the above information (3.4)-(3.8) in (3.2) we obtain (1.7) for any $r \in (3/4, 1)$, $v > 0$, and $\epsilon > 0$, and that for any $v > 0$, $\epsilon > 0$,

$$\int_0^T \int_{\epsilon - vt}^{R-vt} (J^s \partial_x u)^2 dx dt < \infty,$$

from which using the results and Remark 2.4, one obtains (1.8).

Case. $s \in (m, m+1)$, $m \in Z^+$.

We assume (1.7) and (1.8) with $s \leq m$. Hence, from the results in section 2 it follows that for any $\epsilon > 0$, $R > 0$, and $r \in [0, m]$,

$$\int_0^T \int_{\epsilon - vt}^{R-vt} (J^r \partial_x u)^2 dx dt < \infty. \tag{3.9}$$

Again the starting point is the energy estimate identity (3.2). After integrating in time, the terms A_1 and A_2 can be easily bounded using (3.9). So it suffices to consider A_3. Thus, using the notation introduced in (3.1) we have

$$\begin{aligned} \chi J^s(u\partial_x u) &= J^s(u\chi\partial_x u) - \frac{1}{2}[J^s;\chi]\partial_x(u^2) \\ &= u\chi J^s\partial_x u + [J^s; u\chi]\partial_x u - \frac{1}{2}[J^s;\chi]\partial_x(u^2) \\ &= u\chi J^s\partial_x u + [J^s; u\chi]\partial_x(u(\chi + \phi + \psi)) \\ &\quad - \frac{1}{2}[J^s;\chi]\partial_x((u^2)(\chi^2 + (\widetilde{\phi})^2 + \psi)) \\ &= E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7. \end{aligned} \tag{3.10}$$

Inserting E_1 in (3.2) one obtains a term which can be estimated by integration by parts, Gronwall's inequality, and (1.2). From (2.11) we see that

$$\|E_2\|_2 \leq c\|\partial_x(u\chi)\|_\infty \|J^s(u\chi)\|_2 \tag{3.11}$$

and

$$\|E_3\|_2 \leq c(\|\partial_x(u\chi)\|_\infty \|J^s(u\phi)\|_2 + \|\partial_x(u\phi)\|_\infty \|J^s(u\chi)\|_2). \tag{3.12}$$

For E_4 it follows from Corollary 2.1 that

$$\|E_4\|_2 = \|u\chi J^s\partial_x(u\psi)\|_2 \leq c\|u\|_\infty \|u\|_2. \tag{3.13}$$

For E_5 and E_6 a combination of (2.10) and (2.12) yields the estimates

$$\begin{aligned} \|E_5\|_2 &\leq \|[J^s;\chi]\partial_x((u\chi)^2)\|_2 \\ &\leq c\|J^s((u\chi)^2)\|_2 \leq c\|u\|_\infty \|J^s(u\chi)\|_2 \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} \|E_6\|_2 &\leq \|[J^s;\chi]\partial_x((u\widetilde{\phi})^2)\|_2 \leq \|J^s((u\widetilde{\phi})^2)\|_2 \\ &\leq c\|u\|_\infty \|J^s(u\widetilde{\phi})\|_2. \end{aligned} \tag{3.15}$$

Finally, using Corollary 2.1 we write

$$\|E_7\|_2 \leq \|[J^s;\chi]\partial_x(u^2\psi)\|_2 = \|\chi J^s\partial_x(u^2\psi)\|_2 \leq c\|u\|_\infty \|u\|_2. \tag{3.16}$$

To complete the estimates in (3.11), (3.12), (3.14), and (3.15) we observe that

$$J^s(u\chi) = J^s u\chi + [J^s;\chi](u(\chi + \phi + \psi)) = G_1 + G_2,$$

where G_1 is the term whose L^2-norm we are estimating and G_2 is of lower order (hence bounded by assumption), and $\|J^2(u\phi)\|_2$ is bounded by (1.8) (assumption).

Collecting the above information in (3.2) we obtain the desired result.

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