PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 9, September 2018, Pages 3935–3942
http://dx.doi.org/10.1090/proc/14059
Article electronically published on April 18, 2018

SPECTRAL BOUNDS FOR SINGULAR INDEFINITE STURM-LIOUVILLE OPERATORS WITH L^1-POTENTIALS

JUSSI BEHRNDT, PHILIPP SCHMITZ, AND CARSTEN TRUNK

(Communicated by Wenxian Shen)

Abstract. The spectrum of the singular indefinite Sturm-Liouville operator

$$A = \operatorname{sgn}(\cdot)\big(-\tfrac{d^2}{dx^2} + q\big)$$

with a real potential $q \in L^1(\mathbb{R})$ covers the whole real line, and, in addition, non-real eigenvalues may appear if the potential q assumes negative values. A quantitative analysis of the non-real eigenvalues is a challenging problem, and so far only partial results in this direction have been obtained. In this paper the bound

$$|\lambda| \leq \|q\|_{L^1}^2$$

on the absolute values of the non-real eigenvalues λ of A is obtained. Furthermore, separate bounds on the imaginary parts and absolute values of these eigenvalues are proved in terms of the L^1-norm of the negative part of q.

1. Introduction

The aim of this paper is to prove bounds on the absolute values of the non-real eigenvalues of the singular indefinite Sturm-Liouville operator

$$\begin{aligned} Af &= \operatorname{sgn}(\cdot)\big(-f'' + qf\big), \\ \operatorname{dom} A &= \big\{ f \in L^2(\mathbb{R}) : f, f' \in AC(\mathbb{R}), -f'' + qf \in L^2(\mathbb{R}) \big\}, \end{aligned}$$

where $AC(\mathbb{R})$ stands for the space of all locally absolutely continuous functions. It will always be assumed that the potential q is real-valued and belongs to $L^1(\mathbb{R})$.

The operator A is not symmetric nor self-adjoint in an L^2-Hilbert space due to the sign change of the weight function $\operatorname{sgn}(\cdot)$. However, A can be interpreted as a self-adjoint operator with respect to the Krein space inner product $(\operatorname{sgn}\cdot,\cdot)$ in $L^2(\mathbb{R})$. We summarize the qualitative spectral properties of A in the next theorem, which follows from [4, Theorem 4.2] or [16, Proposition 2.4] and the well-known spectral properties of the definite Sturm-Liouville operator $-\frac{d^2}{dx^2} + q$; cf. [23–25].

Theorem 1.1. *The essential spectrum of A coincides with $\mathbb{R}$, and the non-real spectrum of A consists of isolated eigenvalues with finite algebraic multiplicity which are symmetric with respect to $\mathbb{R}$.*

Indefinite Sturm-Liouville operators have been studied for more than a century and have again attracted a lot of attention in the recent past. Early works in this context usually deal with the regular case; that is, the operator A is studied on a

Received by the editors September 14, 2017, and, in revised form, December 7, 2017.
2010 *Mathematics Subject Classification.* Primary 34L15, 47E05.
Key words and phrases. Non-real eigenvalue, indefinite Sturm-Liouville operator, Krein space, Birman-Schwinger principle.

finite interval with appropriate boundary conditions at the endpoints; cf. [15, 22] and, e.g., [11, 18, 26]. In this situation the spectrum of A is purely discrete, and various estimates on the real and imaginary parts of the non-real eigenvalues were obtained in the last few years; cf. [2, 9, 10, 14, 17, 21]. The singular case is much less studied due to the technical difficulties which, very roughly speaking, are caused by the presence of continuous spectrum.

Explicit bounds on non-real eigenvalues for singular Sturm-Liouville operators with L^∞-potentials were obtained with Krein space perturbation techniques in [5] and under additional assumptions for L^1-potentials in [6, 7]; see also [3] for the absence of real eigenvalues and [19] for the accumulation of non-real eigenvalues of a very particular family of potentials. In this paper we substantially improve the earlier bounds in [6, 7] and relax the conditions on the potential. More precisely, here we prove for arbitrary real $q \in L^1(\mathbb{R})$ the following bound.

Theorem 1.2. *Let $q \in L^1(\mathbb{R})$ be real. Every non-real eigenvalue λ of the indefinite Sturm-Liouville operator A satisfies*

$$|\lambda| \leq \|q\|_{L^1}^2. \tag{1.1}$$

Moreover, we prove two bounds in terms of the negative part q_- of q.

Theorem 1.3. *Let $q \in L^1(\mathbb{R})$ be real. Every non-real eigenvalue λ of the indefinite Sturm-Liouville operator A satisfies*

$$|\operatorname{Im}\lambda| \leq 24 \cdot \sqrt{3}\|q_-\|_{L^1}^2 \quad \text{and} \quad |\lambda| \leq (24 \cdot \sqrt{3} + 18)\|q_-\|_{L^1}^2. \tag{1.2}$$

The bound (1.1) is proved in Section 2. Its proof is based on the Birman-Schwinger principle using similar arguments as in [1, 13] and [12, Chapter 14.3]; see also [8]. The bounds in (1.2) are obtained in Section 3 by adapting the techniques from the regular case in [2, 9, 21] to the present singular situation.

2. Proof of Theorem 1.2

In this section we prove the bound (1.1) for the non-real eigenvalues of A. We adapt a technique similar to the Birman-Schwinger principle in [12] and apply it to the indefinite operator A. The main ingredient is a bound for the integral kernel of the resolvent of the operator

$$B_0 f = \operatorname{sgn}(\cdot)\big(-f''\big), \quad \operatorname{dom} B_0 = \big\{f \in L^1(\mathbb{R}) : f, f' \in AC(\mathbb{R}), -f'' \in L^1(\mathbb{R})\big\},$$

in $L^1(\mathbb{R})$.

Lemma 2.1. *The operator B_0 is closed in $L^1(\mathbb{R})$, and for all λ in the open upper half-plane $\mathbb{C}^+$ the resolvent of B_0 is an integral operator*

$$\big[(B_0 - \lambda)^{-1} g\big](x) = \int_{\mathbb{R}} K_\lambda(x, y) g(y)\, \mathrm{d}y, \quad g \in L^1(\mathbb{R}),$$

where the kernel $K_\lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is bounded by $|K_\lambda(x, y)| \leq |\lambda|^{-\frac{1}{2}}$ for all $x, y \in \mathbb{R}$.

Proof. Here and in the following we define $\sqrt{\lambda}$ for $\lambda \in \mathbb{C}^+$ as the principal value of the square root, which ensures that $\operatorname{Im}\sqrt{\lambda} > 0$ and $\operatorname{Re}\sqrt{\lambda} > 0$. For $\lambda \in \mathbb{C}^+$ consider the integral operator

$$(T_\lambda g)(x) = \int_{\mathbb{R}} K_\lambda(x, y) g(y)\, \mathrm{d}y, \quad g \in L^1(\mathbb{R}), \tag{2.1}$$

with the kernel $K_\lambda(x,y) = C_\lambda(x,y) + D_\lambda(x,y)$ of the form

$$C_\lambda(x,y) = \frac{1}{2\alpha\sqrt{\lambda}} \begin{cases} \alpha e^{i\sqrt{\lambda}(x+y)}, & x \geq 0,\ y \geq 0, \\ -e^{\sqrt{\lambda}(ix+y)}, & x \geq 0,\ y < 0, \\ e^{\sqrt{\lambda}(x+iy)}, & x < 0,\ y \geq 0, \\ -\overline{\alpha} e^{\sqrt{\lambda}(x+y)}, & x < 0,\ y < 0, \end{cases}$$

and

$$D_\lambda(x,y) = \frac{1}{2\alpha\sqrt{\lambda}} \begin{cases} \overline{\alpha} e^{i\sqrt{\lambda}|x-y|}, & x \geq 0,\ y \geq 0, \\ 0, & x \geq 0,\ y < 0, \\ 0, & x < 0,\ y \geq 0, \\ -\alpha e^{-\sqrt{\lambda}|x-y|}, & x < 0,\ y < 0, \end{cases}$$

where $\alpha := \frac{1-i}{2}$. Hence,

$$|K_\lambda(x,y)| = |C_\lambda(x,y) + D_\lambda(x,y)| \leq \frac{1}{\sqrt{|\lambda|}},$$

and the integral in (2.1) converges for every $g \in L^1(\mathbb{R})$. We have

$$\sup_{y\geq 0} \int_{\mathbb{R}} |C_\lambda(x,y)|\,\mathrm{d}x = \frac{1}{2\sqrt{|\lambda|}} \left(\frac{1}{\operatorname{Im}\sqrt{\lambda}} + \frac{\sqrt{2}}{\operatorname{Re}\sqrt{\lambda}} \right)$$

and

$$\sup_{y<0} \int_{\mathbb{R}} |C_\lambda(x,y)|\,\mathrm{d}x = \frac{1}{2\sqrt{|\lambda|}} \left(\frac{\sqrt{2}}{\operatorname{Im}\sqrt{\lambda}} + \frac{1}{\operatorname{Re}\sqrt{\lambda}} \right).$$

For $y \geq 0$ we estimate

$$\int_0^\infty |D_\lambda(x,y)|\,\mathrm{d}x = \frac{1}{2\sqrt{|\lambda|}} \int_0^\infty e^{-\operatorname{Im}\sqrt{\lambda}|x-y|}\,\mathrm{d}x = \frac{2 - e^{-\operatorname{Im}\sqrt{\lambda} y}}{2\sqrt{|\lambda|}\operatorname{Im}\sqrt{\lambda}} \leq \frac{1}{\sqrt{|\lambda|}\operatorname{Im}\sqrt{\lambda}},$$

and analogously for $y < 0$,

$$\int_{-\infty}^0 |D_\lambda(x,y)|\,\mathrm{d}x = \frac{1}{2\sqrt{|\lambda|}} \int_{-\infty}^0 e^{-\operatorname{Re}\sqrt{\lambda}|x-y|}\,\mathrm{d}x = \frac{2 - e^{\operatorname{Re}\sqrt{\lambda} y}}{2\sqrt{|\lambda|}\operatorname{Re}\sqrt{\lambda}} \leq \frac{1}{\sqrt{|\lambda|}\operatorname{Re}\sqrt{\lambda}}.$$

Hence,

$$c := \sup_{y\in\mathbb{R}} \int_{\mathbb{R}} |K_\lambda(x,y)|\,\mathrm{d}x < \infty,$$

and Fubini's theorem yields

$$\|T_\lambda g\|_{L^1} \leq \int_{\mathbb{R}} |g(y)| \int_{\mathbb{R}} |K_\lambda(x,y)|\,\mathrm{d}x\,\mathrm{d}y \leq c\|g\|_{L^1}.$$

Therefore T_λ in (2.1) is an everywhere defined bounded operator in $L^1(\mathbb{R})$.

We claim that T_λ is the inverse of $B_0 - \lambda$. In fact, consider the functions u, v given by

$$u(x) = \begin{cases} e^{i\sqrt{\lambda}x}, & x \geq 0, \\ \overline{\alpha} e^{\sqrt{\lambda}x} + \alpha e^{-\sqrt{\lambda}x}, & x < 0, \end{cases} \quad \text{and} \quad v(x) = \begin{cases} \alpha e^{i\sqrt{\lambda}x} + \overline{\alpha} e^{-i\sqrt{\lambda}x}, & x \geq 0, \\ e^{\sqrt{\lambda}x}, & x < 0, \end{cases}$$

which solve the differential equation $\operatorname{sgn}(\cdot)(-f'') = \lambda f$; that is, u and v, and their derivatives, belong to $AC(\mathbb{R})$ and satisfy the differential equation almost everywhere. Since the Wronskian equals $2\alpha\sqrt{\lambda}$, these solutions are linearly independent. Note that $u, v \notin L^1(\mathbb{R})$, and one concludes that $B_0 - \lambda$ is injective. A simple calculation shows the identity

$$K_\lambda(x,y) = C_\lambda(x,y) + D_\lambda(x,y) = \frac{1}{2\alpha\sqrt{\lambda}} \begin{cases} u(x)v(y)\operatorname{sgn}(y), & y < x, \\ v(x)u(y)\operatorname{sgn}(y), & x < y, \end{cases}$$

and hence we have

$$(T_\lambda g)(x) = \frac{1}{2\alpha\sqrt{\lambda}} \left(u(x) \int_{-\infty}^{x} v(y)\operatorname{sgn}(y) g(y)\,\mathrm{d}y + v(x) \int_{x}^{\infty} u(y)\operatorname{sgn}(y) g(y)\,\mathrm{d}y \right).$$

One verifies that $T_\lambda g, (T_\lambda g)' \in AC(\mathbb{R})$ and $T_\lambda g$ is a solution of $\operatorname{sgn}(\cdot)(-f'') - \lambda f = g$. This implies that $(T_\lambda g)'' \in L^1(\mathbb{R})$, and hence $T_\lambda g \in \operatorname{dom} B_0$ satisfies

$$(B_0 - \lambda) T_\lambda g = g \quad \text{for all } g \in L^1(\mathbb{R}).$$

Therefore, $B_0 - \lambda$ is surjective, and we have $T_\lambda = (B_0 - \lambda)^{-1}$. It follows that B_0 is a closed operator in $L^1(\mathbb{R})$ and that λ belongs to the resolvent set of B_0. □

Proof of Theorem 1.2. Since the non-real point spectrum of A is symmetric with respect to the real line (see Theorem 1.1) it suffices to consider eigenvalues in the upper half-plane. Let $\lambda \in \mathbb{C}^+$ be an eigenvalue of A with a corresponding eigenfunction $f \in \operatorname{dom} A$. Since $q \in L^1(\mathbb{R})$ and $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q$ is in the limit point case at $\pm\infty$ (see, e.g., [23, Lemma 9.37]) the function f is unique up to a constant multiple. As $-f'' + qf = \lambda f$ on $\mathbb{R}^+$ and $f'' - qf = \lambda f$ on $\mathbb{R}^-$ with q integrable, one has the well-known asymptotical behaviour

$$\begin{aligned} f(x) &= \alpha_+ \big(1 + o(1)\big) e^{i\sqrt{\lambda}x}, \quad x \to +\infty, \\ f'(x) &= \alpha_+ i\sqrt{\lambda} \big(1 + o(1)\big) e^{i\sqrt{\lambda}x}, \quad x \to +\infty, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} f(x) &= \alpha_- \big(1 + o(1)\big) e^{\sqrt{\lambda}x}, \quad x \to -\infty, \\ f'(x) &= \alpha_- \sqrt{\lambda} \big(1 + o(1)\big) e^{\sqrt{\lambda}x}, \quad x \to -\infty, \end{aligned} \tag{2.3}$$

for some $\alpha_+, \alpha_- \in \mathbb{C}$; see, e.g., [20, §24.2, Example a] or [23, Lemma 9.37]. These asymptotics yield $f, qf \in L^1(\mathbb{R})$, and $-f'' = \lambda \operatorname{sgn}(\cdot) f - qf \in L^1(\mathbb{R})$, and therefore $f \in \operatorname{dom} B_0$. Thus, f satisfies

$$0 = (A - \lambda) f = \operatorname{sgn}(\cdot)(-f'') - \lambda f + \operatorname{sgn}(\cdot) qf = (B_0 - \lambda) f + \operatorname{sgn}(\cdot) qf,$$

and since λ is in the resolvent set of B_0 we obtain

$$-qf = q(B_0 - \lambda)^{-1} \operatorname{sgn}(\cdot) qf.$$

Note that $\|qf\|_{L^1} \neq 0$ as otherwise λ would be an eigenvalue of B_0. With the help of Lemma 2.1 we then conclude that

$$0 < \|qf\|_{L^1} \leq \int_{\mathbb{R}} |q(x)| \int_{\mathbb{R}} |K_\lambda(x,y)| |q(y) f(y)|\,\mathrm{d}y\,\mathrm{d}x \leq \frac{1}{\sqrt{|\lambda|}} \|qf\|_{L^1} \|q\|_{L^1},$$

and this yields the desired bound (1.1). □

3. Proof of Theorem 1.3

In this section we prove the bounds in (1.2) for the non-real eigenvalues of A in Theorem 1.3, which depend only on the negative part $q_-(x) = \max\{0, -q(x)\}$, $x \in \mathbb{R}$, of the potential. The following lemma will be useful.

Lemma 3.1. *Let $\lambda \in \mathbb{C}^+$ be an eigenvalue of A and let f be a corresponding eigenfunction. Define*

$$U(x) := \int_x^\infty \operatorname{sgn}(t)|f(t)|^2\,\mathrm{d}t \quad \textit{and} \quad V(x) := \int_x^\infty |f'(t)|^2 + q(t)|f(t)|^2\,\mathrm{d}t$$

for $x \in \mathbb{R}$. Then the following assertions hold:

(a) $\lambda U(x) = f'(x)\overline{f(x)} + V(x)$;
(b) $\lim_{x\to-\infty} U(x) = 0$ *and* $\lim_{x\to-\infty} V(x) = 0$;
(c) $\|f'\|_{L^2} \le 2\|q_-\|_{L^1}\|f\|_{L^2}$;
(d) $\|f\|_\infty \le 2\sqrt{\|q_-\|_{L^1}}\|f\|_{L^2}$;
(e) $\|qf^2\|_{L^1} \le 8\|q_-\|_{L^1}^2\|f\|_{L^2}^2$.

Proof. Note that f satisfies the asymptotics (2.2)–(2.3) and hence f and f' vanish at $\pm\infty$ and $f' \in L^2(\mathbb{R})$. In particular, $V(x)$ is well defined. We multiply the identity $\lambda f(t) = \operatorname{sgn}(t)(-f''(t) + q(t)f(t))$ by $\operatorname{sgn}(t)\overline{f(t)}$, and integration by parts yields

$$\lambda U(x) = \int_x^\infty -f''(t)\overline{f(t)} + q(t)|f(t)|^2\,\mathrm{d}t = f'(x)\overline{f(x)} + V(x)$$

for all $x \in \mathbb{R}$. This shows (a). Moreover, we have

$$\lambda \int_{\mathbb{R}} \operatorname{sgn}(t)|f(t)|^2\,\mathrm{d}t = \lim_{x\to-\infty} \lambda U(x) = \lim_{x\to-\infty} V(x) = \int_{\mathbb{R}} |f'(t)|^2 + q(t)|f(t)|^2\,\mathrm{d}t.$$

Taking the imaginary part shows that $\lim_{x\to-\infty} U(x) = 0$ and, hence, $\lim_{x\to-\infty} V(x) = 0$. This proves (b).

As f is continuous and vanishes at $\pm\infty$ we have $\|f\|_\infty < \infty$. Let $q_+(x) := \max\{0, q(x)\}$, $x \in \mathbb{R}$. Making use of $\lim_{x\to-\infty} V(x) = 0$ and $q = q_+ - q_-$ we find that

$$\begin{aligned} 0 \le \|f'\|_{L^2}^2 &= -\int_{\mathbb{R}} q(t)|f(t)|^2\,\mathrm{d}t = -\int_{\mathbb{R}} \big(q_+(t) - q_-(t)\big)|f(t)|^2\,\mathrm{d}t \\ &\le \int_{\mathbb{R}} q_-(t)|f(t)|^2\,\mathrm{d}t \le \|q_-\|_{L^1}\|f\|_\infty^2. \end{aligned} \tag{3.1}$$

This implies that $\|q_+f^2\|_{L^1} \le \|q_-f^2\|_{L^1} \le \|q_-\|_{L^1}\|f\|_\infty^2$ and, thus,

$$\|qf^2\|_{L^1} = \int_{\mathbb{R}} |q(t)||f(t)|^2\,\mathrm{d}t = \int_{\mathbb{R}} \big(q_+(t) + q_-(t)\big)|f(t)|^2\,\mathrm{d}t \le 2\|q_-\|_{L^1}\|f\|_\infty^2. \tag{3.2}$$

In order to verify (d) let $x, y \in \mathbb{R}$ with $x > y$. Then

$$|f(x)|^2 - |f(y)|^2 = \int_y^x \big(|f|^2\big)'(t)\,\mathrm{d}t \le 2\int_y^x |f(t)f'(t)|\,\mathrm{d}t \le 2\|f\|_{L^2}\|f'\|_{L^2},$$

together with $f(y) \to 0$, $y \to -\infty$, leads to $\|f\|_\infty^2 \le 2\|f\|_{L^2}\|f'\|_{L^2}$. Since f is an eigenfunction $\|f\|_\infty$ does not vanish and we have with (3.1)

$$\|f\|_\infty \le \frac{2\|f\|_{L^2}\|f'\|_{L^2}}{\|f\|_\infty} \le 2\sqrt{\|q_-\|_{L^1}}\|f\|_{L^2},$$

which shows (d). Moreover, the estimate in (d) applied to (3.1) and (3.2) yields (c) and (e). □

Proof of Theorem 1.3. Let $\lambda \in \mathbb{C}^+$ be an eigenvalue of A and let $f \in \operatorname{dom} A$ be a corresponding eigenfunction. We can assume $\|q_-\|_{L^1} > 0$, as otherwise $f = 0$ by Lemma 3.1(d). Let U and V be as in Lemma 3.1, let $\delta := (24\|q_-\|_{L^1})^{-1}$, and define the function g on $\mathbb{R}$ by

$$g(x) = \begin{cases} \operatorname{sgn}(x), & |x| > \delta, \\ \frac{x}{\delta}, & |x| \le \delta. \end{cases}$$

From Lemma 3.1(a) we have

$$\lambda \int_{\mathbb{R}} g'(x)U(x)\,\mathrm{d}x = \int_{\mathbb{R}} g'(x)\big(f'(x)\overline{f(x)} + V(x)\big)\,\mathrm{d}x. \tag{3.3}$$

Since g is bounded and $U(x)$ vanishes for $x \to \pm\infty$, integration by parts leads to the estimate

$$\begin{aligned} \int_{\mathbb{R}} g'(x)U(x)\,\mathrm{d}x &= \int_{\mathbb{R}} g(x)\operatorname{sgn}(x)|f(x)|^2\,\mathrm{d}x \ge \int_{\mathbb{R}\setminus[-\delta,\delta]} |f(x)|^2\,\mathrm{d}x \\ &= \|f\|_{L^2}^2 - \int_{-\delta}^{\delta} |f(x)|^2\,\mathrm{d}x \ge \|f\|_{L^2}^2 - 2\delta\|f\|_\infty^2 \\ &\ge \|f\|_{L^2}^2 - 8\delta\|q_-\|_{L^1}\|f\|_{L^2}^2 = \frac{2}{3}\|f\|_{L^2}^2; \end{aligned} \tag{3.4}$$

here we have used Lemma 3.1(d) in the last line of (3.4). Further we see with Lemma 3.1(c)–(d) that

$$\begin{aligned} \left|\int_{\mathbb{R}} g'(x)f'(x)\overline{f(x)}\,\mathrm{d}x\right| &\le \|f\|_\infty \|f'\|_{L^2}\|g'\|_{L^2} \le 4\|q_-\|_{L^1}^{\frac{3}{2}}\|f\|_{L^2}^2\sqrt{\frac{2}{\delta}} \\ &\le 16\cdot\sqrt{3}\|q_-\|_{L^1}^2\|f\|_{L^2}^2. \end{aligned} \tag{3.5}$$

Since $\|g\|_\infty = 1$ and $V(x)$ vanishes for $x \to \pm\infty$, integration by parts together with Lemma 3.1(c) and (e) yields

$$\begin{aligned} \left|\int_{\mathbb{R}} g'(x)V(x)\,\mathrm{d}x\right| &= \left|\int_{\mathbb{R}} g(x)\left(|f'(x)|^2 + q(x)|f(x)|^2\right)\,\mathrm{d}x\right| \\ &\le \|g\|_\infty\left(\|f'\|_{L^2}^2 + \|qf^2\|_{L^1}\right) \le 12\|q_-\|_{L^1}^2\|f\|_{L^2}^2. \end{aligned} \tag{3.6}$$

Comparing the imaginary parts in (3.3) we have with (3.4) and (3.5)

$$\begin{aligned} \frac{2}{3}|\operatorname{Im}\lambda|\|f\|_{L^2}^2 \le& |\operatorname{Im}\lambda|\left|\int_{\mathbb{R}} g'(x)U(x)\,\mathrm{d}x\right| \le \left|\int_{\mathbb{R}} g'(x)f'(x)\overline{f(x)}\,\mathrm{d}x\right| \\ \le& 16\cdot\sqrt{3}\|q_-\|_{L^1}^2\|f\|_{L^2}^2. \end{aligned}$$

In the same way we obtain from (3.4), (3.3), and (3.5)–(3.6) that

$$\begin{aligned} \frac{2}{3}|\lambda|\|f\|_{L^2}^2 &\le \left|\lambda\int_{\mathbb{R}} g'(x)U(x)\,\mathrm{d}x\right| = \left|\int_{\mathbb{R}} g'(x)\big(f'(x)\overline{f(x)} + V(x)\big)\,\mathrm{d}x\right| \\ &\le \left(16\cdot\sqrt{3} + 12\right)\|q_-\|_{L^1}^2\|f\|_{L^2}^2. \end{aligned}$$

This shows the bounds in (1.2). □

References

[1] Aleksandr A. Abramov, Anna Aslanyan, and E. Brian Davies, *Bounds on complex eigenvalues and resonances*, J. Phys. A **34** (2001), no. 1, 57–72. MR1819914

[2] Jussi Behrndt, Shaozhu Chen, Friedrich Philipp, and Jiangang Qi, *Estimates on the non-real eigenvalues of regular indefinite Sturm-Liouville problems*, Proc. Roy. Soc. Edinburgh Sect. A **144** (2014), no. 6, 1113–1126. MR3283062

[3] Jussi Behrndt, Qutaibeh Katatbeh, and Carsten Trunk, *Non-real eigenvalues of singular indefinite Sturm-Liouville operators*, Proc. Amer. Math. Soc. **137** (2009), no. 11, 3797–3806. MR2529889

[4] Jussi Behrndt and Friedrich Philipp, *Spectral analysis of singular ordinary differential operators with indefinite weights*, J. Differential Equations **248** (2010), no. 8, 2015–2037. MR2595713

[5] Jussi Behrndt, Friedrich Philipp, and Carsten Trunk, *Bounds on the non-real spectrum of differential operators with indefinite weights*, Math. Ann. **357** (2013), no. 1, 185–213. MR3084346

[6] Jussi Behrndt, Philipp Schmitz, and Carsten Trunk, *Bounds on the non-real spectrum of a singular indefinite Sturm-Liouville operator on* $\mathbb{R}$, Proc. Appl. Math. Mech. **16** (2016), 881–882.

[7] Jussi Behrndt, Bernhard Gsell, Philipp Schmitz, and Carsten Trunk, *An estimate on the non-real spectrum of a singular indefinite Sturm-Liouville operator*, Proc. Appl. Math. Mech. **17** (2017), 859–860.

[8] B. Malcolm Brown and Michael S. P. Eastham, *Analytic continuation and resonance-free regions for Sturm-Liouville potentials with power decay*, J. Comput. Appl. Math. **148** (2002), no. 1, 49–63. On the occasion of the 65th birthday of Professor Michael Eastham. MR1946186

[9] Jiangang Qi and Shaozhu Chen, *A priori bounds and existence of non-real eigenvalues of indefinite Sturm-Liouville problems*, J. Spectr. Theory **4** (2014), no. 1, 53–63. MR3181385

[10] Jiangang Qi, Bing Xie, and Shaozhu Chen, *The upper and lower bounds on non-real eigenvalues of indefinite Sturm-Liouville problems*, Proc. Amer. Math. Soc. **144** (2016), no. 2, 547–559. MR3430833

[11] Branko Ćurgus and Heinz Langer, *A Kreĭn space approach to symmetric ordinary differential operators with an indefinite weight function*, J. Differential Equations **79** (1989), no. 1, 31–61. MR997608

[12] E. Brian Davies, *Linear operators and their spectra*, Cambridge Studies in Advanced Mathematics, vol. 106, Cambridge University Press, Cambridge, 2007. MR2359869

[13] E. Brian Davies and Jiban Nath, *Schrödinger operators with slowly decaying potentials*, on the occasion of the 65th birthday of Professor Michael Eastham, J. Comput. Appl. Math. **148** (2002), no. 1, 1–28. MR1946184

[14] Bing Xie, Huaqing Sun, and Xinwei Guo, *Non-real eigenvalues of symmetric Sturm-Liouville problems with indefinite weight functions*, Electron. J. Qual. Theory Differ. Equ. (2017), Paper No. 18, 14pp. MR3633248

[15] Otto Haupt, *Über eine Methode zum Beweise von Oszillationstheoremen* (German), Math. Ann. **76** (1914), no. 1, 67–104. MR1511814

[16] Illya Karabash and Carsten Trunk, *Spectral properties of singular Sturm-Liouville operators with indefinite weight* $\operatorname{sgn} x$, Proc. Roy. Soc. Edinburgh Sect. A **139** (2009), no. 3, 483–503. MR2506783

[17] Mervis Kikonko and Angelo B. Mingarelli, *Bounds on real and imaginary parts of non-real eigenvalues of a non-definite Sturm-Liouville problem*, J. Differential Equations **261** (2016), no. 11, 6221–6232. MR3552563

[18] Angelo B. Mingarelli, *A survey of the regular weighted Sturm-Liouville problem—the non-definite case*, International workshop on applied differential equations (Beijing, 1985), World Sci. Publishing, Singapore, 1986, pp. 109–137. MR901329

[19] Michael Levitin and Marcello Seri, *Accumulation of complex eigenvalues of an indefinite Sturm-Liouville operator with a shifted Coulomb potential*, Oper. Matrices **10** (2016), no. 1, 223–245. MR3460063

[20] Mark A. Naimark, *Linear differential operators. Part I: Elementary theory of linear differential operators*, Frederick Ungar Publishing Co., New York, 1967. MR0216050

[21] Bing Xie and Jiangang Qi, *Non-real eigenvalues of indefinite Sturm-Liouville problems*, J. Differential Equations **255** (2013), no. 8, 2291–2301. MR3082462
[22] Roland G. D. Richardson, *Contributions to the study of oscillation properties of the solutions of linear differential equations of the second order*, Amer. J. Math. **40** (1918), no. 3, 283–316. MR1506360
[23] Gerald Teschl, *Mathematical methods in quantum mechanics*: *With applications to Schrödinger operators*, Graduate Studies in Mathematics, vol. 99, American Mathematical Society, Providence, RI, 2009. MR2499016
[24] Joachim Weidmann, *Spectral theory of ordinary differential operators*, Lecture Notes in Mathematics, vol. 1258, Springer-Verlag, Berlin, 1987. MR923320
[25] Joachim Weidmann, *Lineare Operatoren in Hilberträumen. Teil II*: *Anwendungen. [Applications]* (German), Mathematische Leitfäden. [Mathematical Textbooks], B. G. Teubner, Stuttgart, 2003. MR2382320
[26] Anton Zettl, *Sturm-Liouville theory*, Mathematical Surveys and Monographs, vol. 121, American Mathematical Society, Providence, RI, 2005. MR2170950

Institut für Angewandte Mathematik, Technische Universität Graz, Steyrergasse 30, A-8010 Graz, Austria
Email address: `behrndt@tugraz.at`

Institut für Mathematik, Technische Universität Ilmenau, Postfach 100565, D-98684 Ilmenau, Germany
Email address: `philipp.schmitz@tu-ilmenau.de`

Institut für Mathematik, Technische Universität Ilmenau, Postfach 100565, D-98684 Ilmenau, Germany – and – Instituto Argentino de Matemática “Alberto P. Calderón” (CONICET), Saavedra 15, (1083) Buenos Aires, Argentina
Email address: `carsten.trunk@tu-ilmenau.de`