

EINSTEIN SUBMANIFOLDS WITH FLAT NORMAL BUNDLE IN SPACE FORMS ARE HOLONOMIC

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ABSTRACT. A well-known result asserts that any isometric immersion with flat normal bundle of a Riemannian manifold with constant sectional curvature into a space form is (at least locally) holonomic. In this note, we show that this conclusion remains valid for the larger class of Einstein manifolds. As an application, when assuming that the index of relative nullity of the immersion is a positive constant we conclude that the submanifold has the structure of a generalized cylinder over a submanifold with flat normal bundle.

A remarkable class of submanifolds in space forms are those that enjoy the property of being holonomic. An isometric immersion $f: M^n \rightarrow \mathbb{Q}_c^N$ of a Riemannian manifold into a space form of constant sectional curvature c is said to be *holonomic* if M^n carries a global system of orthogonal coordinates such that at any point the coordinate vector fields diagonalize its second fundamental form $\alpha: TM \times TM \rightarrow N_f M$ with values in the normal bundle.

Among several interesting facts regarding holonomic submanifolds, we recall that they are a natural playground for the Ribaucour transformation [3]. As an application of the so-called vectorial Ribaucour transformation as given in [5], one can locally parametrically generate any proper holonomic submanifold in terms of a set of smooth functions whose Hessians are all diagonal with respect to the coordinate vector fields of a given orthogonal system of coordinates; see [4] for details.

Since holonomic submanifolds have flat normal bundle, it is a standard fact (see [7]) that at each point $x \in M^n$ there exists a set of unique pairwise distinct normal vectors $\eta_i \in N_f M(x)$, $1 \leq i \leq s = s(x)$, and an associated orthogonal splitting of the tangent space as

$$T_x M = E_1(x) \oplus \cdots \oplus E_s(x),$$

where

$$E_i(x) = \{X \in T_x M : \alpha(X, Y) = \langle X, Y \rangle \eta_i \text{ for all } Y \in T_x M\}.$$

Hence, the second fundamental form of f acquires the form

$$(1) \quad \alpha(X, Y)(x) = \sum_{i=1}^s \langle X^i, Y^i \rangle \eta_i,$$

where $X \mapsto X^i$ is the orthogonal projection from $T_x M$ onto $E_i(x)$.

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A submanifold $f: M^n \rightarrow \mathbb{Q}_c^N$ with flat normal bundle is said to be *proper* if $s(x) = k$ is constant on M^n . If this is the case, then the maps $x \in M^n \mapsto \eta_i(x)$, $1 \leq i \leq k$, are smooth vector fields, called the *principal normal vector fields* of f , and the distributions $x \in M^n \mapsto E_i(x)$, $1 \leq i \leq k$, are also smooth.

There are several conditions that imply that a submanifold of a space form has to be locally holonomic. By locally we mean along connected components of an open dense subset of the manifold. For instance, this is the case of any isometric immersion $f: M_c^n \rightarrow \mathbb{Q}_c^N$ with flat normal bundle of a manifold with the same constant sectional curvature as the ambient space form provided that index of relative nullity vanishes at any point; see [3] for a more general result. Recall that the *index of relative nullity* $\nu(x)$ of $f: M^n \rightarrow \mathbb{Q}_c^N$ at $x \in M^n$ is the dimension of the *relative nullity subspace* $\Delta(x) \subset T_x M$ given by

$$\Delta(x) = \{X \in T_x M : \alpha(X, Y) = 0 \text{ for all } Y \in T_x M\}.$$

Isometric immersions $f: M_c^n \rightarrow \mathbb{Q}_{\tilde{c}}^{n+p}$ with sectional curvatures $c < \tilde{c}$ and in the least possible codimension $p = n - 1$ have flat normal bundle and thus are always locally holonomic. This was already known to Cartan [1, 2], who made an exhaustive study of the subject and determined the degree of generality of such submanifolds. Moreover, being holonomic is also necessarily the case for $c > \tilde{c}$ but now under the extra condition that the submanifold is free of weak-umbilic points; see [6].

In this note, we show that results discussed above still hold for isometric immersions of the larger class of Einstein manifolds. In fact, this turns out to be the case even in the presence of a constant positive index of relative nullity. Thus in the case of submanifolds of manifolds with the same constant sectional curvature the restriction mentioned above can be removed.

Theorem 1. *Any isometric immersion $f: M^n \rightarrow \mathbb{Q}_c^N$ with flat normal bundle and proper of an Einstein manifold is locally holonomic.*

Proof. It holds that M^n is Einstein with $\text{Ric}_M = \lambda I$ if and only if the vector fields

$$\hat{\eta}_i = \eta_i - \frac{n}{2}H, \quad 1 \leq i \leq k,$$

satisfy

$$(2) \quad \|\hat{\eta}_i\|^2 = \frac{n^2}{4}\|H\|^2 + c(n - 1) - \lambda,$$

where H denotes the mean curvature vector field of f . To see this, we first observe that for any isometric immersion $f: M^n \rightarrow \mathbb{Q}_c^N$ a straightforward computation of the Ricci tensor using the Gauss equation yields

$$\text{Ric}_M(X, Y) = c(n - 1)\langle X, Y \rangle + n\langle \alpha(X, Y), H \rangle - \sum_{j=1}^n \langle \alpha(X, X_j), \alpha(Y, X_j) \rangle,$$

where X_1, \dots, X_n is an orthonormal tangent base. It follows easily using (1) that

$$(3) \quad \text{Ric}_M(X, Y) = c(n - 1)\langle X, Y \rangle + n\langle \alpha(X, Y), H \rangle - \sum_{i=1}^s \langle X^i, Y^i \rangle \|\eta_i\|^2$$

for all $X, Y \in TM$. From (1) and (3) we have that $\text{Ric}_M = \lambda I$ is equivalent to

$$c(n - 1) - \lambda = \|\eta_i\|^2 - n\langle H, \eta_i \rangle, \quad 1 \leq i \leq k,$$

and this in turn is equivalent to (2).

We claim that at any point the vectors $\eta_i - \eta_j$ and $\eta_i - \eta_\ell$ are linearly independent if $i \neq j \neq \ell \neq i$. Assume to the contrary that

$$\eta_i - \eta_j = \mu(\eta_i - \eta_\ell)$$

for some $\mu \neq 0$. Then

$$(1 - \mu)\hat{\eta}_i = \hat{\eta}_j - \mu\hat{\eta}_\ell$$

yields

$$\|\hat{\eta}_i\|^2 - 2\mu\|\hat{\eta}_i\|^2 + \mu^2\|\hat{\eta}_i\|^2 = \|\hat{\eta}_j\|^2 - 2\mu\langle\hat{\eta}_j, \hat{\eta}_\ell\rangle + \mu^2\|\hat{\eta}_\ell\|^2.$$

We have from (2) that the $\hat{\eta}_j$'s are of equal length. Thus

$$\langle\hat{\eta}_j, \hat{\eta}_\ell\rangle = \|\hat{\eta}_j\|\|\hat{\eta}_\ell\|,$$

and hence $\hat{\eta}_j = \hat{\eta}_\ell$. This is a contradiction that proves the claim.

In order to conclude holonomicity it is a standard fact that it suffices to show that the distributions E_j^\perp are integrable for $1 \leq j \leq k$. The Codazzi equation is easily seen to yield

$$(4) \quad \langle\nabla_X Y, Z\rangle(\eta_i - \eta_j) = \langle X, Y\rangle\nabla_Z^\perp \eta_i$$

and

$$(5) \quad \langle\nabla_X V, Z\rangle(\eta_j - \eta_\ell) = \langle\nabla_V X, Z\rangle(\eta_j - \eta_i)$$

for all $X, Y \in E_i, Z \in E_j$, and $V \in E_\ell$, where $1 \leq i \neq j \neq \ell \neq i \leq k$.

It follows from (4) that the E_i 's are integrable. Thus, it is sufficient to argue for the case $k \geq 3$. In fact, it suffices to show that if $X \in E_i$ and $Y \in E_j$, then $[X, Y] \in E_\ell^\perp$ if $i \neq j \neq \ell \neq i$. We have from (5) that

$$\langle\nabla_X Y, Z\rangle(\eta_\ell - \eta_j) = \langle\nabla_Y X, Z\rangle(\eta_\ell - \eta_i)$$

for any $Z \in E_\ell$. We obtain using the result of the claim that

$$\langle\nabla_X Y, Z\rangle = \langle\nabla_Y X, Z\rangle = 0,$$

and this completes the proof. □

Let $g: L^{n-s} \rightarrow \mathbb{Q}_c^N, 1 \leq s \leq n-1$, be an isometric immersion carrying a parallel flat normal subbundle $\mathcal{L} \subset N_g L$ of rank s . The *generalized cylinder* determined by the subbundle $\pi: \mathcal{L} \rightarrow L^{n-s}$ is the n -dimensional submanifold $f: M^n \rightarrow \mathbb{Q}_c^N$ parametrized (at regular points) by means of the exponential map of \mathbb{Q}_c^N as

$$\gamma \in \mathcal{L} \mapsto \exp_{g(\pi(\gamma))} \gamma.$$

We have that $\gamma \in \mathcal{L}$ is a regular point if and only if $P = I - A_\gamma^g$ is nonsingular, where A_γ^g stands for the shape operator of g corresponding to γ . Also $N_f M = \mathcal{L}^\perp$, up to parallel identification along the fibers of \mathcal{L} that are contained in the relative nullity subspaces of f . Moreover, the relation between the second fundamental forms of f and g is given by

$$\alpha_f(X, Y) = (\alpha_g(X, PY))_{\mathcal{L}^\perp}$$

for all $X, Y \in TL$. It follows that f has flat normal bundle if and only if g has flat normal bundle.

The following result obtained in [4] asserts that any submanifold with a relative nullity distribution $x \in M^n \mapsto \Delta(x)$ of constant dimension whose *conullity* distribution $x \in M^n \mapsto \Delta^\perp(x)$ is integrable has to be a generalized cylinder.

Proposition 2. *Let $g: L^{n-s} \rightarrow \mathbb{Q}_c^N$ be an isometric immersion carrying a parallel flat normal subbundle $\mathcal{L} \subset N_g L$ of rank s such that*

$$\{Y \in T_x L : (\alpha_g(Y, X))_{\mathcal{L}^\perp} = 0 \text{ for all } X \in T_x L\} = 0$$

for any point $x \in M^n$. Then the generalized cylinder over g determined by \mathcal{L} has constant index of relative nullity $\nu = s$ and integrable conullity.

Conversely, any submanifold $f: M^n \rightarrow \mathbb{Q}_c^N$ with relative nullity distribution Δ , constant index of relative nullity $\nu = s$, and integrable conullity arises this way locally. This means that $\mathcal{L} = \Delta|_L$ is a parallel flat normal subbundle of $g = f|_L$ for any integral leaf L^{n-s} of the conullity and f is locally an open neighborhood of $g(L)$ in the generalized cylinder over g determined by \mathcal{L} .

We have the following consequence of Theorem 1.

Corollary 3. *Let $f: M^n \rightarrow \mathbb{Q}_c^N$ be an isometric immersion with flat normal bundle of an Einstein manifold that is proper and has constant index of relative nullity $\nu = s \geq 1$. Then f is locally a generalized cylinder over a submanifold $g: L^{n-s} \rightarrow \mathbb{Q}_c^N$ with flat normal bundle.*

Proof. Notice that if the index of relative nullity is $\nu \geq 1$ at any point and $\nu > 1$ at some point, then it has to be constant since f is proper. The proof follows easily from Theorem 1 and Proposition 2. \square

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