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THE CLASSIFICATION OF QUASI-ALTERNATING MONTESINOS LINKS

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Abstract. In this paper, we complete the classification of quasi-alternating Montesinos links. We show that the quasi-alternating Montesinos links are precisely those identified independently by Qazaqzeh-Chbili-Qublan and Champanerkar-Ording. A consequence of our proof is that a Montesinos link L is quasi-alternating if and only if its double branched cover is an L-space, and bounds both a positive definite and a negative definite 4-manifold with vanishing first homology.

1. Introduction

Quasi-alternating links were defined by Ozsváth-Szabó [OS05, Definition 3.1] as a natural generalisation of the class of alternating links.

Definition 1. The set $\mathcal{Q}$ of quasi-alternating links is the smallest set of links satisfying the following:

- The unknot U belongs to $\mathcal{Q}$.
- If L is a link with a diagram containing a crossing c such that
  1. both smoothings L_0 and L_1 of the link L at the crossing c, as in Figure 1, belong to $\mathcal{Q}$,
  2. $\det(L_0), \det(L_1) \geq 1$, and
  3. $\det(L) = \det(L_0) + \det(L_1)$,

  then L is in $\mathcal{Q}$. The crossing c is called a quasi-alternating crossing.

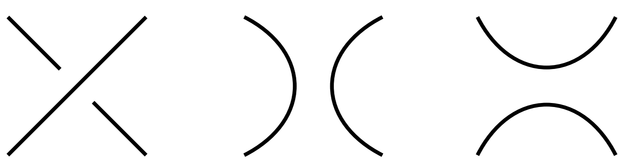

Figure 1. L and its two resolutions L_0 and L_1 in a neighbourhood of c

Ozsváth-Szabó showed that the class of nonsplit alternating links is contained in $\mathcal{Q}$ [OS05, Lemma 3.2]. Moreover, quasi-alternating links share a number of properties with alternating links; we list a few of these. For a quasi-alternating link L:

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(i) L is homologically thin for both Khovanov homology and knot Floer homology [MO08].
(ii) The double branched cover $\Sigma(L)$ of L is an L-space [OS05, Proposition 3.3].
(iii) The 3-manifold $\Sigma(L)$ bounds a smooth negative definite 4-manifold W with $H_1(W) = 0$ [OS05, Proof of Lemma 3.6].

For some further properties see [LO15], [QC15], [Ter15] and [ORS13, Remark after Proposition 5.2].

Due to their recursive definition, it is difficult in general to determine whether or not a link is quasi-alternating. For example, there still remain examples of 12-crossing knots with unknown quasi-alternating status [Jab14]. Champanerkar-Kofman [CK09] showed that the quasi-alternating property is preserved by replacing a quasi-alternating crossing with an alternating rational tangle. They used this to determine an infinite family of quasi-alternating pretzel links, which Greene later showed is the complete set of quasi-alternating pretzel links [Gre10].

Qazaqzeh-Chbili-Qublan [QCQ15] and Champanerkar-Ording [CO15] independently generalised the sufficient conditions on pretzel links to obtain an infinite family of quasi-alternating Montesinos links. This family includes all examples of quasi-alternating Montesinos links found by Widmer [Wid09]. Furthermore, it was conjectured by Qazaqzeh-Chbili-Qublan that this family is the complete set of quasi-alternating Montesinos links. We mention that Watson [Wat11] gave an iterative surgical construction for constructing all quasi-alternating Montesinos links.

Some necessary conditions to be quasi-alternating in terms of the rational parameters of a Montesinos link were obtained in [QCQ15] and [CO15] based on the fact that a quasi-alternating link is homologically thin. Further conditions are described in [CO15] coming from the fact that the double branched cover of a quasi-alternating link is an L-space. Some additional restrictions were found in [QC15].

Our main result is the following theorem which states that the quasi-alternating Montesinos links are precisely those found by Qazaqzeh-Chbili-Qublan [QCQ15] and Champanerkar-Ording [CO15].

Theorem 1. *Let $L = M(e; t_1, \ldots, t_p)$ be a Montesinos link in standard form, that is, where $t_i = \frac{\alpha_i}{\beta_i} > 1$ and $\alpha_i, \beta_i > 0$ are coprime for all $i = 1, \ldots, p$. Then L is quasi-alternating if and only if*

(1) *$e < 1$, or*
(2) *$e = 1$ and $\frac{\alpha_i}{\alpha_i - \beta_i} > \frac{\alpha_j}{\beta_j}$ for some i, j with $i \neq j$, or*
(3) *$e > p - 1$, or*
(4) *$e = p - 1$ and $\frac{\alpha_i}{\alpha_i - \beta_i} < \frac{\alpha_j}{\beta_j}$ for some i, j with $i \neq j$.*

As a corollary of our proof we obtain the following characterisation of the Montesinos links L which are quasi-alternating in terms of their double branched covers $\Sigma(L)$.

Corollary 1. *A Montesinos link L is quasi-alternating if and only if*

(1) *$\Sigma(L)$ is an L-space, and*
(2) *there exist a smooth negative definite 4-manifold W_1 and a smooth positive definite 4-manifold W_2 with $\partial W_i = \Sigma(L)$ and $H_1(W_i) = 0$ for $i = 1, 2$.*

Note that in Corollary 1 and throughout, we assume all homology groups have $\mathbb{Z}$ coefficients.

In light of this corollary, Theorem 2 can also be seen as a classification of the L-space Seifert fibered spaces over S^2 which bound both positive and negative definite 4-manifolds with vanishing first homology. To what extent Corollary 1 generalises to non-Montesinos links remains an interesting question.

This work also gives a classification of the Seifert fibered space formal L-spaces. The notion of a formal L-space was defined by Greene and Levine [GL16] as a 3-manifold analogue of quasi-alternating links. In fact, the double branched cover of a quasi-alternating link is an example of a formal L-space. In [LS17], Lidman and Sivek classified the quasi-alternating links of determinant at most 7. In fact, they showed that the formal L-spaces M^3 with $|H_1(M)| \leq 7$ are precisely the double branched covers of quasi-alternating links with determinant at most 7. In this same direction, as a consequence of Corollary 1, we have the following.

Corollary 2. *A Seifert fibered space over S^2 is a formal L-space if and only if it is the double branched cover of a quasi-alternating link.*

Corollary 1 also seems significant given the recent independent characterisations of alternating knots by Greene [Gre17] and Howie [How17]. A nonsplit link is alternating if and only if it bounds negative definite and positive definite spanning surfaces (which are the checkerboard surfaces). The double branched cover of B^4 over such a surface is a definite 4-manifold of the appropriate sign. Generalising this, a quasi-alternating link has the property that it bounds a pair of surfaces in B^4 with double branched covers a positive definite and a negative definite 4-manifold (these surfaces cannot be embedded in S^3 in general). Corollary 1 shows that among Montesinos links with double branched covers which are L-spaces, this property characterises those which are quasi-alternating.

Our approach to proving Theorem 2 follows that of Greene [Gre10] on the determination of quasi-alternating pretzel links. One of Greene's main strategies is as follows. Suppose L is a quasi-alternating Montesinos link such that $\Sigma(L)$ is the oriented boundary of the standard negative definite plumbing X^4. Since the property of being quasi-alternating is closed under reflection, by property (iii) above, $-\Sigma(L) = \Sigma(\overline{L})$ bounds a negative definite 4-manifold W with $H_1(W) = 0$. By Donaldson's theorem [Don87], the smooth closed negative definite 4-manifold $X \cup W$ has diagonalisable intersection form. Hence, $H_2(X)/\text{Tors} \hookrightarrow H_2(X \cup W)/\text{Tors}$ is an embedding of the intersection lattice of X into the standard negative diagonal lattice. Moreover, using the fact that $H_1(W)$ is torsion free, it is shown that if A is a matrix representing the lattice embedding, then A^T must be surjective.

When L is a pretzel link of a certain form, Greene analyses the possible embeddings of the intersection lattice of X into a negative diagonal lattice and shows that the aforementioned surjectivity condition cannot hold, and hence the link cannot be quasi-alternating. Our main contribution is to argue for more general Montesinos links L that there is no lattice embedding for which A^T is surjective. Key to our argument are some results on lattice embeddings by Lecuona-Lisca [LL11]. The condition we obtain combined with an obstruction based on $\Sigma(L)$ being an L-space leads to the precise necessary conditions to complete the determination of quasi-alternating Montesinos links.

2. Preliminaries

We briefly recall some material on Montesinos links and plumbings. See [CO15] or [BZH14] for further detail on Montesinos links and [NR78] for more on plumbings.

The Montesinos link $M(e; t_1, \dots, t_p)$, where $t_i = \frac{\alpha_i}{\beta_i} \in \mathbb{Q}$ with $\alpha_i > 1$ and β_i coprime integers, and e is an integer, is given by the diagram in Figure 2. In the figure, each box labelled t_i represents the corresponding rational tangle. The 0 rational tangle is shown in Figure 3. Introducing an additional positive (resp., negative) half-twist to the bottom of an a/b rational tangle produces a rational tangle represented by $a/b+1$ (resp., $a/b-1$); see Figure 3. Rotating (in either direction) a rational tangle represented by $t \in \mathbb{Q} \cup \{1/0\}$ by 90 degrees produces the rational tangle represented by $-1/t$. The rational tangle represented by any $a/b \in \mathbb{Q} \cup \{1/0\}$ can be obtained from the 0 rational tangle by a sequence of these two operations. See [Cro04] for a more thorough treatment of rational links. Note, however, that an a/b rational tangle with our conventions corresponds to a b/a rational tangle in [Cro04].

We also note that with our conventions for a Montesinos link $M(e; t_1, \dots, t_p)$, the integer e has opposite sign to that used by Champanerkar-Ording [CO15] and agrees with that of Qazaqzeh-Chbili-Qublan [QCQ15] and Greene [Gre10].

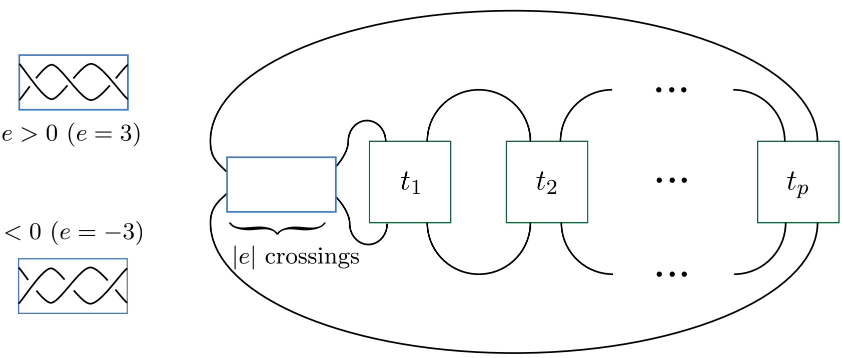

FIGURE 2. The Montesinos link $M(e; t_1, \dots, t_p)$, where a box labelled t_i represents a rational tangle corresponding to t_i. The crossing type of the $|e|$ crossings depends on the sign of e, with the two possibilities shown on the left.

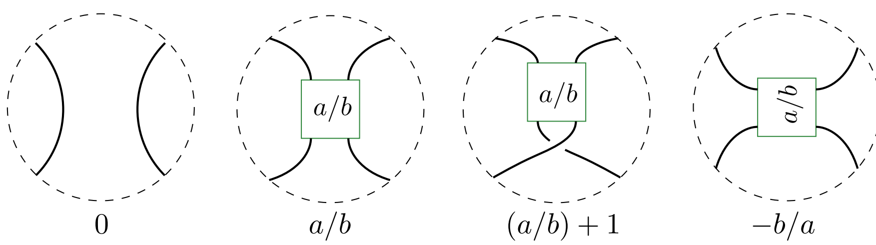

FIGURE 3. From left to right: the 0 rational tangle, an abstract representation of a a/b rational tangle, the $\frac{a}{b} + 1$ rational tangle, and the $-b/a$ rational tangle.

Montesinos link $M(e; t_1, \dots, t_p)$ is isotopic to $M(e+1; t_1, \dots, t_{i-1}, t_i', t_{i+1}, \dots, t_p)$, where $t_i' = \frac{\alpha_i}{\beta_i + \alpha_i}$, and is also isotopic to $M(e-1; t_1, \dots, t_{i-1}, t_i', t_{i+1}, \dots, t_p)$, where

$t_i' = \frac{\alpha_i}{\beta_i - \alpha_i}$. Hence, a Montesinos link is isotopic to one in *standard form*, that is, of the form $M(e; t_1, \ldots, t_p)$ where $t_i > 1$ for all i.

Let $L = M(e; t_1, \ldots, t_p)$ where $t_i < -1$ for all i. Note that any Montesinos link can be put into this form. For each i, there is a unique continued fraction expansion

$$t_i = [a_1^i, \ldots, a_{h_i}^i] := a_1^i - \cfrac{1}{a_2^i - \cfrac{1}{\ddots \, a_{h_i-1}^i - \cfrac{1}{a_{h_i}^i}}},$$

where $h_i \geq 1$ and $a_j^i \leq -2$ for all $j \in \{1, \ldots, h_i\}$.

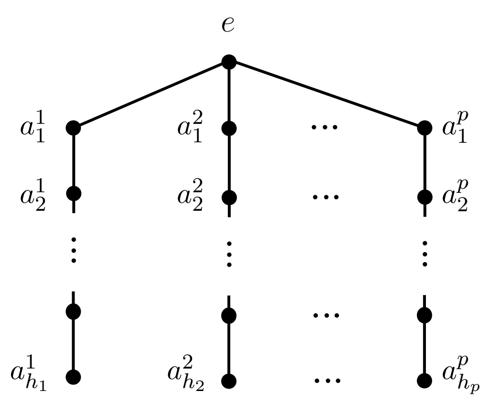

FIGURE 4. The weighted star-shaped plumbing graph Γ

The double branched cover $\Sigma(L)$ of L is the oriented boundary of the 4-dimensional plumbing X_Γ of D^2-bundles over S^2 described by the weighted star-shaped graph Γ shown in Figure 4. We call Γ the standard star-shaped plumbing graph for L. The ith leg of Γ corresponding to t_i is the linear subgraph generated by the vertices labelled with weights $a_1^i, \ldots, a_{h_i}^i$. The degree p vertex labelled with weight e is called the central vertex. Denote the vertices of Γ by $v_1, v_2, \ldots, v_k$. The zero-sections of the D^2-bundles over S^2 corresponding to each of $v_1, \ldots, v_k$ in the plumbing together form a natural spherical basis for $H_2(X_\Gamma)$. With respect to this basis, the intersection form of X_Γ is given by the weighted adjacency matrix Q_Γ with entries Q_{ij}, $1 \leq i, j \leq k$, given by

$$Q_{ij} = \begin{cases} \mathrm{w}(v_i), & \text{if } i = j, \\ 1, & \text{if } v_i \text{ and } v_j \text{ are connected by an edge}, \\ 0, & \text{otherwise}, \end{cases}$$

where $\mathrm{w}(v_i)$ is the weight of vertex v_i. We call $(\mathbb{Z}^k, Q_\Gamma)$ the intersection lattice of X_Γ (or of Γ).

3. Results

Equivalent sufficient conditions for a Montesinos link to be quasi-alternating were given in [CO15, Theorem 5.3] and [QCQ15, Theorem 3.5]. The goal of this section is to prove Theorem 2 which states that these sufficient conditions for a Montesinos link to be quasi-alternating are also necessary conditions.

Lemma 1. *Let $L = M(e; t_1, \ldots, t_p)$, $p \geq 3$, be a Montesinos link in standard form, i.e., where $t_i = \frac{\alpha_i}{\beta_i} > 1$ and $\alpha_i, \beta_i > 0$ are coprime for all i. Suppose that $e \leq p-2$ and $e - \sum_{i=1}^{p} \frac{1}{t_i} > 0$ (in particular $e \geq 1$). Then $\Sigma(L)$ is not an L-space, and therefore L is not quasi-alternating.*

Proof. The reflection of L is given by $\overline{L} = M(e'; t'_1, \ldots, t'_p) = M(-e; -t_1, \ldots, -t_p)$. The space $\Sigma(\overline{L})$ is the oriented boundary of a plumbing X_Γ corresponding to the standard star-shaped plumbing graph Γ for $\overline{L}$. Since $e' - \sum_{i=1}^{p} \frac{1}{t'_i} = -\left(e - \sum_{i=1}^{p} \frac{1}{t_i}\right) < 0$, by [NR78, Theorem 5.2], X_Γ has negative definite intersection form.

Since X_Γ is negative definite and Γ is almost-rational, by [Ném05, Theorem 6.3] we have that $\Sigma(\overline{L})$ is an L-space if and only if X_Γ is a rational surface singularity (more generally, see [Ném15]). Note that Γ is almost-rational since by sufficiently decreasing the weight of the central vertex we obtain a plumbing graph satisfying $-\mathrm{w}(v) \geq \deg(v)$ for all vertices v, where $\mathrm{w}(v)$ denotes the weight of v, and that such a graph is rational (for details see [Ném05, Example 8.2(3)]).

Laufer's algorithm [Lau72, Section 4] can be used to determine whether the negative definite plumbing X_Γ is a rational surface singularity as follows. Let $v_1, \ldots, v_k$ be the vertices of Γ, and for $i \in \{1, \ldots, k\}$ let $[\Sigma_{v_i}] \in H_2(X_\Gamma)$ be the spherical class naturally associated to v_i. The algorithm is as follows (see [Sti08, Section 3] for a similar formulation):

(1) Let $K_0 = \sum_{i=1}^{k} [\Sigma_{v_i}] \in H_2(X_\Gamma)$.
(2) In the ith step, consider the pairings $\langle PD[K_i], [\Sigma_{v_j}] \rangle$, for $j \in \{1, \ldots, k\}$. Note that these pairings may be evaluated using the adjacency matrix Q. If for some j the pairing is at least 2, then the algorithm stops and X_Γ is not a rational surface singularity. If for some j the pairing is equal to 1, then set $K_{i+1} = K_i + [\Sigma_{v_j}]$ and go to the next step. Otherwise all pairings are nonpositive, the algorithm stops, and X_Γ is a rational surface singularity.

Applying Laufer's algorithm to X_Γ, we claim that the algorithm terminates at the 0th step. To see this, note that for v the central vertex of Γ, $\langle PD[K_0], [\Sigma_v] \rangle = p - e$ (each vertex adjacent to v contributes 1 and the central vertex contributes $-e$). By assumption $e \leq p - 2$, so $\langle PD[K_0], [\Sigma_v] \rangle = p - e \geq 2$. Hence, the algorithm terminates, and we conclude that X_Γ is not a rational surface singularity and hence $\Sigma(\overline{L})$ is not an L-space. Therefore $\Sigma(L)$ is not an L-space. □

The following lemma will provide an obstruction to a Montesinos link being quasi-alternating.

Lemma 2 ([Gre10, Lemma 2.1])**.** *Suppose that X and W are a pair of 4-manifolds, $\partial X = -\partial W = Y$ is a rational homology sphere, and $H_1(W)$ is torsion-free. Express the map $H_2(X)/Tors \to H_2(X \cup W)/Tors$ with respect to a pair of bases by the matrix A. This map is an inclusion, and A^T is surjective. In particular, if some k rows of A contain all the nonzero entries of some k of its columns, then the induced $k \times k$ minor has determinant ± 1.*

The following two technical lemmas will be useful when we apply the obstruction to being quasi-alternating based on Lemma 2.

Lemma 3 ([LL11, Lemma 3.1])**.** *Suppose* $-1/r=[a_1,\dots,a_n]$ *and* $-1/s=[b_1,\dots,b_m]$*, where* $r+s=1$*. Consider a weighted linear graph* Ψ *having two connected components,* Ψ_1 *and* Ψ_2*, where* Ψ_1 *consists of* n *vertices* $v_1,\dots,v_n$ *with weights* $a_1,\dots,a_n$ *and* Ψ_2 *of* m *vertices* $w_1,\dots,w_m$ *with weights* $b_1,\dots,b_m$*. Moreover, suppose that there is an embedding of the lattice* $(\mathbb{Z}^{n+m},Q_\Psi)$ *into* $(\mathbb{Z}^k,-Id)$*, with basis* $e_1,\dots,e_k$*. For* S *a subset of vertices of* Ψ*, define*

$$U_S=\{e_i\,|\,e_i\cdot v\neq 0 \text{ for some } v\in S\}.$$

Suppose further that $e_1\in U_{v_1}\cap U_{w_1}$ *and* $U_\Psi=\{e_1,\dots,e_k\}$*. Then* $U_{\Psi_1}=U_{\Psi_2}$ *and* $k=n+m$*.*

Lemma 4 ([LL11, Lemma 3.2])**.** *Let* $-1/r=[a_1,\dots,a_n]$ *and* $-1/s=[b_1,\dots,b_m]$ *be such that* $r+s\geq 1$*. Then there exists* $n_0\leq n$ *and* $m_0\leq m$ *such that* $-1/r_0=[a_1,\dots,a_{n_0}]$ *and* $-1/s_0=[b_1,\dots,b_{m_0}]$ *satisfy* $r_0+s_0=1$*.*

Theorem 2. *Let* $L=M(e;t_1,\dots,t_p)$ *be a Montesinos link in standard form, that is, where* $t_i=\frac{\alpha_i}{\beta_i}>1$ *and* $\alpha_i,\beta_i>0$ *are coprime for all* $i=1,\dots,p$*. Then* L *is quasi-alternating if and only if*

(1) $e<1$*, or*
(2) $e=1$ *and* $\frac{\alpha_i}{\alpha_i-\beta_i}>\frac{\alpha_j}{\beta_j}$ *for some* i,j *with* $i\neq j$*, or*
(3) $e>p-1$*, or*
(4) $e=p-1$ *and* $\frac{\alpha_i}{\alpha_i-\beta_i}<\frac{\alpha_j}{\beta_j}$ *for some* i,j *with* $i\neq j$*.*

Proof. If one of the conditions (1)–(4) is satisfied, then L is quasi-alternating by either of [CO15, Theorem 5.3] or [QCQ15, Theorem 3.5]. Thus it suffices to show that if none of the conditions are satisfied, then L is not quasi-alternating. Thus, assume none of the conditions are satisfied, in particular $p\geq 2$.

By [Sav02, Section 1.2.3] (see also [CO15, Proposition 4.1]), we have that

$$\det(L)=\left|\alpha_1\dots\alpha_p\left(e-\sum_{i=1}^{p}\frac{\beta_i}{\alpha_i}\right)\right|.$$

If $p=2$, since none of the conditions are satisfied we must have $e=1$ and $\frac{\alpha_1}{\alpha_1-\beta_1}=\frac{\alpha_2}{\beta_2}$. Hence, $\det(L)=|\alpha_1\alpha_2(1-\frac{\beta_1}{\alpha_1}-\frac{\beta_2}{\alpha_2})|=0$, and so L is not quasi-alternating (in fact L must be the two component unlink). For the remainder of the argument we assume that $p\geq 3$, and $\det(L)\neq 0$, that is, $e-\sum_{i=1}^{p}\frac{\beta_i}{\alpha_i}\neq 0$.

First consider the case $1<e<p-1$. The reflection of L is given by

$$\overline{L}=M\left(-e,-\frac{\alpha_1}{\beta_1},\dots,-\frac{\alpha_p}{\beta_p}\right)=M\left(p-e,\frac{\alpha_1}{\alpha_1-\beta_1},\dots,\frac{\alpha_p}{\alpha_p-\beta_p}\right),$$

where the latter is written in standard form and $1<p-e<p-1$. Moreover, we see that a reflection reverses the sign of $e-\sum_{i=1}^{p}\frac{\beta_i}{\alpha_i}$, and thus by a reflection if necessary we may assume that $e-\sum_{i=1}^{p}\frac{\beta_i}{\alpha_i}>0$. Then by Lemma 1, $\Sigma(L)$ is not an L-space, so L is not quasi-alternating.

It remains to consider the cases $e=1$ and $e=p-1$. By a reflection if necessary we may assume that $e=1$. Note that conditions (2) and (4) are equivalent under a reflection. We assume that condition (2) is not satisfied. We need to prove that this implies that L is not quasi-alternating. If $e-\sum_{i=1}^{p}\frac{\beta_i}{\alpha_i}>0$, then by Lemma 1, $\Sigma(L)$ is not an L-space, and therefore L is not quasi-alternating.

Otherwise $e-\sum_{i=1}^{p}\frac{\beta_i}{\alpha_i}<0$. We have that

$$L=M\left(1;\frac{\alpha_1}{\beta_1},\dots,\frac{\alpha_p}{\beta_p}\right)=M\left(1-p;\frac{\alpha_1}{\beta_1-\alpha_1},\dots,\frac{\alpha_p}{\beta_p-\alpha_p}\right),$$

where $\frac{\alpha_i}{\beta_i-\alpha_i}<-1$ for all i.

The double branched cover $\Sigma(L)$ of L is therefore the boundary of a plumbing 4-manifold X_Γ on the standard star-shaped planar graph Γ with central vertex of weight $-(p-1)$ and legs corresponding to the fractions $\frac{\alpha_i}{\beta_i-\alpha_i}$, $i\in\{1,\dots,p\}$. Our assumption that $e-\sum_{i=1}^{p}\frac{\beta_i}{\alpha_i}<0$ implies that X_Γ is negative definite [NR78, Theorem 5.2]. Suppose for the sake of contradiction that L is quasi-alternating. Then $\overline{L}$ is quasi-alternating and $-\Sigma(L)=\Sigma(\overline{L})$ bounds a negative definite 4-manifold W with $H_1(W)=0$ [OS05, Proof of Lemma 3.6]. By Donaldson's theorem [Don87], the smooth closed negative definite 4-manifold $X_\Gamma\cup W$ has diagonalisable intersection form. Thus, the map $H_2(X_\Gamma)/\mathrm{Tors}\hookrightarrow H_2(X_\Gamma\cup W)/\mathrm{Tors}$ induced by the inclusion map is an embedding of the intersection lattice $(\mathbb{Z}^k,Q_\Gamma)$ of X_Γ into the standard negative diagonal lattice $(\mathbb{Z}^n,-\mathrm{Id})$ for some n. Denote by $e_1,\dots,e_n$ a basis for $(\mathbb{Z}^n,-\mathrm{Id})$.

We use the lattice embedding to identify elements of $(\mathbb{Z}^k,Q_\Gamma)$ with their image in $(\mathbb{Z}^n,-\mathrm{Id})$. For convenience, we will not distinguish between a vertex of Γ and the vector it corresponds to in the lattice. The central vertex v of Γ has weight $-(p-1)$, and so $v\cdot e_i\neq 0$ for at most $p-1$ values of $i\in\{1,\dots,n\}$. Thus, by applying an automorphism if necessary, we may assume that v pairs nontrivially with precisely $e_1,\dots,e_m$, where $m\le p-1$. Since there are p legs, by the pigeonhole principle there must exist some e_j, where $j\in\{1,\dots,m\}$, and two distinct vertices v_1,v_2 adjacent to v with $v_1\cdot e_j\neq 0$ and $v_2\cdot e_j\neq 0$. Without loss of generality we assume that $j=1$ and that for $i\in\{1,2\}$, the vertex v_i belongs to the ith leg of Γ, i.e., corresponding to the fraction $\frac{\alpha_i}{\beta_i-\alpha_i}$.

Since we are assuming condition (2) does not hold, we have that $\frac{\alpha_i}{\alpha_i-\beta_i}\le\frac{\alpha_j}{\beta_j}$ for all i,j with $i\neq j$. In particular, we have $\frac{\alpha_1}{\alpha_1-\beta_1}\le\frac{\alpha_2}{\beta_2}$. Rearranging this gives $\frac{\beta_1}{\alpha_1}+\frac{\beta_2}{\alpha_2}\le 1$. Note that the two legs correspond to the fractions $-1/r:=-\frac{\alpha_1}{\alpha_1-\beta_1}=[a_1^1,\dots,a_{h_1}^1]$ and $-1/s:=-\frac{\alpha_2}{\alpha_2-\beta_2}=[a_1^2,\dots,a_{h_2}^2]$, where $r,s\in\mathbb{Q}$, and where our notation is as in Section 2. Thus, we have that $r+s=2-\frac{\beta_1}{\alpha_1}-\frac{\beta_2}{\alpha_2}\ge 1$. Since $r+s\ge 1$, by Lemma 4 there exist $h_1'\le h_1$ and $h_2'\le h_2$ such that $-1/r_0=[a_1^1,\dots,a_{h_1'}^1]$ and $-1/s_0=[a_1^2,\dots,a_{h_2'}^2]$ with $r_0+s_0=1$.

Let Ψ be the union of the linear graph containing the first h_1' vertices of the first leg (where we count vertices in a leg starting away from the central vertex) and the linear graph containing the first h_2' vertices of the second leg. By restricting our embedding of $(\mathbb{Z}^k,Q_\Gamma)$, we have an embedding of the sublattice corresponding to Ψ into $(\mathbb{Z}^n,-\mathrm{Id})$. The image of this embedding is contained in a sublattice $(\mathbb{Z}^d,-\mathrm{Id})$ of $(\mathbb{Z}^n,-\mathrm{Id})$ spanned by $\{e_i\in\mathbb{Z}^n \mid e_i\cdot v\neq 0 \text{ for some vertex } v \text{ of } \Psi\}$. Hence U_Ψ consists of d elements (see Lemma 3 for definition of U_Ψ). Let v_1,w_1 be the two vertices of Ψ adjacent to the central vertex in Γ. By our choice of the two legs of Γ which contain the vertices of Ψ, we know that $e_j\in U_{v_1}\cap U_{w_1}$ for some

$j \in \{1, \ldots, n\}$. This shows that the hypotheses of Lemma 3 are satisfied, hence we conclude that $d = h_1' + h_2'$.

Let A be the matrix representing the embedding $(\mathbb{Z}^k, Q_\Gamma)$ into $(\mathbb{Z}^n, -\mathrm{Id})$. Then the $h_1' + h_2'$ columns of A corresponding to the vertices of Ψ are supported in $d = h_1' + h_2'$ rows of A corresponding to the d-dimensional sublattice of $(\mathbb{Z}^n, -\mathrm{Id})$. Denote this $d \times d$ minor by B. Then $-B^T B$ is a matrix for the intersection form of the plumbing corresponding to Ψ. Hence $-B^T B$ is a presentation matrix for $H_1(Y)$, where Y is the boundary of the (disconnected) plumbing corresponding to Ψ. The 3-manifold Y is the disjoint union of two lens spaces, each given by surgery on the unknot with framings $-1/r_0 < -1$ and $-1/s_0 < -1$, respectively. Therefore $|\det(B)|^2 = |H_1(Y)| > 1$, contradicting Lemma 2. Thus, L is not quasi-alternating. □

Corollary 1. *A Montesinos link L is quasi-alternating if and only if*

(1) *$\Sigma(L)$ is an* L*-space, and*
(2) *there exist a smooth negative definite 4-manifold W_1 and a smooth positive definite 4-manifold W_2 with $\partial W_1 = \Sigma(L)$ and $H_1(W_i) = 0$ for $i = 1, 2$.*

Proof. This is a corollary of the proof of Theorem 2. Suppose first that L is quasi-alternating. By [OS05, Proposition 3.3], $\Sigma(L)$ is an L-space. Furthermore, $\Sigma(L)$ must bound a negative definite 4-manifold W_1 with $H_1(W_1) = 0$ [OS05, Proof of Lemma 3.6]. Applying this to the reflection of L which is also quasi-alternating, we get that $\Sigma(L)$ also bounds a positive definite 4-manifold W_2 with $H_1(W_2) = 0$. For the converse, note that these two necessary conditions are the only conditions used to obstruct a Montesinos link from being quasi-alternating in the proof of Theorem 2. □

As a consequence, we obtain a classification of the Seifert fibered spaces which are formal L-spaces. Before stating it, we recall the definition of a formal L-space. We say that a triple (Y_1, Y_2, Y_3) of closed, oriented 3-manifolds forms a *triad* if there is a 3-manifold M with torus boundary, and three oriented curves $\gamma_1, \gamma_2, \gamma_3 \subset \partial M$ at pairwise distance 1, such that Y_i is the result of Dehn filling M along γ_i, for $i = 1, 2, 3$.

Definition 2. The set $\mathcal{F}$ of formal L-spaces is the smallest set of rational homology 3-spheres such that

(1) $S^3 \in \mathcal{F}$ and
(2) if (Y, Y_0, Y_1) is a triad with $Y_0, Y_1 \in \mathcal{F}$ and
$$|H_1(Y)| = |H_1(Y_0)| + |H_1(Y_1)|,$$
then $Y \in \mathcal{F}$.

Corollary 2. *A Seifert fibered space over S^2 is a formal* L*-space if and only if it is the double branched cover of a quasi-alternating link.*

Proof. Let L be a quasi-alternating Montesinos link. Then the double branched cover of L is a Seifert fibered space over S^2. Ozsváth and Szabó show that the double branched cover of a quasi-alternating link is an L-space [OS05, Proposition 3.3]. Their proof in fact shows that the double branched cover of a quasi-alternating link is a formal L-space. Hence $\Sigma(L)$ is a formal L-space Seifert fibered space over S^2.

Now let M be a formal L-space Seifert fibered space over S^2. Then M is the double branched cover of a Montesinos link L. Ozsváth and Szabó in [OS05, Proof of Lemma 3.6] show that the double branched cover of a quasi-alternating link bounds both a positive definite and a negative definite 4-manifold with vanishing first homology. However, their proof in fact shows this for all formal L-spaces. Hence $M = \Sigma(L)$ is a formal L-space bounding positive and negative definite 4-manifolds with vanishing first homology. Thus, Corollary 1 implies that L is quasi-alternating. □

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