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AN ALGEBRAIC CONSTRUCTION OF A SOLUTION TO THE MEAN FIELD EQUATIONS ON HYPERELLIPTIC CURVES AND ITS ADIABATIC LIMIT

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Abstract. In this paper, we give an algebraic construction of the solution to the following mean field equation:

$$\Delta\psi + e^{\psi} = 4\pi \sum_{i=1}^{2g+2} \delta_{P_i}$$

on a genus $g \geq 2$ hyperelliptic curve (X, ds^2), where ds^2 is a canonical metric on X and $\{P_1, \cdots, P_{2g+2}\}$ is the set of Weierstrass points on X.

1. Introduction

Let $f(x)$ be a complex polynomial in x with $2g+2$ distinct complex roots $\{e_1, \cdots, e_{2g+2}\}$. The affine plane curve $C_0 = \{(x,y) \in \mathbb{C}^2 : y^2 = f(x)\}$ defines a noncompact Riemann surface with respect to the complex analytic topology on $\mathbb{C}^2$. To compactifiy C_0 in the category of Riemann surfaces, we introduce another smooth affine plane curve C_0'. Let $g(z)$ be the complex polynomial $g(z) = \prod_{i=1}^{2g+2}(1-e_i z)$ and C_0' be the smooth affine plane curve defined by $w^2 = g(z)$, i.e., $C_0' = \{(z,w) \in \mathbb{C}^2 : w^2 = g(z)\}$. Let U_0 be the open subset of C_0 consisting of points (x,y) so that $x \neq 0$ and U_0' be the open subset of C_0' consisting of points (z,w) such that $z \neq 0$. The map $\varphi : U_0 \to U_0'$ defined by $\varphi(x,y) = (1/x, y/x^{g+1})$ is an isomorphism of Riemann surfaces. It is well known that the gluing $C_0 \cup_\varphi C_0'$ of C_0 and C_0' along φ is a connected compact Riemann surface of genus g; see [3] or [4]. The compact Riemann surface $C_0 \cup_\varphi C_0'$ is called the hyperelliptic curve of genus g defined by $y^2 = f(x)$ and is denoted by X in this paper. The holomorphic map $\pi : X \to \mathbb{P}^1$ defined by

$$\pi(P) = \begin{cases} (x(P):1) & \text{if } P \in C_0, \\ (1:z(P)) & \text{if } P \in C_0', \end{cases}$$

is a degree two ramified covering map of $\mathbb{P}^1$, where $(z_0 : z_1)$ is the homogeneous coordinate on $\mathbb{P}^1$. The Weierstrass points of X are the $2g+2$ ramification points $\{P_1, \cdots, P_{2g+2}\}$ of π such that $(x(P_k), y(P_k)) = (e_k, 0)$ for $1 \leq k \leq 2g+2$.

The space $H^0(X, \Omega_X^1)$ of holomorphic one forms on X has a simple basis of the form $\{x^{i-1}dx/y : 1 \leq i \leq g\}$ and the integral homology group $H_1(X)$ of X has a (symplectic) $\mathbb{Z}$-basis $\{a_i, b_j : 1 \leq i, j \leq g\}$ such that $\int_{a_j} \omega_i = \delta_{ij}$ for $1 \leq i, j \leq g$. Denote $\tau_{ij} = \int_{b_j} \omega_i$ for $1 \leq i, j \leq g$ and let τ be the complex $g \times g$ matrix $[\tau_{ij}]_{i,j=1}^g$.

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Then τ is a symmetric matrix with a positive definite imaginary part. Let Λ_τ be the lattice in $\mathbb{C}^g$ generated by the column vectors of the $g \times 2g$ matrix $\Omega = [I_g, \tau]$. Let Ω_i be the i-th column vector of Ω and let $\{dx_1, \cdots, dx_{2g}\}$ be the real basis dual to $\{\Omega_i : 1 \leq i \leq 2g\}$. The complex torus $\mathrm{Jac}(X) = \mathbb{C}^g/\Lambda_\tau$ together with the class $[\omega]$, where $\omega = \sum_{i=1}^g dx_i \wedge dx_{g+i}$ is a principally polarized abelian variety called the Jacobian variety of X. Fixing a point P_0 on X, we define a holomorphic map

$$\mu : X \to \mathrm{Jac}(X), \quad \mu(P) = \left(\int_{P_0}^P \frac{dx}{y}, \cdots, \int_{P_0}^P \frac{x^{g-1}dx}{y} \right) \mod \Lambda.$$

Let $(z_1, \cdots, z_g)$ be the standard holomorphic coordinate on $\mathrm{Jac}(X)$ and let $d\widetilde{s}_H^2$ be the flat hermitian metric $\sum_{i,j=1}^g h_{ij} dz^i \otimes d\overline{z}^j$ on $\mathrm{Jac}(X)$, where $H = [h_{ij}]$ is a $g \times g$ positive definite hermitian matrix. The canonical metric ds_H^2 on X is defined by $ds_H^2 = \mu^* d\widetilde{s}_H^2$ and has the form

$$ds_H^2 = \frac{1}{|y^2|} \sum_{i,j=1}^g h_{ij} x^{i-1} \overline{x}^{j-1}.$$

Let Δ_H be the Laplace operator associated with the metric ds_H^2. In this paper, we study the following mean field equation:

$$\Delta_H \psi + e^\psi = 4\pi \sum_{i=1}^{2g+2} \delta_{P_i}, \tag{1.1}$$

where $\{P_1, \cdots, P_{2g+2}\}$ is the set of all Weierstrass points on X and $\delta_P : C^\infty(X) \to \mathbb{C}$ is the Dirac measure centered at P for $P \in X$.

In [2], we discovered that when X has genus two, the Gaussian curvature function K of a canonical metric determines a solution ψ to (1.1). This paper is a continuation of [2]; we give an algebraic construction of a solution to (1.1) involving the study of solutions to the formal nonlinear ordinary differential equation

$$(tQ''(t) + Q'(t))Q(t) - t(Q'(t))^2 = S(t)Q(t), \tag{1.2}$$

and the study of solutions to the formal nonlinear partial differential equation

$$u \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} = \sigma u. \tag{1.3}$$

Here $S(t)$ is a complex formal power series and σ is a complex polynomial in x, y. We call S the data for (1.2) and σ the data for (1.3), respectively. In Section 2, we define a sequence of polynomials to solve (1.2) and give a necessary and sufficient condition for (1.2) possesing a polynomial solution. In Section 3, we show that the existence of solutions to (1.3) is equivalent to the nonemptiness of a certain (ind) affine algebraic set. Solutions to (1.2) would allow us to construct solutions to (1.3) for a certain type of polynomials σ. In Section 4, using the method developed in Section 2 and Section 3, we give a construction of a solution to (1.1) and the closed form of a solution to (1.1) when H is a diagonal matrix with positive diagonals. In Section 5, we introduce a real positive parameter γ into (1.1) and generalize the solutions constructed in previous sections. The parameter arises from a rescaling of a canonical metric by γ and we discuss the adiabatic limit of solutions as $\gamma \to 0$.

2. A formal nonlinear ordinary differential equation

Let $\{\lambda_i : i \geq 0\}$ be an infinite sequence of variables and $\mathbb{K}[\Lambda]$ be the ring of polynomials in $\{\lambda_i : i \geq 0\}$ over a field $\mathbb{K}$. We define a sequence of polynomials $\{f^i_\lambda : i \geq 1\}$ in $\mathbb{Q}[\Lambda][t]$ by $f^1_\lambda(t) = \lambda_0$ and
(2.1)

$$f^{k+1}_\lambda(t) = \frac{\lambda_k}{(k+1)^2} + \frac{t}{(k+1)^2}\sum_{i=0}^{k-1}(\lambda_i - (i+1)(2i+1-k)f^{i+1}_\lambda(t))f^{k-i}_\lambda(t), \quad k \geq 2.$$

By definition,

$$f^2_\lambda(t) = \frac{\lambda_0^2 t + \lambda_1}{4}, \quad f^3_\lambda(t) = \frac{\lambda_0\lambda_1 t + \lambda_2}{9}, \quad f^4_\lambda(t) = -\lambda_0^2\lambda_1 t^2 + (3\lambda_1^2 + 8\lambda_0\lambda_2)t + \frac{\lambda_3}{16}.$$

By induction, the degree of $f^i_\lambda(t)$ in t is $i-2$ for $i \geq 3$ and the t^0 term of $f^i_\lambda(t)$ is $\lambda_{i-1}/(i-1)^2$ for $i \geq 2$. Let us denote f^i_λ by

$$f^i_\lambda(t) = \sum_{j=0}^{i-2}\beta_{ij}(\lambda)t^j, \quad \beta_{ij}(\lambda) \in \mathbb{Q}[\Lambda].$$

If $S(t) = \sum_{i=0}^\infty s_i t^i$ is a complex formal power series, we set $f^1_S(t) = s_0$ and

$$f^i_S(t) = \sum_{j=0}^{i-2}\beta_{ij}(s_0, s_1, \cdots)t^j.$$

Notice that when $S(t)$ is a polynomial of degree at most n, then $f^i_S(t)$ is divisible by t for all $i \geq n+2$.

Lemma 2.1. *Let $f(t)$ and $g(t)$ be complex power series such that $g(0) \neq 0$ and $f(t)g(t)$ is divisible by t^m for some $m \geq 1$. Then $f(t)$ is divisible by t^m.*

Proof. We assume that $f(t)g(t) = t^m h(t)$ for some $h(t) \in \mathbb{C}[[t]]$. Then $f(0) = 0$. Taking the (formal) derivatives of the equation $f(t)g(t) = t^m h(t)$ with respect to t and using the fact that $g(0) \neq 0$, we prove by induction that $f^{(i)}(0) = 0$ for $1 \leq i \leq m-1$. This implies that $f(t) = t^m f_1(t)$ with $f_1 \in \mathbb{C}[[t]]$. □

Assume that $Q(t)$ is a solution to (1.2) and $Q(t)$ is divisible by t^m but not by t^{m+1}. Define $Q_1(t) \in \mathbb{C}[[t]]$ such that $Q(t) = t^m Q_1(t)$. ($Q_1(t)$ is defined since $\mathbb{C}[[t]]$ is a unique factorization domain.) Then $Q_1(0) \neq 0$. By an elementary computation,

$$t^m\left((tQ_1''(t) + Q_1'(t)Q_1(t)) - t(Q_1'(t))^2\right) = S(t)Q_1(t).$$

By Lemma 2.1, $S(t)$ is divisible by t^m. Define $S_1(t)$ by $S(t) = t^m S_1(t)$. Then

$$(tQ_1''(t) + Q_1'(t)Q_1(t)) - t(Q_1'(t))^2 = S_1(t)Q_1(t).$$

This shows that $Q_1(t)$ is a solution to (1.2) with the data $S_1(t)$ and with the initial condition $Q_1(0) \neq 0$. Owing to this observation, it suffices to consider the solutions $Q(t)$ to (1.2) for a given data $S(t)$ under the assumption $Q(0) \neq 0$.

Proposition 2.2. *Let $S(t) = \sum_{i=0}^\infty s_i t^i$ be a complex formal power series and let a be a nonzero complex number. A formal power series $Q(t) = \sum_{i=1}^\infty q_i t^i$ with $Q(0) = 1/a$ solves (1.2) for the data $S(t)$ if and only if $q_i = f^i_S(a)$ for $i \geq 1$.*

Proof. After some basic computation, we know

$$(tQ''(t)+Q'(t))Q(t)-t(Q'(t))^2=\sum_{k=0}^{\infty}\left(\sum_{i=0}^{k}(i+1)(2i+1-k)q_{i+1}q_{k-i}\right)t^k,$$
$$S(t)Q(t)=\sum_{k=0}^{\infty}\left(\sum_{i=0}^{k}s_iq_{k-i}\right)t^k.$$

If $Q(t)$ is a solution to (1.2), then

$$\sum_{i=0}^{k}(i+1)(2i+1-k)q_{i+1}q_{k-i}=\sum_{i=0}^{k}s_iq_{k-i},\quad k\geq 0. \tag{2.2}$$

Hence $q_0q_1=q_0s_0$. Since $q_0\neq 0$, $q_1=s_0$. Then $q_1=f_S^1(a)$ holds. Furthermore, (2.2) can be rewritten as:

$$(k+1)^2q_{k+1}q_0=s_kq_0+\sum_{i=0}^{k-1}(s_i-(i+1)(2i+1-k)q_{i+1})q_{k-i},\quad k\geq 1, \tag{2.3}$$

which implies that (by $q_0=1/a$)

$$q_{k+1}=\frac{s_k}{(k+1)^2}+\frac{a}{(k+1)^2}\sum_{i=0}^{k-1}(s_i-(i+1)(2i+1-k)q_{i+1})q_{k-i},\quad k\geq 1.$$

By (2.1) and induction, $q_{k+1}=f_S^{k+1}(a)$ for $k\geq 1$. For the converse, since $q_i=f_S^i(a)$, (q_i) satisfies (2.2). Define $Q(t)=\sum_{i=0}^{\infty}q_it^i$. By (2.2), $Q(t)$ satisfies (1.2). We complete the proof of our assertion. □

This proposition implies that the solution to (1.2) is uniquely determined by the initial condition $Q(0)=a$ with $a\neq 0$ and the solution can be constructed by the numbers $f_S^i(a)$ for $i\geq 1$.

When $S(t)$ is a polynomial of degree m, we would like to find the necessary and the sufficient condition for (1.2) possessing polynomial solutions.

Lemma 2.3. *Let $S(t)$ be a polynomial of degree m. If the solution $Q(t)$ to* (1.2) *for the data $S(t)$ is a polynomial of degree n in t, then $n\geq m+2$.*

Proof. The polynomial $(tQ''(t)+Q'(t))Q(t)-t(Q'(t))^2$ has degree at most $2n-2$ while the degree of $S(t)Q(t)$ is $n+m$. Hence $n+m\leq 2n-2$ implies that $n\geq m+2$. □

Proposition 2.4. *Let $S(t)$ be a polynomial of degree m. The solution $Q(t)$ to* (1.2) *for the data $S(t)$ constructed in Proposition* 2.2 *is a polynomial in t if and only if there exists $N\in\mathbb{N}$ with $N\geq m+2$ such that q_0^{-1} is the common root of the polynomials $\{f_S^i : N+1\leq i\leq 2N-1\}$. Here $q_0=Q(0)$.*

Proof. Suppose that $Q(t)=\sum_{i=0}^{\infty}q_it^i$ is a polynomial of degree n. Then $q_i=0$ for all $i\geq n+1$. We choose $N=n$. By Lemma 2.3, $N\geq m+2$. Since $q_i=f_S^i(q_0^{-1})$, q_0^{-1} is a root of f_S^i for all $i\geq N+1$ and hence $f_S^i(q_0^{-1})=0$ for all $i\geq N+1$. Therefore, q_0^{-1} is a root of $f_S^i(t)$ for $i\geq N+1$ and thus for $N+1\leq i\leq 2N-1$.

Let us prove the converse. Assume that q_0^{-1} is a common root of $f_S^i(t)$ for $N+1\leq i\leq 2N-1$. Let us prove the statement $q_{2N-1+j}=0$ for $j\geq 1$ by induction on j. For $j = 1$,

$$q_{2N} = \frac{1}{(2N)^2 q_0} \sum_{i=0}^{2N-2} (s_i - (i+1)(2N - 2i + 2)q_{i+1})q_{2N-1-i}.$$

For $0 \le i \le N-2$, $N+1 \le 2N-1-i \le 2N-1$. Hence $q_{2N-1-i} = 0$ for $0 \le i \le N-2$. For $i \ge N-1$, $i \ge m+1$ and hence $s_i = 0$ for $i \ge N-1$. For $N \le i \le 2N-2$, $N+1 \le i+1 \le 2N-1$. Hence $q_{i+1} = 0$ for $N \le i \le 2N-2$ by assumption. Notice that when $i = N-1$, $2i - 2N + 2 = 0$. We conclude that $q_{2N} = 0$. This proves that the statement holds for $j = 1$. We assume that the statement is true for $0 \le j \le l$. If $j = l+1$, $2N - 1 + j \ge N \ge m+2$ and hence $s_{2N-1+j} = 0$ which implies that

$$q_{2N+l} = \frac{1}{(2N+l)^2 q_0} \sum_{i=0}^{2N+l-2} (s_i - (i+1)(2i+1-k)q_{i+1})q_{2N+l-1-i}.$$

For $0 \le i \le N-1$, $N+1 \le N+l < 2N-1+l-i \le 2N-1+l-i \le 2N-1+l$. By induction hypothesis and the assumption, we obtain that $q_{2N-1+l-i} = 0$ for $0 \le i \le N-1$. For $N \le i \le 2N+l-2$, $N+1 \le i+1 \le 2N-1+l$ and hence $s_i = q_{i+1} = 0$. We conclude that $q_{2N+l} = 0$. We prove that $q_{2N-1+j} = 0$ holds for $j = l+1$. By mathematical induction, $q_{2N-1+j} = 0$ for all $j \ge 1$. Combining with the assumption, one has $q_i = 0$ for all $i \ge N+1$. Therefore, $Q(t)$ is a polynomial. □

In fact, we can prove that:

Lemma 2.5. *Let $x_0, \cdots, x_m$ be a set of formal variables for $m \ge 1$. For each i, define a polynomial over $\mathbb{Q}$ in $x_0, \cdots, x_m, t$ by*

$$F^i(x_0, \cdots, x_m, t) = f^i_{x_0 + x_1 t + \cdots + x_m t^m}(t), \tag{2.4}$$

where f^i_S is the polynomial defined in (2.1). *Let N be a natural number so that $N \ge m+2$. The set of polynomials $\{F^i : N+1 \le i \le 2N-1\}$ is divisible by $G(x_0, \cdots, x_{g-1}, t) \in \mathbb{Q}[x_0, \cdots, x_{g-1}, t]$ if and only if $\{F^i : i \ge N+1\}$ is divisible by G.*

Proof. The proof follows from the recursive relation

$$F^{k+1} = \frac{t}{(k+1)^2} \sum_{i=0}^{k-1} (x_i - (i+1)(2i+1-k)F^{i+1})F^{k-i}, \quad k \ge m+1,$$

and is similar to that given in Proposition 2.4. We leave it to the readers. □

Corollary 2.6. *Let $S(t)$ be a complex polynomial of degree m and let $Q(t)$ be a solution to* (1.2) *for the data $S(t)$ such that $Q(0)^{-1}$ is a common zero of $\{f^i_S : m+3 \le i \le 2m+3\}$. Then $Q(t)$ is a polynomial of degree $m+2$.*

Proof. By assumption and Proposition 2.4, $q_i = 0$ for $i \ge m+3$. Then $Q(t)$ is a polynomial of degree at most $m+2$. By Lemma 2.3, the degree of $Q(t)$ is at least $m+2$. We conclude that $Q(t)$ is a polynomial of degree $m+2$. □

Lemma 2.7. *For any $i \ge 1$, and any $S(t) \in \mathbb{C}[[t]]$,*

$$f^i_{\lambda S}(t) = \lambda f^i_S(\lambda t)$$

for any $\lambda \in \mathbb{C}$.

Proof. When $i=1$, the statement is obvious. One can also verify that the statement is true for $i=2$ and 3. Assume that the statement is true for $i=k$. For $i=k+1$, we use the recursive relation:

$$\begin{aligned}
f_{\lambda S}^{k+1}(t) &= \frac{\lambda s_k}{(k+1)^2} + \frac{t}{(k+1)^2}\sum_{i=0}^{k-1}(\lambda s_i - (i+1)(2i+1-k)f_{\lambda S}^{i+1}(t))f_{\lambda S}^{k-i}(t) \\
&= \frac{\lambda s_k}{(k+1)^2} + \frac{t}{(k+1)^2}\sum_{i=0}^{k-1}(\lambda s_i - (i+1)(2i+1-k)\lambda f_{S}^{i+1}(\lambda t))\lambda f_{\lambda S}^{k-i}(\lambda t) \\
&= \lambda\left(\frac{s_k}{(k+1)^2} + \frac{\lambda t}{(k+1)^2}\sum_{i=0}^{k-1}(s_i - (i+1)(2i+1-k)f_{S}^{i+1}(\lambda t))f_{S}^{k-i}(\lambda t)\right) \\
&= \lambda f_S^i(\lambda t).
\end{aligned}$$

□

This lemma implies that:

Corollary 2.8. *Let $S(t)$ be a complex polynomial of degree m. Then* (1.2) *has a polynomial solution for data $S(t)$ if and only if* (1.2) *has polynomial solution for data $\lambda S(t)$ for $\lambda \in \mathbb{C}^*$.*

Given any complex polynomial $B(t) = \sum_{i=0}^n b_i t^i$, we define a new polynomial $\widetilde{B}(t)$ by

$$\widetilde{B}(t) = t^{\deg B} B(t^{-1})$$

and write $\widetilde{B}(t) = \sum_{i=0}^n \widetilde{b}_i t^i$, where $n = \deg B(t)$. Then $\widetilde{b}_i = b_{n-i}$ for $0 \le i \le n$.

Proposition 2.9. *Let $S(t)$ be a complex polynomial of degree m. Suppose that* (1.2) *has a polynomial solution $Q(t)$ of degree n for the data $S(t)$. Then $\widetilde{Q}(t)$ solves* (1.2) *for the data $t^{n-m-2}\widetilde{S}(t)$.*

Proof. One uses the chain rules to prove the statement while the calculation is elementary. □

This proposition implies that

Corollary 2.10. *Let $S(t)$ be a complex polynomial of degree m. Suppose that* (1.2) *has a polynomial solution $Q(t)$ of degree $m+2$ for the data $S(t)$. Then $\widetilde{Q}(t)$ solves* (1.2) *for the data $\widetilde{S}(t)$.*

3. A formal nonlinear partial differential equation

Let $M_n(\mathbb{C})$ be the algebra of $n\times n$ complex matrices. For each $n \ge 1$, we consider the algebra monomorphism $\psi_{n,n+1} : M_n(\mathbb{C}) \to M_{n+1}(\mathbb{C})$ defined by

$$\psi_{n,n+1}(A) = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}.$$

The direct limit of the directed system $\{(M_n(\mathbb{C}), \psi_{n,m})\}$ is denoted by $M_\infty(\mathbb{C})$, where the algebra monomorphism $\psi_{n,m} : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ for $n < m$ is defined by

$$\psi_{n,m} = \psi_{m,m-1} \circ \cdots \circ \psi_{n+1,n}.$$

Denote the canonical map $M_n(\mathbb{C}) \to M_\infty(\mathbb{C})$ by ψ_n and identify $M_n(\mathbb{C})$ with its image in $M_\infty(\mathbb{C})$. Then $M_\infty(\mathbb{C})$ can be realized as a union $\bigcup_{n=1}^\infty M_n(\mathbb{C})$; $M_\infty(\mathbb{C})$ is an ind-variety over $\mathbb{C}$.

By an ind-variety over a field k, we mean that a set X together with a filtration $X_0 \subset X_1 \subset X_2 \subset \cdots$ such that $\bigcup_{n\geq 0} X_n = X$ and each X_n is a finite dimensional variety over k such that the inclusion $X_n \to X_{n+1}$ is a closed embedding. An ind-variety has a natural topology defined as follows. A subset U of X is said to be open if and only if $U \cap X_n$ is open in X_n for each $n \geq 0$. The ring of regular functions on X denoted by $k[X]$ is defined to be $k[X] = \varprojlim_n k[X_n]$. An ind-variety is said to be projective, resp., affine, if each X_n is projective, resp., affine. For more details about ind-varieties, see [5].

For each $A \in M_\infty(\mathbb{C})$, we may write $A = (a_{ij})_{i,j=1}^\infty$ with $a_{ij} = 0$ for all but finitely many i, j. We associate to A a complex polynomial $\mathfrak{p}(A)(x, y)$ in x, y by

$$\mathfrak{p}(A)(x,y) = \sum_{i,j=0}^{\infty} a_{i+1,j+1} x^i y^j.$$

We obtain a linear monomorphism $\mathfrak{p} : M_\infty(\mathbb{C}) \to \mathbb{C}[x, y]$. The image of $\mathfrak{p}$ is denoted by $\mathfrak{P}_\infty[x, y]$. Given $\sigma \in \mathfrak{P}_\infty[x, y]$, we would like to solve for the formal nonlinear differential equation (1.3) in $\mathfrak{P}_\infty[x, y]$. To solve for (1.3) in $\mathfrak{P}_\infty[x, y]$, let us assume that

$$u(x,y) = \sum_{\alpha,\beta=0}^{\infty} a_{\alpha+1,\beta+1} x^\alpha y^\beta \quad \text{and} \quad \sigma(x,y) = \sum_{i,j=0}^{\infty} c_{i+1,j+1} x^i y^j.$$

By simple computation,

$$uu_{xy} - u_x u_y = \sum_{\alpha,\beta=0}^{\infty} \left(\sum_{i=0}^{\alpha+1} \sum_{j=0}^{\beta+1} i(2j-\beta-1) a_{i+1,j+1} a_{\alpha-i+2,\beta-j+2} \right) x^\alpha y^\beta,$$

$$\sigma u = \sum_{\alpha,\beta=0}^{\infty} \left(\sum_{i=0}^{\alpha} \sum_{j=0}^{\beta} a_{i+1,j+1} c_{\alpha-i+1,\beta-j+1} \right) x^\alpha y^\beta.$$

Then u solves (1.3) if and only if

$$\sum_{i=0}^{\alpha+1} \sum_{j=0}^{\beta+1} i(2j-\beta-1) a_{i+1,j+1} a_{\alpha-i+2,\beta-j+2} = \sum_{i=0}^{\alpha} \sum_{j=0}^{\beta} a_{i+1,j+1} c_{\alpha-i+1,\beta-j+1}$$

for any $\alpha, \beta \geq 0$. For each α, β, we define

$$\varphi_\sigma^{\alpha,\beta}(A) = \sum_{i=0}^{\alpha+1} \sum_{j=0}^{\beta+1} i(2j-\beta-1) a_{i+1,j+1} a_{\alpha-i+2,\beta-j+2} - \sum_{i=0}^{\alpha} \sum_{j=0}^{\beta} a_{i+1,j+1} c_{\alpha-i+1,\beta-j+1}.$$

Then $u = \mathfrak{p}(A)$ for some $A \in M_\infty(\mathbb{C})$ solves (1.3) for data σ if and only if $\varphi_\sigma^{\alpha,\beta}(A) = 0$ for all α, β, i.e., A satisfies a family of quadratic polynomials. The subset

$$V_\sigma = \{A \in M_\infty(\mathbb{C}) : \varphi_\sigma^{\alpha,\beta}(A) = 0\}$$

of $M_\infty(\mathbb{C})$ is called the ind-affine algebraic variety associated with σ. The equation (1.3) has a solution for σ if and only if V_σ is nonempty.

For each $u \in \mathfrak{P}_\infty[x, y]$, we define $M_{xy}u$ by

$$(M_{xy}u)(x,y) = (xy)u(x,y).$$

Then M_{xy} defines a linear endomorphism on $\mathfrak{P}_\infty[x, y]$.

Lemma 3.1. *Suppose that* $u \in \mathfrak{P}_\infty[x,y]$ *is a solution to* (1.3) *for data* σ. *Then* $M_{xy}u$ *is a solution to* (1.3) *for data* $M_{xy}\sigma$.

Proof. Let $v = M_{xy}u$. Then $v(x,y) = (xy)u(x,y)$. Hence $v_x = yu + (xy)u_x$, and $v_y = xu + (xy)u_y$, and $v_{xy} = u + yu_y + xu_x + (xy)u_{xy}$. We discover that

$$vv_{xy} - v_x v_y = (xy)^2(uu_{xy} - u_x u_y) = (xy)^2\sigma u = (M_{xy}\sigma)v.$$

This proves our assertion. □

By making use of the fact that $\mathbb{C}[x,y]$ is a unique factorization domain, we prove the following fact:

Proposition 3.2. *Let* $v \in \mathfrak{P}_\infty[x,y]$ *be a solution to* (1.3) *for a data* $\sigma \in \mathfrak{P}_\infty[x,y]$. *Assume that there exists* $m \in \mathbb{N}$ *such that* v *is divisible by* $(xy)^m$ *but not by* $x^{m+1}y^m$ *and not by* $x^m y^{m+1}$. *Then* σ *is divisible by* $(xy)^m$. *Furthermore, if* $u \in \mathfrak{P}_\infty[x,y]$ *and* $\gamma \in \mathfrak{P}_\infty[x,y]$ *are polynomials so that* $v = M_{xy}^m u$ *and* $\sigma = M_{xy}^m\gamma$, *then* u *is a solution to* (1.3) *for the data* γ.

Proof. Since v is divisible by $(xy)^m$, we write $v = M_{xy}^m u$ for some $u \in \mathfrak{P}_\infty[x,y]$. We can show that

$$vv_{xy} - v_x v_y = (xy)^{2m}(uu_{xy} - u_x u_y).$$

Since $vv_{xy} - v_x v_y = \sigma v = (xy)^m\sigma u$, we find

$$\sigma u = (xy)^m(uu_{xy} - u_x u_y).$$

Since v is not divisible by $x^{m+1}y^m$ and not by $x^m y^{m+1}$, u is not divisible by x and y. We see that σ is divisible by $(xy)^m$. Let $\sigma = M_{xy}^m\gamma$ for $\gamma \in \mathfrak{P}_\infty[x,y]$. Then

$$uu_{xy} - u_x u_y = \gamma u.$$

This proves our assertion. □

Definition 3.3. A solution $u \in \mathfrak{P}_\infty[x,y]$ to (1.3) for a given data is called a prime solution to (1.3) if u is not divisible by xy.

Let us denote the image of $M_n(\mathbb{C})$ in $\mathbb{C}[x,y]$ via $\mathfrak{p}$ by $\mathfrak{P}_n[x,y]$. Then $\mathfrak{P}_\infty[x,y] = \bigcup_{n\geq 1}\mathfrak{P}_n[x,y]$.

Lemma 3.4. *Let* $\sigma \in \mathfrak{P}_m[x,y]$ *with* $\deg\sigma = 2m-2$. *If* $u \in \mathfrak{P}_\infty[x,y]$ *is a solution to* (1.3) *for the data* σ *of degree* $2n-2$, *then* $n \geq m+2$.

Proof. We observe that the coefficients of $x^{2n-3}y^{2n-3}$ and of $x^{2n-3}y^{2n-4}$ and of $x^{2n-4}y^{2n-3}$ in $uu_{xy} - u_x u_y$ all vanish. Then $uu_{xy} - u_x u_y$ is a polynomial of degree at most $4n-8$. On the other hand, the degree of σu is $2n+2m-4$. We conclude that $n \geq m+2$. □

Let us write a remark that V_σ is an ind-affine variety. Given $\sigma \in \mathfrak{P}_m[x,y]$ with degree $2m-2$, the intersection $V_\sigma^n = V_\sigma \cap M_n(\mathbb{C})$ is an affine algebraic subvariety of $M_n(\mathbb{C}) \cong \mathbb{A}^{n^2}(\mathbb{C})$ for $n \geq m+2$ and $V_\sigma = \bigcup_{n\geq m+2} V_\sigma^n$.

Let $q \in \mathfrak{P}_n[x,y]$. Formally, we define

$$\widetilde{q}(x,y) = (xy)^n q(x^{-1}, y^{-1}).$$

Lemma 3.5. *Let* $\sigma \in \mathfrak{P}_m[x,y]$ *be given with* $\deg\sigma = 2m-2$. *If* $u \in \mathfrak{P}_n[x,y]$ *is a solution to* (1.3) *for data* σ, *then* $\widetilde{u}$ *is a solution to* (1.3) *with data* $M_{xy}^{n-m-2}\widetilde{\sigma}$.

Proof. Let $v = \widetilde{u}$. Then $v(x,y) = (xy)^n u(x^{-1}, y^{-1})$. Then

$$\begin{aligned} v_x &= nx^{n-1}y^n u(x^{-1},y^{-1}) - x^{n-2}y^n u_x(x^{-1},y^{-1}),\\ v_y &= nx^n y^{n-1}u(x^{-1},y^{-1}) - x^n y^{n-2}u_y(x^{-1},y^{-1}),\\ v_{xy} &= n^2x^{n-1}y^{n-1}u(x^{-1},y^{-1}) - nx^{n-1}y^{n-2}u_y(x^{-1},y^{-1})\\ &\quad - nx^{n-2}y^{n-1}u_x(x^{-1},y^{-1}) + x^{n-2}y^{n-2}u_{xy}(x^{-1},y^{-1}). \end{aligned}$$

This implies that

$$\begin{aligned} vv_{xy} - v_xv_y &= (xy)^{2n-2}(u(x^{-1},y^{-1})u_{xy}(x^{-1},y^{-1}) - u_x(x^{-1},y^{-1})u_y(x^{-1},y^{-1}))\\ &= (xy)^{2n-2}\sigma(x^{-1},y^{-1})u(x^{-1},y^{-1})\\ &= (xy)^{n-m-2}(xy)^m\sigma(x^{-1},y^{-1})\cdot(xy)^n u(x^{-1},y^{-1})\\ &= M_{xy}^{n-m-2}\widetilde{\sigma}(x,y)v(x,y). \end{aligned}$$

This proves our assertion. ☐

This lemma leads to:

Corollary 3.6. *Let $\sigma \in \mathfrak{P}_m[x,y]$ be given with $\deg\sigma = 2m-2$. If $u \in \mathfrak{P}_{m+2}[x,y]$ is a solution to (1.3) for data σ, then $\widetilde{u}$ is a solution to (1.3) with data $\widetilde{\sigma}$.*

Apparently, it is not simple to determine whether the set V_σ is empty or not. For the main purpose of this paper, we give only a partial solution to this question.

A polynomial u in $\mathfrak{P}_\infty[x,y]$ is called diagonal if $u = \mathfrak{p}(A)$ for some diagonal matrix $A \in M_\infty(\mathbb{C})$. If a polynomial u is diagonal, we can find a polynomial $Q(t) \in \mathbb{C}[t]$ such that $u(x,y) = Q(xy)$. Here comes a natural question: given a diagonal polynomial σ as a data of (1.3), can we find a solution u to (1.3) such that u is also diagonal? From now on, we only consider prime solutions to (1.3).

Theorem 3.7. *Let $\sigma \in \mathfrak{P}_m[x,y]$ be a diagonal polynomial of degree $2m-2$ with $\sigma(x,y) = S(xy)$ for some $S(t) \in \mathbb{C}[t]$. Then (1.3) has a solution u that is also diagonal if and only if there exists $N \in \mathbb{N}$ with $N \geq m+2$ such that the family of polynomial $\{f_S^i : N+1 \leq i \leq 2N-1\}$ has a nonzero common root. Furthermore, if $N = m+2$, then $u \in \mathfrak{P}_{m+2}[x,y]$ with $\deg u = 2m+2$.*

Proof. Assume that $v(x,y) = q(xy)$ for some $q \in \mathbb{C}[t]$. Then

$$\begin{aligned} vv_{xy} - v_xv_y - \sigma v &= (xy)q''(xy)q(xy) + q'(xy)q(xy)\\ &\quad - (xy)(q'(xy))^2 - S(xy)q(xy). \end{aligned} \tag{3.1}$$

If $u(x,y)$ is a diagonal polynomial that solves (1.3) for data σ, and if we write $u(x,y) = Q(xy)$ for some $Q(t) \in \mathbb{C}[t]$, then by (3.1), $Q(t)$ solves (1.2) for data $S(t)$ with $t = xy$. Since $Q(t)$ is a polynomial solution to (1.2) with data $S(t)$, Proposition 2.4 implies the result.

Let us prove the converse. Let a be a nonzero common root of $\{f_S^i : N+1 \leq i \leq 2N-1\}$. Define q_i by $q_0 = 1/a$ and $q_i = f_S^i(a)$ for $i \geq 1$. By Proposition 2.4, the polynomial $Q(t) = \sum_{i=0}^\infty q_i t^i$ solves (1.2) with data $S(t)$. Define $u(x,y) = Q(xy)$. Then $u(x,y)$ is a polynomial. By (3.1), u solves (1.3) for data σ. The rest follows from Corollary 2.6. ☐

This theorem enables us to find a class of polynomials σ in $\mathfrak{P}_\infty[x,y]$ such that V_σ is nonempty. It would be interesting to find criterions to know when V_σ is nonempty for any $\sigma \in \mathfrak{P}_\infty[x,y]$.

4. An explicit construction of a solution to the mean field equation for hyperelliptic curves

Let $H = (h_{ij})_{i,j=1}^g$ be a $g \times g$ positive definite hermitian matrix and consider the corresponding canonical metric ds_H^2 on the hyperelliptic curve X of genus g defined in the introduction. If we let $\sigma_H(x,y)$ be the complex polynomial $\sigma_H(x,y) = \sum_{i,j=1}^g h_{ij}x^{i-1}y^{j-1}$, then the canonical metric ds_H^2 on X has the local expression

$$ds_H^2 = \begin{cases} \dfrac{\sigma_H(x,\overline{x})}{|y^2|} dx \otimes d\overline{x} & \text{on } C_0, \\ \dfrac{\widetilde{\sigma}_H(z,\overline{z})}{|w^2|} dz \otimes d\overline{z} & \text{on } C_0'. \end{cases}$$

Theorem 4.1. *Suppose* (1.3) *has a solution* $u = \mathfrak{p}(A) \in P_{g+1}[x,y]$ *for the data* σ_H *with* $A \in M_{g+1}(\mathbb{C})$ *being positive definite. Then the function*

$$\varphi = \begin{cases} \dfrac{4|f(x)|}{u(x,\overline{x})} & \text{on } C_0, \\ \dfrac{4|g(z)|}{\widetilde{u}(z,\overline{z})} & \text{on } C_0', \end{cases}$$

is a globally defined nonnegative smooth function whose zero set coincides with the set of Weierstrass points of X *and* $\psi = \log\varphi$ *defines smooth function on* $X \setminus \{P_1, \cdots, P_{2g+2}\}$ *satisfying* (1.1)

Proof. The proof is the same as that given in our previous paper; we give a sketch of the proof. For more details, see [2]. Let us verify that $\Delta\psi + e^\psi = 0$ on $U = X \setminus \{P_1, \cdots, P_{2g+2}\}$. We will prove this equation on $U \cap C_0$. Since u satisfies (1.3), on $U \cap C_0$,

$$\frac{\partial^2}{\partial x \partial \overline{x}} \log\varphi = -\frac{u_{x\overline{x}}u - u_x u_{\overline{x}}}{u^2} = -\frac{\sigma u}{u^2} = -\frac{\sigma}{u}.$$

As a consequence,

$$\Delta_H \psi = 4\frac{|f(x)|}{\sigma(x,\overline{x})} \frac{\partial^2}{\partial x \partial \overline{x}} \log\varphi = -4\frac{|f(x)|}{u(x,\overline{x})} = -\varphi = -e^\psi.$$

Similarly, the equation holds on $U \cap C_0'$.

Let $P = P_k$ be a Weierstrass point of X. In a coordinate neighborhood (U_P, ζ) of $P = P_k$, where $\zeta = \sqrt{x - e_k}$, the function ψ has a local expression $\psi = 2\log|\zeta| + \alpha$, where α is a nonzero smooth function on U_P. By classical analysis, the action of the Laplace operator Δ on ψ creates a Dirac delta measure $4\pi\delta_{P_k}$. We complete the proof of our assertion. □

Since H is a $g \times g$ positive definite hermitian matrix, there exists a $g \times g$ unitary matrix U such that U^*HU is a diagonal matrix Λ with positive diagonals. We assume that $\Lambda = \operatorname{diag}(\lambda_1, \cdots, \lambda_g)$ with $\lambda_i > 0$ for $1 \le i \le g$. Let us denote $S_\Lambda(t) = \sum_{i=0}^{g-1} \lambda_{i+1} t^i$; then the polynomial $\sigma_\Lambda(x,y) = S_\Lambda(xy)$ is diagonal. In other words, we consider the canonical metric on X of the form

$$ds_\Lambda^2 = \begin{cases} \dfrac{\sum_{i=1}^g \lambda_i (x\overline{x})^{i-1}}{|y^2|} dx \otimes d\overline{x} & \text{on } C_0, \\ \dfrac{\sum_{i=1}^g \lambda_i (z\overline{z})^{g-i}}{|w^2|} dz \otimes d\overline{z} & \text{on } C_0'. \end{cases}$$

One can use Theorem 3.7 to determine diagonal solutions to (1.3) for σ_H in this case and to obtain "positive definite" solutions to (1.3) for σ_H. We need further analysis, i.e., solutions $u = \mathfrak{p}(A)$ so that A is a $g \times g$ positive definite hermitian matrix. For $g \geq 2$, let $\{F^i(x_0, \cdots, x_{g-1}, t) : i \geq 1\}$ be the sequence of polynomials defined in (2.4). Let V be the affine algebraic subset of $\mathbb{C}^{g+1}$ defined by the zero set of the polynomials $\{F^{g+2}, \cdots, F^{2g-1}\}$ and let D_+^{g+1} be the set of all n-tuples of real numbers $(a_1, \cdots, a_{g+1})$ such that $a_i > 0$ for all $1 \leq i \leq g+1$ and let Q_+^{g+1} be the subset of all D_+^{g+1} consisting of points $(a_0, \cdots, a_{g+1})$ so that $F^i(a_0, \cdots, a_{g+1}) > 0$ for $1 \leq i \leq g+1$. If there exists a positive real number a such that $(\Lambda, a) \in V \cap Q_+^{g+1}$, then the polynomial

$$u_{(\Lambda,a)}(x,y) = \frac{1}{a} + \sum_{i=1}^{g+1} F^i(\Lambda, a)(xy)^i \tag{4.1}$$

solves for (1.3) and equals $\mathfrak{p}(A)$ for $A = \mathrm{diag}(1/a, F^1(\Lambda, a), \cdots, F^{g+1}(\Lambda, a))$ and hence determines a solution to (1.1) by

$$\psi_{(\Lambda,a)} = \begin{cases} \log \dfrac{|f(x)|}{u_{(\Lambda,a)}(x, \overline{x})} & \text{on } C_0, \\ \log \dfrac{|g(z)|}{\widetilde{u}_{(\Lambda,a)}(z, \overline{z})} & \text{on } C_0', \end{cases} \tag{4.2}$$

for $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_g)$. Let us take a look at the case when X is of genus two and of genus three.

Example 4.2. Let X be the hyperelliptic curve defined by the equation $y^2 = f(x)$ with metric ds^2, where $f(x)$ is a degree six polynomial with six distinct roots and

$$ds^2 = \frac{1 + |x|^2}{|y^2|} dx \otimes d\overline{x}.$$

In this case, $S(t) = 1 + t$. Then $f_S^1(t) = 1$ and $f_S^2(t) = (t+1)/4$ and $f_S^3(t) = t/9$ and

$$f_S^4(t) = -\frac{1}{192}t^2 + \frac{1}{64}t,$$
$$f_S^5(t) = \frac{1}{1800}t^3 - \frac{1}{600}t^2.$$

One sees that 3 is the common root of the polynomials $f_S^4(t)$ and $f_S^5(t)$. Then $h_0 = 1/3$ and $h_1 = f_S^1(3) = 1$ and $h_2 = f_S^2(3) = 1$ and $h_3 = f_S^3(3) = 1/3$. We obtain a polynomial $u(x, y)$ by

$$u(x,y) = \frac{1}{3} + xy + (xy)^2 + \frac{1}{3}(xy)^3$$

which solves (1.3) for data $\sigma(x, y) = 1 + xy$. This gives us a solution ψ to (1.1) by the construction of Theorem 4.1 for the genus two hyperelliptic curve:

$$\psi = \begin{cases} \log \dfrac{12|f(x)|}{(1+|x|^2)^3} & \text{on } C_0, \\ \log \dfrac{12|g(z)|}{(1+|z|^2)^3} & \text{on } C_0', \end{cases}$$

The result coincides with that obtained in our previous paper.

Example 4.3. Let X be the hyperelliptic curve defined by the equation $y^2 = f(x)$ with the metric ds^2, where $f(x)$ is a polynomial with eight distinct roots and

$$ds^2 = \frac{1+|x|^2+|x|^4}{|y^2|} dx \otimes d\overline{x}.$$

In this case, $S(t) = 1+t+t^2$. Then $f_S^1(t) = 1$ and $f_S^2(t) = (t+1)/4$ and $f_S^3(t) = (t+1)/9$ and $f_S^4(t) = (-t^2+11t)/192$ and

$$\begin{aligned} f_S^5(t) &= \frac{1}{1800}t^3 - \frac{11}{1800}t^2 + \frac{1}{75}t, \\ f_S^6(t) &= -\frac{1}{11520}t^4 + \frac{11}{11520}t^3 - \frac{1}{405}t^2 + \frac{1}{324}t, \\ f_S^7(t) &= \frac{1}{58800}t^5 - \frac{401}{2116800}t^4 + \frac{373}{705600}t^3 - \frac{43}{52920}t^2. \end{aligned}$$

One sees that 8 is the common root of the polynomials $f_S^5(t)$ and $f_S^6(t)$ and $f_S^7(t)$. We see that $h_0 = 1/8$ and $h_1 = f_S^1(8) = 1$ and $h_2 = f_S^2(8) = 9/4$ and $h_3 = f_S^3(8) = 1$ and $h_4 = f_S^4(8) = 1/8$. We obtain a polynomial

$$u(x,y) = \frac{1}{8} + xy + \frac{9}{4}(xy)^2 + (xy)^3 + \frac{1}{8}(xy)^4$$

that solves (1.3) for the data $\sigma(x,y) = 1 + xy + (xy)^2$. This gives us a solution ψ to (1.1) by Theorem 4.1 for the genus three hyperelliptic curve:

$$\psi = \begin{cases} \log \dfrac{12|f(x)|}{\left(\frac{1}{8} + |x|^2 + \frac{9}{4}|x|^4 + |x|^6 + \frac{1}{8}|x|^8\right)} & \text{on } C_0, \\ \log \dfrac{12|g(z)|}{\left(\frac{1}{8} + |z|^2 + \frac{9}{4}|z|^4 + |z|^6 + \frac{1}{8}|z|^8\right)} & \text{on } C_0', \end{cases}$$

5. Adiabatic limit of solutions to mean field equations

We propose a possible direction following the results above. Rescale the canonical metric by $\gamma \in \mathbb{R}^+$, i.e., we consider the rescaling of the canonical metric $ds^2_{\Lambda,\gamma} = \gamma ds^2_\Lambda$. With respect to this metric, the mean field equation is equivalent to

$$\Delta\psi_\gamma + \gamma e^{\psi_\gamma} = 4\pi\gamma \sum_{i=1}^{2g+2} \delta_{P_i} \tag{5.1}$$

with respect to ds^2_Λ.[1] Following from the analysis in [1], we study the existence of a solution to this equation for small γ, as well as the limit of the solutions $\{\psi_\gamma\}$ as $\gamma \to 0$. Directly observing (5.1), we naturally expect $\Delta\psi_\gamma \to 0$ as $\gamma \to 0$, or that ψ_γ approaches to a constant function since X is a connected closed manifold. Classical analysis from [1] confirms both expectations. We normalize the metrics so that the area of X is 1. Let $W^{k,p}(X)$ be the completion of $C^\infty(X)$ with respect to the (k,p)-norm:

$$\|u\|_{W^{k,p}(X)} = \sum_{j=0}^{k} \left(\int_X |\nabla^j u|^p d\nu \right)^{1/p},$$

[1]For convenience, we use Δ instead of Δ_Λ in this section.

where $\nabla^j u$ is the j-th covariant derivative of u. We call $W^{k,p}(X)$ the Sobolev (k,p)-space on X.[2] A technical analytic statement is needed to conclude the asymptotic behaviors:

Proposition 5.1. *If $u_j \to u$ weakly in $W^{1,2}(X)$, then $e^{u_j} \to e^u$ strongly in $L^2(X)$.*

Proof. For the proof, see (3.7) in [1]. □

Theorem 5.2 (Adiabatic limit)**.** *A solution to* (5.1) *exists for all γ small enough and approaches a constant in $W^{2,2}(X)$ as $\gamma \to 0$.*

Proof. We only sketch the existence part of the proof since it is a replica of the proof from Theorem 7.2 in [1]. Let

$$\psi_\gamma := v_\gamma + 4\pi\gamma \sum_{i=1}^{2g+2} G_i, \tag{5.2}$$

where G_i is the Green's function satisfying $\Delta G_i = -\delta_{P_i} + 1$. Solving (5.1) is then equivalent to solving the following equation:

$$\Delta v_\gamma + \gamma h e^{v_\gamma} = 8\pi\gamma(g+1), \tag{5.3}$$

where the function $h = \exp\left(4\pi \sum_{i=1}^{2g+2} G_i\right) \in C^\infty(X)$ is nonnegative with zero set precisely the Weierstrass points. This is a Kazdan-Warner equation of the type discussed in section 7 from [1], which is solved by a variational method. One notes that (5.3) is the minimizing equation to the functional

$$J(u) = \int_X \left(\frac{1}{2}|\nabla u|^2 + 8\pi\gamma(g+1)u\right) d\nu \tag{5.4}$$

on the subset $B \subset W^{1,2}(X)$ satisfying the constraint equation

$$\int_X h e^u d\nu = 8\pi(g+1). \tag{5.5}$$

Following identical reasoning, we have the following estimate for J:

$$J(u) \geq \frac{1}{4\beta}(2\beta - 8\pi\gamma(g+1))\|\nabla u\|^2_{L^2(X)} + \delta, \tag{5.6}$$

where δ is a constant and β is a Trudinger constant for X both independent of γ. More precisely, β is a positive constant so that

$$\int_X e^{\beta v^2} d\nu$$

are uniformly bounded for all $v \in W^{1,2}(X)$ with $\overline{v} = 0$ and $\|\nabla v\|_{L^2(X)} \leq 1$. Such a constant always exists for surfaces (cf. (3.4) in [1]). Therefore, for γ small enough so that $2\beta - 8\pi\gamma(g+1) > 0$, J is bounded below and positive.

For each γ, (5.6) and Sobolev embedding shows that the minimizing sequence $\{v_\gamma^i\}$ of J is contained in a fixed ball of radius R_γ in $W^{1,2}(X)$, which is weakly compact. Passing to a subsequence, let v_γ be the weak limit. Arguments in the proof of Theorem 5.3 in [1] show that v_γ minimizes J in B and, therefore, is a strong limit and solution to (5.3). The proof there also provides a regularity argument,

[2] In some context, people use $H^{k,p}(X)$ for Sobolev (k,p) spaces.

which is applicable to our case here, to show that v_γ is actually smooth. The existence of a smooth solution for each γ is established.

Furthermore, one notices that the radii R_γ are uniformly controlled over γ (in fact proportional to $(2\beta-8\pi\gamma(g+1))^{-1}$) and therefore $\{v_\gamma\}$ are uniformly bounded in $W^{1,2}(X)$. Following identical arguments, let v be the limit of v_γ in $W^{1,2}(X)$. Proposition 5.1 then implies that e^{v_γ} converge to e^v in $L^2(X)$ and, therefore, are uniformly bounded in $L^2(X)$. It then follows from elliptic regularity of Δ in (5.3):

$$\|v_\gamma\|_{W^{2,2}(X)} \leq c(\gamma\|8\pi(g+1) - he^{v_\gamma}\|_{L^2(X)} + \|v_\gamma\|_{L^2(X)}) \tag{5.7}$$

that v_γ are uniformly bounded in $W^{2,2}(X)$. The estimate, together with some Schauder estimates, also imply that $v \in C^\infty(X)$. After taking a subsequence, we conclude that $v_\gamma \to v$ in $W^{2,2}(X)$. Taking the limit $\gamma \to 0$ in (5.3), it then follows that

$$\Delta v = \lim_{\gamma\to 0} \Delta v_\gamma = 0 \tag{5.8}$$

and, therefore, v is a constant function since X is closed. □

It is of great interest, as stated in [1], to study the upper bound of γ:

$$\gamma_0 = \frac{\beta}{4\pi(g+1)}$$

for (5.3) to be solvable, a quantity related to the geometry of X. It is not immediately clear whether $\gamma_0 \geq 1$, despite the explicit solution to (5.3) with $\gamma = 1$ in Section 4. One may attempt to construct a variation of (4.2) depending on γ, and its corresponding mean field equation so that the limiting solution at $\gamma = 0$ coincides with that of Theorem 5.2. Such a conjecture provides significant geometric insight. In the case of Example 4.2 where solutions are precisely the logarithm of Gaussian curvatures, the limiting solution suggests that the manifold deforms into $\mathbb{S}^2$, a sign of topological jumps, or bubbling.

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