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FINITE NUMBERS OF INITIAL IDEALS IN NON-NOETHERIAN POLYNOMIAL RINGS

FELICITAS LINDNER

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Abstract. In this article, we generalize the well-known result that ideals of Noetherian polynomial rings have only finitely many initial ideals to the situation of ideals in the polynomial ring $R = \mathbb{K}[x_{i,j} \mid 1 \leq i \leq c, j \in \mathbb{N}]$ that are invariant under the action of the monoid $\mathrm{Inc}(\mathbb{N})$ of strictly increasing functions on $\mathbb{N}$. This monoid acts on R by shifting the second variable index. We show that for every $\mathrm{Inc}(\mathbb{N})$-invariant ideal, the number of initial ideals with respect to term orders that are compatible with this $\mathrm{Inc}(\mathbb{N})$-action is finite. The article also addresses the question of how many such term orders exist. We give a complete list of the $\mathrm{Inc}(\mathbb{N})$-compatible term orders for the case $c = 1$ and show that there are infinitely many for $c > 1$.

1. Introduction

It has long been known that for ideals of polynomial rings in finitely many variables, the number of initial ideals with respect to arbitrary term orders is finite (e.g. [MR], Lemma 2.6). As this result relies on the Noetherianity of such polynomial rings, it cannot be transferred to ideals of polynomial rings in infinitely many variables in general. However, more recent results show that for certain non-Noetherian polynomial rings, there are classes of ideals satisfying a weaker kind of Noetherianity, namely Noetherianity up to the action of certain monoids. Thus, it seems worthwhile to try to generalize the result on finiteness of numbers of initial ideals in the Noetherian case to this class of ideals.

Let $R := \mathbb{K}[x_{i,j} \mid i \in [c], j \in \mathbb{N}]$ be the polynomial ring over an arbitrary field $\mathbb{K}$ in the variables indexed by $[c] \times \mathbb{N}$, where $\mathbb{N} := \{1, 2, 3, ...\}$ denotes the set of natural numbers, $c \in \mathbb{N}$ is any fixed number, and $[c] := \{1, ..., c\}$. On R, we can define an action of the monoid

$$\mathrm{Inc}(\mathbb{N}) := \{p : \mathbb{N} \to \mathbb{N} \mid p(n) < p(n+1) \text{ for all } n \in \mathbb{N}\}$$

of strictly increasing functions on $\mathbb{N}$ by a $\mathbb{K}$-linear extension of the map

$$x_{i_1,j_1}^{e_1} \cdot ... \cdot x_{i_r,j_r}^{e_r} \mapsto p \cdot (x_{i_1,j_1}^{e_1} \cdot ... \cdot x_{i_r,j_r}^{e_r}) := x_{i_1,p(j_1)}^{e_1} \cdot ... \cdot x_{i_r,p(j_r)}^{e_r}$$

for every $p \in \mathrm{Inc}(\mathbb{N})$. We call an ideal J of R $\mathrm{Inc}(\mathbb{N})$-invariant if $\mathrm{Inc}(\mathbb{N}) \cdot J = J$.

Let $\prec$ be a term order on R, i.e., a total order on the monomials of R respecting multiplication and satisfying $1 \preceq f$ for every monomial f. Note that we do not require $\prec$ to be a well-order. If $\prec$ has the additional property that

$$f \prec g \Rightarrow p \cdot f \prec p \cdot g$$

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for all monomials $f, g \in R$ and every $p \in \operatorname{Inc}(\mathbb{N})$, then we call $\prec$ an $\operatorname{Inc}(\mathbb{N})$-compatible term order on R. In [HS] the authors showed that for any $\operatorname{Inc}(\mathbb{N})$-invariant ideal $J \subseteq R$ and any $\operatorname{Inc}(\mathbb{N})$-compatible term order $\prec$, the ideal J has a finite $\operatorname{Inc}(\mathbb{N})$-Gröbner basis with respect to $\prec$; i.e., there is a finite subset $\mathcal{G} \subseteq J$ such that

$$\operatorname{in}_\prec(J) = \langle \operatorname{Inc}(\mathbb{N}) \cdot \operatorname{in}_\prec(g) \mid g \in \mathcal{G} \rangle_R.$$

This can be seen as a generalization of the statement that in Noetherian polynomial rings, every ideal has finite Gröbner bases with respect to arbitrary term orders. For Noetherian polynomial rings it is also known that for any ideal J, the number of initial ideals of J with respect to arbitrary term orders is finite (e.g. [MR], Lemma 2.6). In this article, we will generalize this result to the case of $\operatorname{Inc}(\mathbb{N})$-invariant ideals of the polynomial ring R in the following sense:

Theorem 1.1. *Let J be an $\operatorname{Inc}(\mathbb{N})$-invariant ideal of R. Then J has only finitely many initial ideals with respect to $\operatorname{Inc}(\mathbb{N})$-compatible term orders on R.*

Theorem 1.1 is a straightforward consequence of certain finiteness results for $\operatorname{Inc}(\mathbb{N})$-invariant ideal chains in R, which are also part of this article. Let $R_n := \mathbb{K}[x_{i,j} \mid i \in [c], j \in [n]]$ and

$$\operatorname{Inc}(\mathbb{N})_{m,n} := \{p \in \operatorname{Inc}(\mathbb{N}) \mid p(m) \leq n\}$$

for each pair of natural numbers $m \leq n$. For every n, let J_n be an ideal of R_n. We call the sequence $J_\circ = J_1 \subseteq J_2 \subseteq ...$ an $\operatorname{Inc}(\mathbb{N})$-invariant ideal chain in R if for every $m \leq n$, we have

$$\operatorname{Inc}(\mathbb{N})_{m,n} \cdot J_m \subseteq J_n.$$

In [HS] it was shown that every $\operatorname{Inc}(\mathbb{N})$-invariant ideal chain $J_\circ$ in R stabilizes up to the action of $\operatorname{Inc}(\mathbb{N})$; i.e., there is an index $N \in \mathbb{N}$ satisfying

$$\langle \operatorname{Inc}(\mathbb{N})_{N,n} \cdot J_N \rangle_{R_n} = J_n \tag{1.1}$$

for every $n \geq N$. We call the minimal N satisfying (1.1) the stability index of $J_\circ$ and denote it by $\operatorname{Ind}(J_\circ)$.

If $J_\circ$ is an $\operatorname{Inc}(\mathbb{N})$-invariant ideal chain and $\prec$ is an $\operatorname{Inc}(\mathbb{N})$-compatible term order on R, then the chain of initial ideals

$$\operatorname{in}_\prec(J_\circ) := \operatorname{in}_\prec(J_1) \subseteq \operatorname{in}_\prec(J_2) \subseteq ...$$

is $\operatorname{Inc}(\mathbb{N})$-invariant, too, and therefore stabilizes. Thus, we can define the set

$$\mathrm{I}(J_\circ) := \{\operatorname{Ind}(\operatorname{in}_\prec(J_\circ)) \mid \prec \text{ is an } \operatorname{Inc}(\mathbb{N})\text{-compatible term order on } R\}$$

of stability indices of initial ideal chains of $J_\circ$ with respect to $\operatorname{Inc}(\mathbb{N})$-compatible term orders. In this article, we will prove the following statement:

Theorem 1.2. *For every $\operatorname{Inc}(\mathbb{N})$-invariant ideal chain $J_\circ$ in R, the set $\mathrm{I}(J_\circ)$ is bounded above (and, thus, finite).*

Note that the global stability index $\operatorname{Ind}(J_\circ)$ of an $\operatorname{Inc}(\mathbb{N})$-invariant ideal chain can be smaller than $\max(\mathrm{I}(J_\circ))$. For instance, let $c = 1$, $J_1 = J_2 = J_3 = \{0\}$, $J_4 = \langle x_1 + x_3 \rangle_{R_4}$, and $J_n = \langle \operatorname{Inc}(\mathbb{N})_{4,n} \cdot J_4 \rangle_{R_n}$ for $n \geq 5$. Let $\prec$ be any $\operatorname{Inc}(\mathbb{N})$-compatible term order satisfying $x_n \prec x_{n+1}$ for all $n \in \mathbb{N}$. As $(x_2+x_4)-(x_1+x_4) = x_2 - x_1$ lies in J_5, we conclude that $x_2 \in \operatorname{in}_\prec(J_5)$. On the other hand, we have $\operatorname{in}_\prec(J_4) = \langle x_3 \rangle_{R_4}$. Thus, x_2 is not an element of $\langle \operatorname{Inc}(\mathbb{N})_{4,5} \cdot \operatorname{in}_\prec(J_4) \rangle_{R_5}$, so $\operatorname{Ind}(\operatorname{in}_\prec(J_\circ)) > 4 = \operatorname{Ind}(J_\circ)$.

Hence, the seemingly obvious idea to prove Theorem 1.2 by showing that $\mathrm{I}(J_\circ)$ is bounded by $\mathrm{Ind}(J_\circ)$ must fail.

We will use Theorem 1.2 to prove that (a) the number of initial ideal chains of $\mathrm{Inc}(\mathbb{N})$-invariant ideal chains in R with respect to $\mathrm{Inc}(\mathbb{N})$-compatible term orders is finite (Corollary 3.1), and (b) the number of initial ideals of $\mathrm{Inc}(\mathbb{N})$-invariant ideals of R with respect to $\mathrm{Inc}(\mathbb{N})$-compatible term orders is finite (Theorem 1.1). Also, Theorem 1.2 will allow us to generalize the notion of universal Gröbner bases to the setting of $\mathrm{Inc}(\mathbb{N})$-invariant ideals of R and prove their existence (Corollary 3.7).

Of course, all of these statements would be trivial if there were only finitely many $\mathrm{Inc}(\mathbb{N})$-compatible term orders on R. However, for $c \geq 2$ it is easy to show that the number of $\mathrm{Inc}(\mathbb{N})$-compatible term orders is infinite: Choose any term order $\prec'$ on the polynomial ring $\mathbb{K}[y_1, ..., y_c]$. For a tuple $\mathbf{a} = (a_1, ..., a_c) \in \mathbb{N}_0^c$, write $\mathbf{x_i^a} := x_{1,i}^{a_1} \cdots x_{c,i}^{a_c}$, $\mathbf{y^a} := y_1^{a_1} \cdots y_c^{a_c}$. Then define the relation $\prec$ by

$$\mathbf{x_1^{a_1}} \cdots \mathbf{x_n^{a_n}} \prec \mathbf{x_1^{b_1}} \cdots \mathbf{x_n^{b_n}} :\Leftrightarrow \mathbf{y^{a_i}} \prec' \mathbf{y^{b_i}} \text{ for } i = \min\{j \mid \mathbf{a_j} \neq \mathbf{b_j}\}.$$

This is obviously an $\mathrm{Inc}(\mathbb{N})$-compatible term order on R, and if we choose two distinct term orders $\prec_1', \prec_2'$ of $\mathbb{K}[y_1, ..., y_c]$, then the corresponding term orders $\prec_1, \prec_2$ on R are distinct, too. As for $c \geq 2$, there are uncountably many distinct term orders on $\mathbb{K}[y_1, ..., y_c]$. There are also uncountably many $\mathrm{Inc}(\mathbb{N})$-compatible term orders on R.

As we will see in Section 3, this is not true for $c = 1$. In this case, the only $\mathrm{Inc}(\mathbb{N})$-compatible term orders on R are the lexicographic, the degree-lexicographic, and the degree-reverse-lexicographic term orders with respect to the $\mathrm{Inc}(\mathbb{N})$-compatible variable orderings $x_1 \prec x_2 \prec \cdots$ or $x_1 \succ x_2 \succ \cdots$ (Corollary 4.5). This answers Question 5.5 in [HKL] in the negative.

The article is organized as follows: We begin with some technical preparations in Section 2 needed for the proof of Theorem 1.2, which will be given in Section 3. In Section 3, we will also discuss the consequences of Theorem 1.2 described above. In Section 4 we will then study the question of what the $\mathrm{Inc}(\mathbb{N})$-compatible term orders are in the case $c = 1$.

2. Technical preparations

Here and in the section that follows, the number c from the definitions of R and R_n is an arbitrary natural number. We start this section with some observations concerning the monoid $\mathrm{Inc}(\mathbb{N})$ and its action on R.

Lemma 2.1 (cf. [NR], Proposition 4.6)**.** *Let $l \leq m \leq n$ be natural numbers. Then*

$$\mathrm{Inc}(\mathbb{N})_{l,n} = \mathrm{Inc}(\mathbb{N})_{m,n} \circ \mathrm{Inc}(\mathbb{N})_{l,m},$$

meaning that for every $p_1 \in \mathrm{Inc}(\mathbb{N})_{l,m}$, $p_2 \in \mathrm{Inc}(\mathbb{N})_{m,n}$ we have $p_2 \circ p_1 \in \mathrm{Inc}(\mathbb{N})_{l,n}$, and every element $p \in \mathrm{Inc}(\mathbb{N})_{l,n}$ has such a decomposition.

The somewhat technical proof of the following lemma is left to the reader.

Lemma 2.2. *Let $N, l \in \mathbb{N}$, $n \geq N$, and $i_1 < \cdots < i_l \leq N$, $j_1 < \cdots < j_l \leq n$ be two ascending sequences of natural numbers. Then there is $p \in \mathrm{Inc}(\mathbb{N})_{N,n}$ such that $p(i_r) = j_r$ for all $r \in [l]$ if and only if $j_1 \geq i_1$, $j_{r+1} - j_r \geq i_{r+1} - i_r$ for $1 \leq r \leq l-1$ and $n - j_l \geq N - i_l$.*

Lemma 2.3. *Let $m \leq n$ be natural numbers, let $f \in R_m$ be a polynomial of degree $\deg(f) > 0$, and let $p \in \mathrm{Inc}(\mathbb{N})$ with $p \cdot f \in R_n$. Let $m', n' \in \mathbb{N}$ be minimal with*

$f \in R_{m'}$ and $p \cdot f \in R_{n'}$. Then there is $p' \in \operatorname{Inc}(\mathbb{N})_{m,n}$ such that $p' \cdot f = p \cdot f$ if and only if $n - n' \geq m - m'$.

Proof. Let $i_1 < \cdots < i_l \leq m$ and $j_1 < \cdots < j_l \leq n$ be the second variable indices occurring in f and $p \cdot f$, respectively. By assumption, we have $l \geq 1$, $i_l = m'$, and $j_l = n'$. As $p(i_r) = j_r$ for all $r \in [l]$, Lemma 2.2 yields $j_1 \geq i_1$ and $j_{r+1} - j_r \geq i_{r+1} - i_r$ for $1 \leq r \leq l-1$. Therefore, again by Lemma 2.2, we can conclude that there is $p' \in \operatorname{Inc}(\mathbb{N})_{m,n}$ with $p'(i_r) = j_r$ for all $r \in [l]$ if and only if $n - n' = n - j_l \geq m - i_l = m - m'$. $\square$

Lemma 2.4. *Let $i_1 \leq i_2 \leq ...$ be an ascending sequence of natural numbers and let $g_{i_n} \in R_{i_n}$ be monomials. Then there are indices $j < k$ such that g_{i_k} is contained in $\langle \operatorname{Inc}(\mathbb{N})_{i_j,i_k} \cdot g_{i_j} \rangle_{R_{i_k}}$.*

Proof. There is nothing to show if $g_{i_n} \in \mathbb{K}$ for some $n \in \mathbb{N}$, so assume $\deg(g_{i_n}) > 0$ for all n. By Theorem 3.1 in [HS], there is an infinite subsequence $(g_{i_{n_k}})_{k \geq 1}$ of $(g_{i_n})_{n \geq 1}$ such that for each $k \in \mathbb{N}$ we have $g_{i_{n_{k+1}}} = f_k(p_k \cdot g_{i_{n_k}})$ for a monomial $f_k \in R_{i_{n_{k+1}}}$ and $p_k \in \operatorname{Inc}(\mathbb{N})$. We claim that one of the p_k can be substituted for an element from $\operatorname{Inc}(\mathbb{N})_{i_{n_k}, i_{n_{k+1}}}$. By contradiction, assume that this is not the case. For each k let $m_k \leq i_{n_k}$ be minimal with $g_{i_{n_k}} \in R_{m_k}$. By Lemma 2.3 we have $i_{n_k} - m_k > i_{n_{k+1}} - m_{k+1}$ for every $k \geq 1$. But this contradicts the fact that there are no infinite, strictly decreasing sequences of natural numbers. $\square$

We now return to our problem of stability indices of initial ideal chains with respect to $\operatorname{Inc}(\mathbb{N})$-compatible term orders. We begin with the remark that if $J_\circ$ is an $\operatorname{Inc}(\mathbb{N})$-invariant ideal chain, then every $N \geq \operatorname{Ind}(J_\circ)$ satisfies the stability condition (1.1).

Lemma 2.5. *Let $J_\circ$ be an $\operatorname{Inc}(\mathbb{N})$-invariant ideal chain in R and let $N \geq \operatorname{Ind}(J_\circ)$. Then*

$$\langle \operatorname{Inc}(\mathbb{N})_{N,n} \cdot J_N \rangle_{R_n} = J_n$$

for all $n \geq N$.

Proof. Let $N \geq \operatorname{Ind}(J_\circ)$. Then by Lemma 2.1 and the $\operatorname{Inc}(\mathbb{N})$-invariance of $J_\circ$, we have

$$\begin{aligned} J_n &= \left\langle \operatorname{Inc}(\mathbb{N})_{\operatorname{Ind}(J_\circ),n} \cdot J_{\operatorname{Ind}(J_\circ)} \right\rangle_{R_n} \\ &= \left\langle \operatorname{Inc}(\mathbb{N})_{N,n} \cdot \left(\operatorname{Inc}(\mathbb{N})_{\operatorname{Ind}(J_\circ),N} \cdot J_{\operatorname{Ind}(J_\circ)} \right) \right\rangle_{R_n} \subseteq \langle \operatorname{Inc}(\mathbb{N})_{N,n} \cdot J_N \rangle_{R_n}. \end{aligned}$$

$\square$

The key to our proof of Theorem 1.2 is the following proposition.

Proposition 2.6. *Let $J_\circ = J_1 \subseteq J_2 \subseteq ...$ be an $\operatorname{Inc}(\mathbb{N})$-invariant ideal chain in R and let $N \geq \operatorname{Ind}(J_\circ)$. Then for every $\operatorname{Inc}(\mathbb{N})$-compatible term order $\prec$, the identity $\operatorname{in}_\prec(J_{2N}) = \langle \operatorname{Inc}(\mathbb{N})_{N,2N} \cdot \operatorname{in}_\prec(J_N) \rangle_{R_{2N}}$ implies that $\operatorname{Ind}(\operatorname{in}_\prec(J_\circ)) \leq 2N$.*

Proof. Let $N \geq \operatorname{Ind}(J_\circ)$ and let $\prec$ be an $\operatorname{Inc}(\mathbb{N})$-compatible term order. Suppose that $\operatorname{in}_\prec(J_{2N}) = \langle \operatorname{Inc}(\mathbb{N})_{N,2N} \cdot \operatorname{in}_\prec(J_N) \rangle_{R_{2N}}$. To prove the proposition, it is enough to show that the corresponding identity holds for every $n \geq 2N$, as this implies that

$$\begin{aligned}
\operatorname{in}_{\prec}(J_n) &= \langle \operatorname{Inc}(\mathbb{N})_{N,n} \cdot \operatorname{in}_{\prec}(J_N)\rangle_{R_n} \\
&= \langle \operatorname{Inc}(\mathbb{N})_{2N,n} \cdot (\operatorname{Inc}(\mathbb{N})_{N,2N} \cdot \operatorname{in}_{\prec}(J_N))\rangle_{R_n} \\
&\subseteq \langle \operatorname{Inc}(\mathbb{N})_{2N,n} \cdot \operatorname{in}_{\prec}(J_{2N})\rangle_{R_n},
\end{aligned}$$

where we used Lemma 2.1 in the second line and the $\operatorname{Inc}(\mathbb{N})$-invariance of $\operatorname{in}_{\prec}(J_\circ)$ in the third line. To this end, it suffices to prove the following.

Claim. If $\mathcal{G}$ is a Gröbner basis of J_N with respect to $\prec$, then $\mathcal{G}' := \operatorname{Inc}(\mathbb{N})_{N,n} \cdot \mathcal{G}$ is a Gröbner basis of J_n with respect to $\prec$ for every $n \geq 2N$.

Clearly, $\operatorname{Inc}(\mathbb{N})_{N,2N} \cdot \mathcal{G}$ is a Gröbner basis for J_{2N}. Let $n > 2N$. As $\mathcal{G}$ generates J_N and $N \geq \operatorname{Ind}(J_\circ)$, $\mathcal{G}'$ is a generating set for J_n by Lemma 2.5. Thus, we only have to show that the S-polynomials of the elements of $\mathcal{G}'$ reduce to zero with respect to $\mathcal{G}'$. Choose $f', g' \in \mathcal{G}'$ and write $f' = p_1 \cdot f$, $g' = p_2 \cdot g$ with $p_1, p_2 \in \operatorname{Inc}(\mathbb{N})_{N,n}$ and $f, g \in \mathcal{G}$. Let $j_1 < \cdots < j_N \leq n$, $k_1 < \cdots < k_N \leq n$ be natural numbers satisfying $p_1([N]) = \{j_1, ..., j_N\}$ and $p_2([N]) = \{k_1, ..., k_N\}$ and let $i_1 < \cdots < i_{2N} \leq n$ be natural numbers such that $\{j_1, ..., j_N\} \cup \{k_1, ..., k_N\} \subseteq \{i_1, ..., i_{2N}\}$. Define the map p by

$$p(j) := \begin{cases} i_j, & j \in [2N], \\ n + j, & j > 2N. \end{cases}$$

Then p is an element of $\operatorname{Inc}(\mathbb{N})_{2N,n}$ satisfying $p_1([N]), p_2([N]) \subseteq p(\mathbb{N})$. Therefore, $p^{-1} \circ p_1 : [N] \to [2N]$ and $p^{-1} \circ p_2 : [N] \to [2N]$ are well-defined, strictly increasing maps which can easily be extended to elements $q_1, q_2 \in \operatorname{Inc}(\mathbb{N})_{N,2N}$. Due to the $\operatorname{Inc}(\mathbb{N})$-invariance of $J_\circ$, this yields $p^{-1} \cdot f' = (p^{-1} \circ p_1) \cdot f = q_1 \cdot f \in J_{2N}$ and $p^{-1} \cdot g' = (p^{-1} \circ p_2) \cdot g = q_2 \cdot g \in J_{2N}$.

Recall that for polynomials $h_1, h_2 \in R$, the S-polynomial $S(h_1, h_2)$ of h_1 and h_2 with respect to the term order $\prec$ is defined as

$$S(h_1, h_2) = \operatorname{lcm}(\operatorname{in}_{\prec}(h_1), \operatorname{in}_{\prec}(h_2)) \left(\frac{h_1}{\operatorname{lt}_{\prec}(h_1)} - \frac{h_2}{\operatorname{lt}_{\prec}(h_2)} \right),$$

where $\operatorname{lcm}(\operatorname{in}_{\prec}(h_1), \operatorname{in}_{\prec}(h_2))$ stands for the least common multiple of $\operatorname{in}_{\prec}(h_1)$ and $\operatorname{in}_{\prec}(h_2)$ and $\operatorname{lt}_{\prec}(h_i)$ denotes the leading term of h_i, i.e., the product of the leading monomial of h_i with respect to $\prec$ and its coefficient in h_i. Due to the $\operatorname{Inc}(\mathbb{N})$-compatibility of $\prec$, the S-polynomial of $p^{-1} \cdot f'$ and $p^{-1} \cdot g'$ satisfies

$$\begin{aligned}
&S(p^{-1} \cdot f', p^{-1} \cdot g') \\
&\quad = \operatorname{lcm}(\operatorname{in}_{\prec}(p^{-1} \cdot f'), \operatorname{in}_{\prec}(p^{-1} \cdot g')) \left(\frac{p^{-1} \cdot f'}{\operatorname{lt}_{\prec}(p^{-1} \cdot f')} - \frac{p^{-1} \cdot g'}{\operatorname{lt}_{\prec}(p^{-1} \cdot g')} \right) \\
&\quad = \operatorname{lcm}(p^{-1} \cdot \operatorname{in}_{\prec}(f'), p^{-1} \cdot \operatorname{in}_{\prec}(g')) \left(\frac{p^{-1} \cdot f'}{p^{-1} \cdot \operatorname{lt}_{\prec}(f')} - \frac{p^{-1} \cdot g'}{p^{-1} \cdot \operatorname{lt}_{\prec}(g')} \right) \\
&\quad = p^{-1} \cdot \left[\operatorname{lcm}(\operatorname{in}_{\prec}(f'), \operatorname{in}_{\prec}(g')) \left(\frac{f'}{\operatorname{lt}_{\prec}(f')} - \frac{g'}{\operatorname{lt}_{\prec}(g')} \right) \right] \\
&\quad = p^{-1} \cdot S(f', g'). \qquad (2.1)
\end{aligned}$$

As both $p^{-1} \cdot f'$ and $p^{-1} \cdot g'$ are contained in J_{2N}, this is also true for $S(p^{-1} \cdot f', p^{-1} \cdot g')$. Therefore, $S(p^{-1} \cdot f', p^{-1} \cdot g')$ reduces to zero with respect to $\operatorname{Inc}(\mathbb{N})_{N,2N} \cdot \mathcal{G}$;

i.e., it can be written as

$$S(p^{-1} \cdot f', p^{-1} \cdot g') = \sum_{i=1}^{r} h_i\,(q_i' \cdot g_i)$$

with $h_i \in R_{2N}$, $q_i' \in \operatorname{Inc}(\mathbb{N})_{N,2N}$, $g_i \in \mathcal{G}$, and $\operatorname{in}_\prec(S(p^{-1}\cdot f', p^{-1}\cdot g')) \succeq \operatorname{in}_\prec(h_i(q_i'\cdot g_i))$ for all $i \in [r]$. This yields

$$\begin{aligned}\operatorname{in}_\prec(p \cdot S(p^{-1} \cdot f', p^{-1} \cdot g')) &= p \cdot \operatorname{in}_\prec(S(p^{-1} \cdot f', p^{-1} \cdot g')) \\ &\succeq p \cdot \operatorname{in}_\prec(h_i(q_i' \cdot g_i)) = \operatorname{in}_\prec((p \cdot h_i)((p \circ q_i') \cdot g_i))\end{aligned}$$

for all $i \in [r]$. As by equation (2.1) we have

$$S(f', g') = p \cdot S(p^{-1} \cdot f', p^{-1} \cdot g') = \sum_{i=1}^{r} (p \cdot h_i)((p \circ q_i') \cdot g_i)$$

and $p \circ q_i' \in \operatorname{Inc}(\mathbb{N})_{N,n}$ by Lemma 2.1, we conclude that $S(f', g')$ reduces to zero with respect to $\mathcal{G}'$. $\square$

3. Proof of Theorem 1.2 and implications

Proof of Theorem 1.2. By contradiction, assume the existence of a sequence $(\prec_n)_{n\geq 1}$ of $\operatorname{Inc}(\mathbb{N})$-compatible term orders on R with $\lim_{n\to\infty} \operatorname{Ind}(\operatorname{in}_{\prec_n}(J_\circ)) = \infty$. Set $N_0 := \operatorname{Ind}(J_\circ)$ and $N_i := 2N_{i-1}$ for $i \geq 1$. We claim that there is a collection $(\prec_n^i)_{n\geq 1}$ of infinite subsequences of $(\prec_n)_{n\geq 1}$, where i ranges over $\mathbb{N}_0$, such that

(1) $(\prec_n^i)_{n\geq 1}$ is a subsequence of $(\prec_n^{i-1})_{n\geq 1}$ for all $i \geq 1$;
(2) $\operatorname{in}_{\prec_n^i}(J_{N_i}) \supsetneq \langle \operatorname{Inc}(\mathbb{N})_{N_{i-1},N_i} \cdot \operatorname{in}_{\prec_n^i}(J_{N_{i-1}})\rangle_{R_{N_i}}$ for all $i, n \geq 1$;
(3) $\operatorname{in}_{\prec_n^i}(J_{N_i}) = \operatorname{in}_{\prec_1^i}(J_{N_i})$ for all $i, n \geq 1$.

Indeed, we can construct these subsequences as follows: Set $(\prec_n^0)_{n\geq 1} := (\prec_n)_{n\geq 1}$. By induction, assume that the subsequence $(\prec_n^i)_{n\geq 1}$ has already been defined for some $i \geq 0$. Then $\lim_{n\to\infty} \operatorname{Ind}(\operatorname{in}_{\prec_n^i}(J_\circ)) = \infty$, so in particular, there are infinitely many indices n such that $\operatorname{Ind}(\operatorname{in}_{\prec_n^i}(J_\circ)) > N_{i+1}$. By Proposition 2.6, these indices satisfy $\operatorname{in}_{\prec_n^i}(J_{N_{i+1}}) \supsetneq \langle \operatorname{Inc}(\mathbb{N})_{N_i,N_{i+1}} \cdot \operatorname{in}_{\prec_n^i}(J_{N_i})\rangle_{R_{N_{i+1}}}$. Hence, we obtain an infinite subsequence of $(\prec_n^i)_{n\geq 1}$ satisfying (2). As the total number of initial ideals of $J_{N_{i+1}}$ is finite, this subsequence contains another infinite subsequence $(\prec_n^{i+1})_{n\geq 1}$ such that $\operatorname{in}_{\prec_n^{i+1}}(J_{N_{i+1}}) = \operatorname{in}_{\prec_1^{i+1}}(J_{N_{i+1}})$ for all n and we are done.

For every $i \geq 1$, choose a monomial $g_i \in \operatorname{in}_{\prec_1^i}(J_{N_i})$ that is not contained in $\langle \operatorname{Inc}(\mathbb{N})_{N_{i-1},N_i} \cdot \operatorname{in}_{\prec_1^i}(J_{N_{i-1}})\rangle_{R_{N_i}}$. Then for any pair $i < j$ of natural numbers, we have

$$\begin{aligned} g_j &\notin \langle \operatorname{Inc}(\mathbb{N})_{N_{j-1},N_j} \cdot \operatorname{in}_{\prec_1^j}(J_{N_{j-1}})\rangle_{R_{N_j}} \\ &\supseteq \langle \operatorname{Inc}(\mathbb{N})_{N_{j-1},N_j} \cdot (\operatorname{Inc}(\mathbb{N})_{N_i,N_{j-1}} \cdot \operatorname{in}_{\prec_1^j}(J_{N_i}))\rangle_{R_{N_j}} \\ &= \langle \operatorname{Inc}(\mathbb{N})_{N_{j-1},N_j} \cdot (\operatorname{Inc}(\mathbb{N})_{N_i,N_{j-1}} \cdot \operatorname{in}_{\prec_1^i}(J_{N_i}))\rangle_{R_{N_j}} \\ &= \langle \operatorname{Inc}(\mathbb{N})_{N_i,N_j} \cdot \operatorname{in}_{\prec_1^i}(J_{N_i})\rangle_{R_{N_j}} \\ &\supseteq \langle \operatorname{Inc}(\mathbb{N})_{N_i,N_j} \cdot g_i\rangle_{R_{N_j}}, \end{aligned}$$

where we used properties (1) and (3) in the third line and Lemma 2.1 in the fourth line. But by Lemma 2.4, such a sequence $(g_i)_{i\geq 1}$ cannot exist, and we have reached a contradiction. $\square$

For any subset $A \subseteq \mathbb{N}$, let R_A denote the polynomial ring over $\mathbb{K}$ in the variables indexed by $[c] \times A$.

Corollary 3.1. *Let $J_\circ$ be an* $\operatorname{Inc}(\mathbb{N})$*-invariant ideal chain in R. Then:*

(1) *The set of ideal chains*

$$\{\operatorname{in}_\prec(J_\circ) \mid \prec \text{ is an } \operatorname{Inc}(\mathbb{N})\text{-compatible term order on } R\}$$

is finite.

(2) *There is $N \in \mathbb{N}$ such that*

$$\operatorname{in}_\prec(J_n) = \sum_{1 \le i_1 < \cdots < i_N \le n} \big\langle \operatorname{in}_\prec(J_n \cap R_{\{i_1,\ldots,i_N\}}) \big\rangle_{R_n} \tag{3.1}$$

for all $n \ge N$ and every $\operatorname{Inc}(\mathbb{N})$*-compatible term order $\prec$ on R. Here, we regard the intersections $J_n \cap R_{\{i_1,\ldots,i_N\}}$ as ideals of $R_{\{i_1,\ldots,i_N\}}$.*

Furthermore, statement (1) *is equivalent to $|\mathrm{I}(J_\circ)| < \infty$.*

Proof. Let $N := \max(\mathrm{I}(J_\circ))$. Then by Lemma 2.5, for $\operatorname{Inc}(\mathbb{N})$-compatible term orders $\prec, \prec'$ we have $\operatorname{in}_\prec(J_\circ) = \operatorname{in}_{\prec'}(J_\circ)$ if and only if $\operatorname{in}_\prec(J_n) = \operatorname{in}_{\prec'}(J_n)$ for all $n \in [N]$. As $J_1, \ldots, J_N$ each have only finitely many initial ideals, there are only finitely many sequences $L_1 \subseteq \cdots \subseteq L_N$ such that $L_n = \operatorname{in}_\prec(J_n)$ for all n for any term order $\prec$ on R. This proves (1). It is clear that conversely, the boundedness of $\mathrm{I}(J_\circ)$ can be deduced from (1).

For the proof of (2), choose any $\operatorname{Inc}(\mathbb{N})$-compatible term order $\prec$ on R. Then by Remark 2.5 and the $\operatorname{Inc}(\mathbb{N})$-compatibility of $\prec$, $\operatorname{in}_\prec(J_n)$ is generated by

$$\{\operatorname{in}_\prec(p \cdot f) \mid p \in \operatorname{Inc}(\mathbb{N})_{N,n}, f \in J_N\}.$$

As $J_\circ$ is $\operatorname{Inc}(\mathbb{N})$-invariant, each of the polynomials $p \cdot f$ in the above set lies in one of the intersections $J_n \cap R_{\{i_1,\ldots,i_N\}}$, where $i_1 < \cdots < i_N$ ranges over all strictly ascending sequences of $[n]$. This proves the inclusion $\subseteq$ in (3.1). The reverse inclusion is obvious. ☐

Remark 3.2. By equation (3.1), for every $\operatorname{Inc}(\mathbb{N})$-invariant ideal chain $J_\circ$ in R, there is a natural number N such that for every $n \ge N$ and every $\operatorname{Inc}(\mathbb{N})$-compatible term order $\prec$ on R, there is a Gröbner basis of J_n with respect to $\prec$ whose elements each contain no more than cN distinct variables. This is not the case for arbitrary ideal chains in R. For instance, set $c = 1$ and consider the ideal chain $J_\circ$ defined by

$$\begin{aligned}
&J_1 := \{0\},\\
&J_{2^n} := \langle J_{2^{n-1}}, x_{2^{n-1}+1} + \cdots + x_{2^n} \rangle_{R_{2^n}} \text{ for } n \ge 1,\\
&J_m := \langle J_{2^n} \rangle_{R_m} \text{ for } 2^n \le m < 2^{n+1}.
\end{aligned}$$

Then for any term order $\prec$ on R and $n \ge 1$, every polynomial $f \in J_{2^n}$ with $\operatorname{in}_\prec(f) \mid \operatorname{in}_\prec(x_{2^{n-1}+1} + \cdots + x_{2^n})$ must contain a non-trivial $\mathbb{K}$-multiple of $x_{2^{n-1}+1} + \cdots + x_{2^n}$ and, hence, at least 2^{n-1} distinct variables.

We can now prove our statement on the number of initial ideals with respect to $\operatorname{Inc}(\mathbb{N})$-compatible term orders for $\operatorname{Inc}(\mathbb{N})$-invariant ideals of R:

Proof of Theorem 1.1. For every term order $\prec$ on R, we have

$$\operatorname{in}_\prec(J) = \bigcup_{n \ge 1} \operatorname{in}_\prec(J \cap R_n).$$

Thus, if $\prec, \prec'$ are term orders on R with $\mathrm{in}_\prec(J\cap R_n) = \mathrm{in}_{\prec'}(J\cap R_n)$ for all n, then $\mathrm{in}_\prec(J) = \mathrm{in}_{\prec'}(J)$. As the ideal chain $J_\circ := J\cap R_1 \subseteq J\cap R_2 \subseteq ...$ is $\mathrm{Inc}(\mathbb{N})$-invariant, Corollary 3.1(1) tells us that there exists a finite number of $\mathrm{Inc}(\mathbb{N})$-compatible term orders $\prec_1, ..., \prec_N$ on R such that for every $\mathrm{Inc}(\mathbb{N})$-compatible term order $\prec$ on R there is $i \in [N]$ with $\mathrm{in}_\prec(J_\circ) = \mathrm{in}_{\prec_i}(J_\circ)$. This proves our claim. □

Remark 3.3. If we only consider initial ideals with respect to $\mathrm{Inc}(\mathbb{N})$-compatible term orders that are also well-orders, there is a more direct way to prove Theorem 1.1 that does not depend on Theorem 1.2. Note that in the case where a term order $\prec$ on R is also a well-order, for any subset $S \subseteq R$ and any $f \in R$ we can find a decomposition $f = f_1 + f_2$, where f_1 is an element of $\langle S\rangle_R$ and f_2 does not contain any monomials from $\langle \mathrm{in}_\prec(g) \,|\, g \in S\rangle_R$. In particular, if $J \subseteq R$ is an ideal, then the residue classes of the monomials of R that are not contained in $\mathrm{in}_\prec(J)$ (i.e., the standard monomials with respect to $\prec$) form a $\mathbb{K}$-basis of the quotient ring R/J. As in the Noetherian setting, this implies that there can be no proper containments $\mathrm{in}_{\prec'}(J) \subsetneq \mathrm{in}_\prec(J)$ for term orders of R that are also well-orders. This is not true for general $\mathrm{Inc}(\mathbb{N})$-compatible term orders: For instance, consider the $\mathrm{Inc}(\mathbb{N})$-invariant ideal $J = \langle \mathrm{Inc}(\mathbb{N}) \cdot (x_1 - x_2)\rangle_R$ and let $\prec, \prec'$ be the lexicographic term orders with respect to the variable orderings $x_i \prec x_{i+1}$ and $x_i \succ' x_{i+1}$ for all $i \geq 1$, respectively. Then $\mathrm{in}_\prec(J) = \langle x_i \,|\, i \geq 2\rangle_R \subsetneq \langle x_i \,|\, i \geq 1\rangle_R = \mathrm{in}_{\prec'}(J)$. Note that as 1 is the only standard monomial with respect to $\prec'$ and $x_1 + c \notin J$ for all $c \in \mathbb{K}$, the residue classes of the standard monomials with respect to $\prec'$ do not form a $\mathbb{K}$-basis of R/J.

Using these observations on term orders on R that are also well-orders, we can modify the proof of Lemma 2.6 in [MR] to show that any $\mathrm{Inc}(\mathbb{N})$-invariant ideal of R has only finitely many initial ideals with respect to $\mathrm{Inc}(\mathbb{N})$-compatible term orders that are also well-orders. For $f \in R$, let $\mathrm{supp}(f)$ be the set of monomials occurring in f with a non-zero coefficient. Let $J \subseteq R$ be an $\mathrm{Inc}(\mathbb{N})$-invariant ideal. By way of contradiction, assume that the number of initial ideals of J with respect to $\mathrm{Inc}(\mathbb{N})$-compatible term orders that are well-orders is infinite. Let Σ be an infinite set of $\mathrm{Inc}(\mathbb{N})$-compatible term orders on R that are also well-orders such that $\mathrm{in}_\prec(J) \neq \mathrm{in}_{\prec'}(J)$ for distinct term orders $\prec, \prec' \in \Sigma$. Choose any polynomial $f_1 \in J$ and let $\Sigma_1 \subseteq \Sigma$ be an infinite subset such that $\mathrm{in}_\prec(f_1) = m_1$ for all $\prec \in \Sigma_1$. Let $\mathbf{m_1} := \langle \mathrm{Inc}(\mathbb{N}) \cdot m_1\rangle_R$ and choose any $\prec \in \Sigma_1$. Then $\mathbf{m_1} \subsetneq \mathrm{in}_\prec(J)$, so there is $f_2 \in J$ with $\mathrm{in}_\prec(f_2) \notin \mathbf{m_1}$. We may assume that $\mathrm{supp}(f_2) \cap \mathbf{m_1} = \emptyset$. Again, let $\Sigma_2 \subseteq \Sigma_1$ be an infinite subset such that $\mathrm{in}_{\prec'}(f_2) = m_2$ for all $\prec' \in \Sigma_2$ and set $\mathbf{m_2} := \langle \mathrm{Inc}(\mathbb{N}) \cdot m_1, \mathrm{Inc}(\mathbb{N}) \cdot m_2\rangle_R$. Iterating this procedure, we obtain an infinite sequence $(m_i)_{i\geq 1}$ of monomials such that m_{i+1} is not contained in $\langle \mathrm{Inc}(\mathbb{N}) \cdot m_j \,|\, j \in [i]\rangle_R$ for all $i \geq 1$. But this contradicts the fact that $\mathrm{Inc}(\mathbb{N})$-divisibility is a well-partial-order on the monomials of R (cf. [HS], Theorem 3.1).

Remark 3.4. One might wonder if it is possible to deduce Theorem 1.2 from Theorem 1.1. Indeed, for any $\mathrm{Inc}(\mathbb{N})$-invariant ideal chain $J_\circ$ in R, the ideal $J := \bigcup_{n\geq 1} J_n$ is an $\mathrm{Inc}(\mathbb{N})$-invariant ideal of R, and for every term order $\prec$ on R, we have $\mathrm{in}_\prec(J) = \bigcup_{n\geq 1} \mathrm{in}_\prec(J_n)$. Hence, Theorem 1.1 yields

$$\#\Big\{ \bigcup_{n\geq 1} \mathrm{in}_\prec(J_n) \,|\, \prec \text{ is } \mathrm{Inc}(\mathbb{N})\text{-compatible} \Big\} < \infty.$$

However, Theorem 1.2 provides more information than that: By Corollary 3.1(1), not only the number of unions of the initial ideals of the ideals J_n with respect to Inc(ℕ)-compatible term orders is finite but also the number of sequences $(\mathrm{in}_{\prec}(J_n))_{n\geq 1}$ giving rise to the same union. Thus, Theorem 1.2 seems to be a stronger result than Theorem 1.1.

Remark 3.5. Theorem 1.1 does not hold for the number of initial ideals with respect to arbitrary term orders on R: Let $c=1$ and let $J := \langle \mathrm{Inc}(\mathbb{N}) \cdot (x_1^2x_2 + x_1x_2^2)\rangle_R$ be the ideal that is generated by the Inc(ℕ)-orbits of the polynomial $x_1^2x_2+x_1x_2^2$. For every $n \in \mathbb{N}$, define the term order $\prec_n$ by

$$x_{\sigma_n(1)}^{a_1}\cdot \ldots \cdot x_{\sigma_n(k)}^{a_k} \prec_n x_{\sigma_n(1)}^{b_1}\cdot \ldots \cdot x_{\sigma_n(k)}^{b_k} :\Leftrightarrow a_i < b_i \text{ for } i = \min\{j \mid a_j \neq b_j\},$$

where the map $\sigma_n \in S_\infty$ is defined by

$$\sigma_n(j) = \begin{cases} n+1-j, & j \leq n, \\ j, & j > n. \end{cases}$$

For example, if $n=3$, then $(\sigma_3(1), \sigma_3(2), \sigma_3(3), \sigma_3(4), \sigma_3(5)) = (3,2,1,4,5)$. We claim that for every pair $n<n'$ of natural numbers, $x_1^2x_{n'} \in \mathrm{in}_{\prec_n}(J) \setminus \mathrm{in}_{\prec_{n'}}(J)$. We have $\mathrm{in}_{\prec_n}(x_1^2x_{n'} + x_1x_{n'}^2) = x_1^2x_{n'}$ as $\sigma_n^{-1}(n') = n' > n = \sigma_n^{-1}(1)$, so $x_1^2x_{n'} \in \mathrm{in}_{\prec_n}(J)$. Let f be a polynomial in J that contains the monomial $x_1^2x_{n'}$. We may assume f to be homogeneous, so $f = \sum_{i=1}^k c_ip_i \cdot (x_1^2x_2 + x_1x_2^2)$ with $c_i \in \mathbb{K}\setminus\{0\}$ and $p_i \in \mathrm{Inc}(\mathbb{N})$, where $p_i \cdot (x_1^2x_2+x_1x_2^2) \neq p_j \cdot (x_1^2x_2+x_1x_2^2)$ for $i \neq j$. As f contains $x_1^2x_{n'}$, there is exactly one i with $p_i \cdot (x_1^2x_2+x_1x_2^2) = x_1^2x_{n'} + x_1x_{n'}^2$. Therefore, f contains the monomial $x_1x_{n'}^2$. But $x_1^2x_{n'} \prec_{n'} x_1x_{n'}^2$, so $x_1^2x_{n'} \notin \mathrm{in}_{\prec_{n'}}(J)$. We conclude that the initial ideals $\mathrm{in}_{\prec_n}(J)$ are pairwise distinct. Thus, J has infinitely many distinct initial ideals with respect to arbitrary term orders.

Definition 3.6. Let $J \subseteq R$ be an Inc(ℕ)-invariant ideal. Then we call a subset $\mathcal{G} \subseteq J$ a universal Inc(ℕ)-Gröbner basis of J if for every Inc(ℕ)-compatible term order on R, we have $\mathrm{in}_{\prec}(J) = \langle \mathrm{Inc}(\mathbb{N}) \cdot \mathrm{in}_{\prec}(g) \mid g \in \mathcal{G}\rangle_R$.

Corollary 3.7. *Every* Inc(ℕ)*-invariant ideal $J \subseteq R$ has a finite universal* Inc(ℕ)*-Gröbner basis.*

Proof. Define the Inc(ℕ)-invariant ideal chain $J_\circ$ by setting $J_n := J \cap R_n$ and let $N := \max(\mathrm{I}(J_\circ))$. Let $\mathcal{G}$ be a finite universal Gröbner basis of J_N. Then for every $n \geq N$ and every Inc(ℕ)-compatible term order $\prec$, we have $\mathrm{in}_{\prec}(J_n) = \langle \mathrm{Inc}(\mathbb{N})_{N,n} \cdot \mathrm{in}_{\prec}(g) \mid g \in \mathcal{G}\rangle_{R_n}$. As $\mathrm{in}_{\prec}(J)$ is the union of all initial ideals $\mathrm{in}_{\prec}(J_n)$, this implies that $\mathrm{in}_{\prec}(J) = \langle \mathrm{Inc}(\mathbb{N}) \cdot \mathrm{in}_{\prec}(g) \mid g \in \mathcal{G}\rangle_R$. □

4. Inc(ℕ)-COMPATIBLE TERM ORDERS FOR $c=1$

In this section, we will always assume $c=1$. Observe that any term order $\prec$ on R can be regarded as a total order of $\mathbb{N}^\infty := \bigcup_{n\geq 1}\mathbb{N}^n$ by setting $(m_i)_{i\geq 1} \prec (n_i)_{i\geq 1}$ if and only if $\prod_{i\geq 1} x_i^{m_i} \prec \prod_{i \geq 1} x_i^{n_i}$. This order relation is compatible with addition on $\mathbb{N}^\infty$; i.e., $(m_i) \prec (n_i)$ implies $(m_i)+(l_i) \prec (n_i)+(l_i)$ for all $(l_i), (m_i), (n_i) \in \mathbb{N}^\infty$. Thus, by Lemma 1.3 in [KTV] the order relation $\prec$ on $\mathbb{N}^\infty$ can be uniquely extended to an order relation on $\mathbb{Q}^\infty := \bigcup_{n\geq 1}\mathbb{Q}^n$ which satisfies

(O1) $(q_i) \prec (r_i) \Rightarrow (q_i)+(s_i) \prec (r_i)+(s_i)$ for all $(q_i),(r_i),(s_i) \in \mathbb{Q}^\infty$;

(O2) $(q_i) \prec (r_i) \Rightarrow a(q_i) \prec a(r_i)$ for all $(q_i),(r_i) \in \mathbb{Q}^\infty$ and $a \in \mathbb{Q}_{>0}$.

Using terminology from [KTV], we call such an order relation a total order of the vector space $\mathbb{Q}^\infty$. The monoid $\mathrm{Inc}(\mathbb{N})$ acts on $\mathbb{Q}^\infty$ by $\mathbb{Q}$-vector space homomorphisms

$$(p\cdot -):\mathbb{Q}^\infty\to\mathbb{Q}^\infty,\ (p\cdot(q_i))_j:=\begin{cases} q_{p^{-1}(j)}, & j\in p(\mathbb{N}),\\ 0, & j\notin p(\mathbb{N}),\end{cases}$$

for every $p\in\mathrm{Inc}(\mathbb{N})$. We call an order of the vector space $\mathbb{Q}^\infty$ $\mathrm{Inc}(\mathbb{N})$-compatible if it is compatible with this action. Hence, the problem of finding all $\mathrm{Inc}(\mathbb{N})$-compatible term orders on R is equivalent to finding all $\mathrm{Inc}(\mathbb{N})$-compatible total orders of the vector space $\mathbb{Q}^\infty$.

In what follows, we will not only consider total orders $\prec$ but also partial orders of $\mathbb{Q}^\infty$ satisfying (O1), (O2) and

(O3) $(q_i)+(s_i)\prec(r_i)+(s_i)\Rightarrow(q_i)\prec(r_i)$ for all $(q_i),(r_i),(s_i)\in\mathbb{Q}^\infty$;

(O4) incomparability with respect to $\prec$ is transitive.

Again using terminology from [KTV], we call every such partial order $\prec$ a weak partial order of the vector space $\mathbb{Q}^\infty$. If $\prec$ has the additional property that for every $p\in\mathrm{Inc}(\mathbb{N})$ the implication $(q_i)\prec(r_i)\Rightarrow p\cdot(q_i)\prec p\cdot(r_i)$ holds, we call $\prec$ an $\mathrm{Inc}(\mathbb{N})$-compatible weak partial order of the vector space $\mathbb{Q}^\infty$.

For any weak partial order $\prec$ of the vector space $\mathbb{Q}^\infty$, the set $E_\prec$ of vectors incomparable to $\mathbf{0}$ forms a $\mathbb{Q}$-subspace. Furthermore, the relation $(q_i)+E_\prec\prec(r_i)+E_\prec$ if and only if $(q_i)\prec(r_i)$ is a well-defined total order of the vector space $\mathbb{Q}^\infty/E_\prec$ (cf. [KTV], Lemma 1.4).

Now let $\prec$ be an $\mathrm{Inc}(\mathbb{N})$-compatible weak partial order of the vector space $\mathbb{Q}^\infty$ and let $\prec_n$ denote its restriction to $\mathbb{Q}^n$. In [KTV] it was shown that there is $m\le n$ and a matrix $A\in\mathbb{R}^{m\times n}$ such that for any $(q_i),(r_i)\in\mathbb{Q}^n$, we have

$$(q_i)\prec(r_i)\Leftrightarrow A\cdot(q_i)\prec_{\mathrm{lex}}A\cdot(r_i),$$

where $\prec_{\mathrm{lex}}$ denotes the lexicographic order on $\mathbb{R}^m$ corresponding to the order $\mathbf{e_i}\succ_{\mathrm{lex}}\mathbf{e_{i+1}}$ of the standard basis of $\mathbb{R}^m$.

Lemma 4.1. *Let $n\ge 4$. If $\prec_n$ is non-trivial (i.e. $E_\prec\cap\mathbb{Q}^n\ne\mathbb{Q}^n$), then $\prec_n$ is represented by a matrix $A\in\mathbb{R}^{m\times n}$ such that for the first row A_{1*}, we have*

$$A_{1*}\in\{\pm\mathbf{e_1},\pm\mathbf{e_n},\pm(1\ \cdots\ 1)\},$$

and, if A_{1*} is constant, either $m=1$ or $m\ge 2$ and the second row A_{2*} satisfies

$$A_{2*}\in\{\pm\mathbf{e_1},\pm\mathbf{e_n}\}.$$

Proof. Let $A\in\mathbb{R}^{m\times n}$ be a matrix representing $\prec$. Note that for any vector $(q_i)\in\mathbb{Q}^{n-1}$, the $\mathrm{Inc}(\mathbb{N})$-compatibility of $\prec$ yields the following implications:

$$(A_{11}\ A_{12}\ \cdots\ A_{1n-1})\cdot(q_i)<0\Rightarrow\begin{cases}(A_{11}\ A_{13}\ A_{14}\ \cdots\ A_{1n})\cdot(q_i)\le 0,\\ (A_{12}\ A_{13}\ A_{14}\ \cdots\ A_{1n})\cdot(q_i)\le 0.\end{cases}$$

This means that the matrices

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n-1}\\ A_{11} & A_{13} & A_{14} & \cdots & A_{1n}\end{pmatrix},$$

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n-1}\\ A_{12} & A_{13} & A_{14} & \cdots & A_{1n}\end{pmatrix}$$

have rank ≤ 2. It is easy to show that this is only the case if A_{1*} is constant or if either $A_{11}=\cdots=A_{1n-1}=0$ or $A_{12}=\cdots=A_{1n}=0$.

Now assume that A_{1*} is constant and $m \geq 2$. Then for any rational numbers q_i, the vector $(q_1 \ \cdots \ q_{n-2} \ (-q_1 - \cdots - q_{n-2}) \ 0)$ lies in the orthogonal complement of A_{1*}. Therefore, setting $(r_i) := (q_1 \ \cdots \ q_{n-2} \ (-q_1 - \cdots - q_{n-2}))$, the Inc($\mathbb{N}$)-compatibility of $\prec$ again yields implications

$$(A_{21} \ A_{22} \ \cdots \ A_{2n-1}) \cdot (r_i) < 0 \Rightarrow \begin{cases} (A_{21} \ A_{23} \ A_{24} \ \cdots \ A_{2n}) \cdot (r_i) \leq 0, \\ (A_{22} \ A_{23} \ A_{24} \ \cdots \ A_{2n}) \cdot (r_i) \leq 0. \end{cases}$$

Hence, the matrices

$$\begin{pmatrix} A_{21} - A_{2n-1} & A_{22} - A_{2n-1} & A_{23} - A_{2n-1} & \cdots & A_{2n-2} - A_{2n-1} \\ A_{21} - A_{2n} & A_{23} - A_{2n} & A_{24} - A_{2n} & \cdots & A_{2n-1} - A_{2n} \end{pmatrix},$$
$$\begin{pmatrix} A_{21} - A_{2n-1} & A_{22} - A_{2n-1} & A_{23} - A_{2n-1} & \cdots & A_{2n-2} - A_{2n-1} \\ A_{22} - A_{2n} & A_{23} - A_{2n} & A_{24} - A_{2n} & \cdots & A_{2n-1} - A_{2n} \end{pmatrix}$$

must have rank ≤ 2. This is only possible for either $A_{21} = \cdots = A_{2n-1}$ or $A_{22} = \cdots = A_{2n}$, and the claim follows. □

Definition 4.2. For weak partial orders $\prec, \prec'$ of the vector space $\mathbb{Q}^\infty$, define their product $\prec' * \prec$ by $(q_i) \prec' * \prec (r_i)$ if and only if $(q_i) \prec (r_i)$ or $[(q_i) - (r_i) \in E_\prec$ and $(q_i) \prec' (r_i)]$. Clearly, this is again a weak partial order of the vector space $\mathbb{Q}^\infty$.

Lemma 4.3. *Define weak partial orders $\prec^{(N)}_{\pm,\max}$, $\prec_{\pm,\min}$, and $\prec_{\pm,\deg}$ of the vector space $\mathbb{Q}^\infty$ by*

- $(q_i) \prec_{+,\min} (r_i) :\Leftrightarrow q_j < r_j$ *for the minimal j such that $q_j \neq r_j$;*
  $(q_i) \prec_{-,\min} (r_i) :\Leftrightarrow (q_i) \succ_{+,\min} (r_i)$*;*
- $(q_i) \prec^{(N)}_{+,\max} (r_i) :\Leftrightarrow$ *for the maximal j such that $q_j \neq r_j$, we have $j \geq N$ and $q_j < r_j$;*
  $(q_i) \prec^{(N)}_{-,\max} (r_i) :\Leftrightarrow (q_i) \succ^{(N)}_{+,\max} (r_i)$*;*
- $(q_i) \prec_{+,\deg} (r_i) :\Leftrightarrow \sum_{i\geq 1} q_i < \sum_{i \geq 1} r_i$*;*
  $(q_i) \prec_{-,\deg} (r_i) :\Leftrightarrow (q_i) \succ_{+,\deg} (r_i)$*.*

Then for $n \geq 4$, the weak partial order $\prec_n$ is either trivial or agrees with one of the following weak partial orders:

- $\prec_{\pm,\deg}$*;*
- $\prec_{\pm,\min}$*;*
- $\prec_{\pm,\min} * \prec_{\pm,\deg}$*;*
- $\prec^{(N)}_{\pm,\max}$ *for some $1 \leq N \leq n$;*
- $\prec^{(N)}_{\pm,\max} * \prec_{\pm,\deg}$ *for some $2 \leq N \leq n$.*

Proof. Assume $\prec_n$ to be non-trivial. As a consequence of the Inc($\mathbb{N}$)-compatibility of $\prec$, $\prec_n$ is fully determined by the first two rows of the matrix A representing it. This can be seen from the following distinction of cases:

- A has only one row and $A_{1*} = \text{const}$: In this case, we have $\prec_n = \prec_{\pm,\deg}$.
- $A_{1*} = \text{const}$, $A_{2*} = \mathbf{e_n}$: For any $(q_i) \in \mathbb{Q}^n$ satisfying $\sum_{i\geq 1} q_i = 0$ and $q_n < 0$, we have $(q_i) \prec \mathbf{0}$. Let $N \in [n]$ be minimal such that $\mathbf{e_1} - \mathbf{e_N} \prec \mathbf{0}$. As every $(q_i) \in \mathbb{Q}^n$ with $\sum_{i\geq 1} q_i = 0$ and $q_N = \cdots = q_n = 0$ can be written as
  $$(q_i) = \sum_{i=2}^{N-1} q_i(-\mathbf{e_1} + \mathbf{e_i}),$$
  the inclusions $-\mathbf{e_1} + \mathbf{e_i} \in E_\prec$ for $2 \leq i \leq N-1$ imply $(q_i) \in E_\prec$. Now assume that there is $j \geq N$ with $q_j \neq 0$ and let j be maximal with this

property. Suppose that $q_j < 0$. If $j = N$, we have

$$(q_i) = \sum_{i=2}^{j-1} q_i(-\mathbf{e_1} + \mathbf{e_l}) + q_N(-\mathbf{e_1} + \mathbf{e_N}) \prec \mathbf{0}.$$

On the other hand, if $j > N$ and $(q_i), \mathbf{0}$ were incomparable with respect to $\prec$, then $(q_i) + (-\mathbf{e_1} + \mathbf{e_N}) \succ \mathbf{0}$, contradicting the Inc(ℕ)-compatibility of $\prec$. Thus, we obtain $\prec_n = \prec^{(N)}_{+,\max} * \prec_{\pm,\deg}$ for some $N \geq 2$. If $A_{2*} = -\mathbf{e_n}$, the same argument shows that $\prec_n = \prec^{(N)}_{-,\max} * \prec_{\pm,\deg}$ for some $N \geq 2$.

- $A_{1*} = \mathbf{e_n}$: In this case, any $(q_i) \in \mathbb{Q}^n$ with $q_n < 0$ satisfies $(q_i) \prec \mathbf{0}$. Again, choose $N \in [n]$ minimal such that $-\mathbf{e_N} \prec \mathbf{0}$. For $(q_i) \in \mathbb{Q}^n$ with $q_N = \cdots = q_n = 0$ we have

$$(q_i) = \sum_{i=1}^{N-1} q_i \mathbf{e_i} \in E_\prec.$$

If on the other hand there is $j \geq N$ with $q_j \neq 0$ and the maximal such j satisfies $q_j < 0$, then an analogous argument to the one above shows that $(q_i) \prec \mathbf{0}$. This yields $\prec_n = \prec^{(N)}_{+,\max}$ for some $N \geq 1$. In the case that $A_{1*} = -\mathbf{e_n}$, we obtain $\prec_n = \prec^{(N)}_{-,\max}$ for some $N \geq 1$.
- $A_{1*} = \text{const}$, $A_{2*} = \mathbf{e_1}$: If $\sum_{i \geq 1} q_i = 0$ and $q_1 < 0$, then $(q_i) \prec \mathbf{0}$. Therefore, the Inc(ℕ)-compatibility of $\prec$ yields $\prec_n = \prec_{+,\min} * \prec_{\pm,\deg}$. If $A_{2*} = -\mathbf{e_1}$, we have $\prec_n = \prec_{-,\min} * \prec_{\pm,\deg}$.
- $A_{1*} = \mathbf{e_1}$: As for every $(q_i) \in \mathbb{Q}^n$ with $q_1 < 0$, we have $(q_i) \prec \mathbf{0}$, the Inc(ℕ)-compatibility of $\prec$ yields $\prec_n = \prec_{+,\min}$. If $A_{1*} = -\mathbf{e_1}$, we obtain $\prec_n = \prec_{-,\min}$.

□

From our results for the restrictions $\prec_n$ of $\prec$ to $\mathbb{Q}^n$, we can now easily deduce:

Theorem 4.4. *If $\prec$ is non-trivial, then it is equal to one of the following* Inc(ℕ)*-compatible weak partial orders:*

- $\prec_{\pm,\deg}$*;*
- $\prec_{\pm,\min}$*;*
- $\prec_{\pm,\min} * \prec_{\pm,\deg}$*;*
- $\prec^{(N)}_{\pm,\max}$ *for some* $N \geq 1$*;*
- $\prec^{(N)}_{\pm,\max} * \prec_{\pm,\deg}$ *for some* $N \geq 2$*.*

Proof. It is easy to see that if there is $n \geq 4$ such that $\prec_n = \prec_{+,\min}$, then $\prec_m = \prec_{+,\min}$ for all $m \geq n$. This implies $\prec = \prec_{+,\min}$. The same is true for $\prec_n = \prec_{-,\min}$ and $\prec_n = \prec_{\pm,\min} * \prec_{\pm,\deg}$. Likewise, if there is $n \geq 4$ such that $\prec_n = \prec^{(N)}_{+,\max}$ for some $N \leq n$, then $\prec_m = \prec^{(N)}_{+,\max}$ for all $m \geq N$ and thus $\prec = \prec^{(N)}_{+,\max}$. The same holds for $\prec_n = \prec^{(N)}_{-,\max}$ and $\prec_n = \prec^{(N)}_{\pm,\max} * \prec_{\pm,\deg}$. If none of the aforementioned cases applies, we must have $\prec_n = \prec_{\pm,\deg}$ for all $n \geq 4$, which implies $\prec = \prec_{\pm,\deg}$. □

The Inc(ℕ)-compatible term orders on R induce Inc(ℕ)-compatible total orders of the vector space $\mathbb{Q}^\infty$ which satisfy $\mathbf{0} \preceq (q_i)$ for all $(q_i) \in \mathbb{Q}^\infty_{\geq 0}$. Thus, we obtain:

Corollary 4.5. *The lexicographic, the degree-lexicographic, and the degree-reverse-lexicographic term orders with respect to $x_1 \prec x_2 \prec \cdots$ or $x_1 \succ x_2 \succ \cdots$ are the only* Inc(ℕ)*-compatible term orders on R.*

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Universität Marburg, Fachbereich Mathematik und Informatik, 35032 Marburg, Germany

Email address: lindner5@mathematik.uni-marburg.de