PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 9, September 2018, Pages 3895–3906
http://dx.doi.org/10.1090/proc/14089
Article electronically published on June 1, 2018

EXPANSIVE MEASURES VERSUS LYAPUNOV EXPONENTS

ALMA ARMIJO AND MARIA JOSÉ PACIFICO

(Communicated by Nimish Shah)

Abstract. In this paper we investigate the relation between measure-expansiveness and hyperbolicity. We prove that non-atomic invariant ergodic measures with all of their Lyapunov exponents positive are positively measure-expansive. We also prove that local diffeomorphisms robustly positively measure-expansive are expanding. Finally, we prove that a C^1-volume-preserving diffeomorphism that cannot be accumulated by positively measure-expansive diffeomorphisms has a dominated splitting.

1. Introduction

The notion of expansiveness was introduced by Utz in the middle of the twentieth century; see [18]. Roughly speaking, expansiveness means that orbits through different points separate when time evolves. This notion is very important in the context of the theory of dynamical systems and is shared by a large class of dynamical systems exhibiting chaotic behavior. Nowadays there is an extensive literature about these systems. See, for instance, [6, 8, 10, 19] and references therein. Examples of expansive systems are hyperbolic diffeomorphisms defined on compact manifolds. This includes Anosov systems and the non-wandering set of Axiom A diffeomorphisms.

Recently the notion of measure-expansiveness that generalizes the concept of expansiveness was introduced in [12]. Roughly speaking, a system is measure-expansive if the set of points whose orbit is near the orbit of a given point is zero. There is already a consistent literature respecting measure-expansive systems, relating this property with expansiveness, ergodicity, and some other properties already established elsewhere. We refer to [3, 5, 15] and references therein for more on this.

The purpose of this work is to exploit more this notion and establish some of its relation with expansiveness, existence of some weak form of hyperbolicity, and existence of positive Lyapunov exponents.

To announce precisely our results, let us introduce some definitions. To this end, let (M, d) be a compact boundaryless Riemannian manifold and let $\mathrm{Diff}^1_{loc}(M)$ be the set of C^1-local diffeomorphisms $f : M \to M$. Let $\mathcal{M}(M)$ be the space of Borel probability measures of M.

Received by the editors February 7, 2017, and, in revised form, September 5, 2017, October 10, 2017, and November 26, 2017.

2010 *Mathematics Subject Classification.* Primary 37C40.

This work was partially supported by CAPES, CNPq, FAPERJ.

Recall that $\mu \in \mathcal{M}(M)$ is atomic if there is a point $x \in M$ such that $\mu(x) > 0$. The set of atomic measures $\mathcal{A}(M)$ of M is dense in $\mathcal{M}(M)$. A measure μ is f-invariant if $\mu(f^{-1}(B)) = \mu(B)$ for every measurable set B and μ is ergodic if the measure of any invariant set is zero or one. We denote $\mathcal{M}_f(M)$ for the set of f-invariant probability measures on M.

The classical Oseledets's theorem [14] asserts that for any f-invariant measure μ, there exist a Df-invariant (measurable) splitting

$$T_xM = E_1(x) \oplus \cdots \oplus E_{k(x)}(x)$$

and numbers $\lambda_1(x) < \lambda_2(x) < \cdots < \lambda_{k(x)}(x)$ so that for all $v \in E_i(x) \setminus \{0\}$, it holds that

$$\lambda_i(x) = \lim_{n\to\infty} \frac{1}{n} \log \|Df^n(x)v\|$$

for μ-almost all $x \in M$. The numbers $\lambda_i(x)$, $1 \le i \le k(x)$, are called the *Lyapunov exponents* of f at x.

We say that $f \in \mathrm{Diff}^1_{loc}(M)$ is *positively measure-expansive for* μ (or *positively* μ*-expansive* for short) if there is a constant $\delta > 0$ such that $\mu(\Gamma^+_\delta(f, x)) = 0$ for all $x \in M$, where

$$\Gamma^+_\delta(f, x) \equiv \{y \in M \,/\, d(f^n(x), f^n(y)) \le \delta, \text{ for all } n \in \mathbb{N}\}. \tag{1}$$

Note that every positively μ-expansive map is positively μ-expansive for any $\mu \in \mathcal{M}(M) \setminus \mathcal{A}(M)$.

The first result in this paper establishes that local diffeomorphisms with positive Lyapunov exponents are positively μ-expansive.

Theorem A. *Let $f \in \mathrm{Diff}^1_{loc}(M)$ and let $\mu \in \mathcal{M}_f \setminus \mathcal{A}(M)$ be an ergodic probability measure such that all of its Lyapunov exponents are positive. Then f is positively μ-expansive.*

An open class of dynamical systems such that every non-atomic measure is positively expansive is the class of expanding endomorphisms. Recall that a map $f \in \mathrm{Diff}^1_{loc}(M)$ is *expanding* if there exists $\beta > 1$ and $K > 0$ such that for every $x \in M$ we have

$$||Df^n(x)|| \ge K\beta^n, \ \ n \in \mathbb{N}.$$

Next we want to relate positive μ-expansiveness to expansiveness for maps $f \in \mathrm{Diff}^1_{loc}(M)$. In this direction we point out that in [16] the author proved that the C^1-interior of the set of positively measure-expansive C^1-diffeomorphisms coincides with the interior of the set of C^1-expansive diffeomorphisms. In [17] it is proved that the interior of the set of measure-expansive diffeomorphisms coincides with the interior of the expansive diffeomorphisms. More recently, in [9], the authors proved that C^1-generically, a differentiable map is positively μ-expansive if and only if it is expanding. The next result is a version for local diffeomorphisms of [9, Theorem A] and establishes that local diffeomorphisms C^1 robustly positively μ-expansive are expanding.

Theorem B. *Let $f \in \mathrm{Diff}^1_{loc}(M)$ and suppose that there is a C^1-open neighborhood $\mathcal{U}$ of f such that every $g \in \mathcal{U}$ is positively μ-expansive for all $\mu \in \mathcal{M}(M) \setminus \mathcal{A}(M)$. Then f is expanding.*

To prove this last result, we first prove a perturbing lemma, Lemma A, that has the same flavor as [7, Lemma (1.1)], establishing that if a map f has a non-expanding periodic point p, then there is $\delta > 0$ such that $\Gamma_\delta^+(f,p)$ contains a manifold S with $\dim(S) \geq 1$. The proof of the theorem follows by contradiction, applying this lemma to a non-expanding periodic point. As another consequence of Lemma A we get a similar result to [1, Theorem 1.1]:

Theorem 1. *There exists an open and dense subset $\mathcal{R} \subset \mathrm{Diff}^1_{loc}(M)$ such that if $f \in \mathcal{R}$ and f is positively μ-expansive, then f is expanding.*

We can also ask if measure-expansiveness for only the *invariant measures* could be enough for the map to have some hyperbolicity. Next, we give a partial answer to this question in the *conservative* context. This means that the manifold is endowed with a smooth volume form ω; then we can speak of conservative (i.e., volume-preserving) diffeomorphisms. In order to announce this last result, denote by $\mathcal{D}iff^1_\omega(M)$ the set of C^1 conservative diffeomorphisms and for easy notation set

$$\mathcal{PIM} = \{f \in \mathcal{D}iff^1(M); f \text{ is positively } \mu\text{-expansive for all } \mu \in \mathcal{M}_f(M) \setminus \mathcal{A}(M)\}.$$

Theorem C. *Let $f \in \mathcal{D}iff^1_\omega(M)$ and assume there is an open neighborhood $\mathcal{U}(f)$ such that $\mathcal{U}(f) \subset \mathcal{PIM} \cup \{f\}$. Then f has a dominated splitting.*

This text is organized as follows. In Section 2 we give the basic definitions and recall previous results proved elsewhere that will be used to obtain our main theorems. In Section 3 we study positive measure-expansive maps and give a sufficient condition for a measure-expansive map to be expanding and prove Theorem A. In Section 4 we give a necessary condition for a local diffeomorphism to be measure-expansive and prove Theorem B and Theorem 1. Finally in Section 5 we prove Theorem C.

2. Preliminaries

In this section we set the notation and recall some definitions and results proved elsewhere that we shall use to obtain our results. For this let M be a compact boundaryless n-dimensional Riemannian manifold, $n \geq 2$. As above $\mathrm{Diff}^1_{loc}(M)$ denotes the set of C^1 local diffeomorphisms on M endowed with the C^1-topology. Denote by d the distance on M induced y the Riemannian metric $||\cdot||$ on the tangent bundle TM.

Let $f \in \mathrm{Diff}^1_{loc}(M)$ and $p \in M$. Recall that p is a *periodic* point if $f^n(p) = p$ for some $n \geq 1$. The minimal number n such that $f^n(p) = p$ is the *period* of p and is denoted by $\tau(p)$. Given a periodic point p with period $\tau(p)$ we denote by $\mathcal{O}(p)$ the orbit of p, i.e., $\mathcal{O}(p) = \{p, f(p), \ldots, f^{\tau(p)-1}(p)\}$. A periodic point p is *hyperbolic* if the eigenvalues of $Df^{\tau(p)}(p)$ do not belong to the unit circle S^1 and is *expanding* if all the eigenvalues of $Df^{\tau(p)}(p)$ have absolute value greater than one.

The stable manifold of $x \in M$, $W^s(x)$, is defined as

$$W^s(x) = \{y | d(f^n(y), f^n(x)) \to 0, n \in \mathbb{N}\}.$$

The unstable manifold of x, $W^u(x)$, is defined as

$$W^u(x) = \{y \,|\, d(f^{-n}(y), f^{-n}(x)) \to 0, n \in \mathbb{N}\}.$$

A *saddle point* is a hyperbolic periodic point whose stable and unstable manifolds have a positive dimension.

A compact invariant set Λ of a diffeomorphism f is *hyperbolic* if there is a Df-invariant continuous splitting $T_\Lambda M = E^s \oplus E^u$ and constants $C > 0$ and $\kappa < 1$ such that for every $x \in \Lambda$ and $n \in \mathbb{N}$ it holds that

$$||Df^{-n}(x)_{|E^u_x}|| \leq C\kappa^n \quad \text{and} \quad ||Df^n(x)_{|E^s_x}|| \leq C\kappa^n.$$

A compact f-invariant set $\Lambda \subset M$ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E_1 \oplus \cdots \oplus E_k$ and there exist constants $C > 0$, $0 < \lambda < 1$, such that for all $i < j$, $\forall x \in \Lambda$, and $n \geq 0$ it holds that

$$||Df^n|E_i(x)|| \cdot ||Df^{-n}|E_j(f^n(x))|| \leq C\lambda^n.$$

Let $C^1(M)$ be the set of C^1 maps $f : M \to M$, endowed with the C^1-topology. A subset $\mathcal{R} \subset C^1(M)$ is a *residual subset* if it contains a countable intersection of open and dense sets. The countable intersection of residual subsets is also a residual subset.

A property (P) holds *generically* if there exists a residual subset $\mathcal{R} \subset C^1(M)$ such that any $f \in \mathcal{R}$ has the property (P).

We finish this section by stating a lemma, due to V. I. Pliss, whose proof can be found in [11, Lemma 11.8, p. 276].

Lemma 2. *Given $A \geq c_2 > c_1 > 0$, let $\theta_0 = \dfrac{(c_2 - c_1)}{(A - c_1)}$. Then, given any real numbers $a_1, \ldots, a_N$ such that*

$$\sum_{j=1}^{N} a_j \geq c_2 N \quad \textit{and} \quad a_j \leq A \textit{ for every } 1 \leq j \leq N,$$

there are $l > \theta_0 N$ and $1 < n_1 < \cdots < n_l \leq N$ so that

$$\sum_{j=n+1}^{n_i} a_j \geq c_1(n_i - n) \quad \textit{for every} \quad 1 \leq n \leq n_i \quad \textit{and} \quad i = 1, \ldots, l.$$

3. Proof of Theorem A

In this section we prove Theorem A. For this, first recall that a map $f \in \mathrm{Diff}^1_{loc}(M)$ is *asymptotically c-expanding* at $x \in M$ if (see [13, p. 1314])

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df(f^i(x))^{-1}\|^{-1} > 4c\,. \tag{2}$$

Now, let μ be an invariant ergodic probability measure such that all of its Lyapunov exponents are positive. By [13, Lemma 3.5] there exist $c > 0$ and $l \in \mathbb{N}$ such that

$$\int_M \frac{1}{l} \log(\|Df^l(x)^{-1}\|) d\mu < -4c < 0. \tag{3}$$

For every $l \in \mathbb{N}$ define the set

$$\mathcal{J}_l := \{x \in M : f^l \text{ is asymptotically } c\text{-expanding at } x\}.$$

Claim 1. *The set $\mathcal{J}_l$ has total measure with respect to μ.*

In fact, since $\mathcal{J}_l$ is f^l-invariant and μ is ergodic then the measure of $\mathcal{J}_l$ is null or total. By contradiction, assume that the measure of $\mathcal{J}_l$ is null. By Birkhoff's Theorem, the limit in (2) defines a measurable and integrable map φ that satisfies

$$\int_M \varphi d\mu = \int_M \log(\|Df^l(x)^{-1}\|^{-1}) d\mu.$$

So,

$$\int_M \log(\|Df^l(x)^{-1}\|^{-1}) d\mu = \int_M \varphi d\mu = \int_{\mathcal{J}_l} \varphi d\mu + \int_{M\setminus\mathcal{J}_l} \varphi d\mu\,.$$

Since we are assuming $\mu(\mathcal{J}_l) = 0$ we have $\int_{\mathcal{J}_l} \varphi d\mu = 0$ and thus

$$\int_M \log(\|Df^l(x)^{-1}\|^{-1}) d\mu = \int_{M\setminus\mathcal{J}_l} \varphi d\mu \geq -4c\,.$$

So, by (3), we get

$$\int_M \log(\|Df^l(x)^{-1}\|) d\mu < -4lc \leq -4c \leq \int_M \log(\|Df^l(x)^{-1}\|) d\mu\,,$$

a contradiction. Thus the measure of $\mathcal{J}_l$ is total and finishes the proof of Claim 1. □

Now, define $g := f^l$. Since g is a C^1-local diffeomorphism, there exists an open cover $\{V_i\}_{i\in\Lambda}$ of M such that $g|_{V_i} : V_i \to g(V_i)$ is a diffeomorphism. We can assume that these sets V_i are connected, and by compactness of M there is $\delta' > 0$ such that

$$\text{dist}(\xi,\eta) < \delta' \quad \text{implies that} \quad \xi,\eta \in V_i \quad \text{for some} \quad i \in \Lambda. \tag{4}$$

Moreover, by uniform continuity there exists $\widehat{\delta} > 0$ such that

$$\text{dist}(x,y) < \widehat{\delta} \quad \text{implies that} \quad \frac{\|Dg(x)^{-1}\|}{\|Dg(y)^{-1}\|} > e^{-c/2}. \tag{5}$$

Fix $\delta = \min\{\widehat{\delta}, \delta'\}$ and consider $\Gamma^+_\delta(f,x)$. Then, if $y \in \Gamma^+_\delta(f,x)$ we have that

$$y \in \bigcap_{j\in\mathbb{N}} f^{-j}(B(f^j(x),\delta)) \subset \bigcap_{k\in\mathbb{N}} g^{-k}(B(g^k(x),\delta)).$$

Hence,

$$\text{dist}(g^k(x), g^k(y)) < \delta, \ \ \forall k \in \mathbb{N}. \tag{6}$$

Pick a point $x \in M$ satisfying condition (2) for g. Then

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Dg(g^i(x))^{-1}\|^{-1} > 4c.$$

For $N \in \mathbb{N}$ sufficiently large we have

$$\frac{1}{N} \sum_{i=1}^{N} \log \|Dg(g^i(x))^{-1}\|^{-1} > 4c. \tag{7}$$

Applying Lemma 2 with

$$A = \max_{\xi\in M} \log \|Dg(\xi)^{-1}\|^{-1}, \qquad c_2 = 4c, \qquad c_1 = 2c, \quad \text{and} \quad \theta_0 = \frac{2c}{A-2c},$$

we obtain that there are $l \in \mathbb{N}$, with $l > \theta_0 N$, and $1 < n_1 < \cdots < n_l < N$, with N as in (7), such that

$$\sum_{j=n+1}^{n_i} \log \|Dg(g^j(x))^{-1}\|^{-1} > (n_i - n)2c \quad \text{for every} \quad 1 \le n \le n_i \quad \text{and} \quad i = 1, \dots, l.$$

Moreover, for all z satisfying $d(z, g^j(x)) < \delta$, for all $j \in \mathbb{N}$, (5) implies that

$$\frac{\|Dg(g^j(x))^{-1}\|}{\|Dg(z)^{-1}\|} > e^{\frac{-c}{2}}.$$

Then

$$\|Dg(z)^{-1}\| < e^{\frac{c}{2}} \|Dg(g^j(x))^{-1}\|. \tag{8}$$

By connectedness, (4), and the Mean Value Theorem applied to an inverse branch g^{-1} which sends $g^{n+1}(x)$ to $g^n(x)$, for $n = 0, \dots, n_i$, we get

$$\begin{aligned} \operatorname{dist}(g^n(x), g^n(y)) &= \operatorname{dist}(g^{-1}(g^{n+1}(x)), g^{-1}(g^{n+1}(y))) \\ &\le \|Dg(z_n)^{-1}\| \operatorname{dist}(g^{n+1}(x), g^{n+1}(y)), \end{aligned}$$

with $\operatorname{dist}(z_n, g^{n+1}(x)) < \delta$. Thus, by (8) we get

$$\begin{aligned} \operatorname{dist}(x, y) &\le \prod_{n=1}^{n_i} \|Dg(g^n(x))^{-1}\| e^{\frac{c}{2} n_i} \delta \\ &< \frac{e^{-2c(n_i - 1)}}{\|Dg(g(x))^{-1}\|} e^{\frac{c}{2} n_i} \delta \\ &< \frac{e^{\frac{-3}{2} c(n_i - \frac{4}{3})}}{\|Dg(g(x))^{-1}\|} \delta. \end{aligned}$$

Since $N\theta_0 < l < n_l$ we obtain that

$$d(x, y) < \frac{e^{\frac{-3}{2} c(N\theta_0 - \frac{4}{3})}}{\|Dg(g(x))^{-1}\|} \delta.$$

As N can be chosen arbitrarily large, we get $\operatorname{dist}(x, y) = 0$. Thus $\Gamma_\delta^+(g, x) = \{x\}$ for μ-almost every $x \in M$. Therefore μ is positively expansive. This completes the proof of Theorem A. □

4. Expanding properties and proof of Theorem B

We start observing that a necessary condition for a local diffeomorphism f to be positively measure-expansive is that for all $\delta > 0$ and all $x \in M$, $\Gamma_\delta^+(f, x)$ doesn't contain any manifold of dimension greater than or equal to one.

In fact, assume that $\Gamma_\delta^+(f, x)$ contains a manifold S, $\dim(S) \ge 1$. Let m be the Lebesgue measure on S and let ν be the normalized Lebesgue measure on S; that is, for any Borel set $A \subset S$,

$$\nu(A) = \frac{m(A)}{m(S)}.$$

Define a measure μ on M in the following way: for any Borel set C of M we set

$$\mu(C) = \nu(C \cap S).$$

Clearly μ is non-atomic and $\mu(\Gamma_\delta^+(f, x)) \ge \mu(S) = 1 > 0$, which implies that f is not positively measure-expansive. □

4.1. **Main Lemma.** Recall that a periodic point p of f with period $\tau(p)$ is *expanding* if all the eigenvalues of $Df^{\tau(p)}(p)$ have absolute value greater than one. The next lemma shows that if there is a non-expanding periodic point p, then there is $\delta > 0$ such that $\Gamma_\delta^+(f,p)$ contains a manifold S with $\dim(S) \geq 1$.

Lemma A (Main Lemma)**.** *Let $g \in \mathrm{Diff}^1_{loc}(M)$, $p \in \mathrm{Per}^{\tau(p)}(g)$ be such that $Dg^{\tau(p)}(p)$ has at least one eigenvalue λ with $|\lambda| \leq 1$, and let $\delta > 0$. Then there are $h \in \mathrm{Diff}^1_{loc}(M)$ C^1-closed to g such that $h = g$ in $\mathcal{O}(p)$ and an $h^{\tau(p)}$-invariant manifold $\mathcal{I}_p \ni p$ such that*

(1) *$\mathcal{I}_p \subset \Gamma_\delta^+(h,p)$.*
(2) *If $Dg^{\tau(p)}(p) = id_{T_pM}$, there exists $\mu_h \in \mathcal{M}_h(M)$ such that $\mu_h(\mathcal{I}_p) > 0$.*

Proof. We endow M with a Riemannian metric. By [7, Lemma (1.1)] there exists a local C^1-diffeomorphism h ϵ-closed to g in the C^1-topology, such that $Dh_x = Dg_x$ for all $x \in \mathcal{O}(p)$. Moreover, there is an ϵ-neighborhood $\mathcal{U}$ of the $\mathcal{O}(p)$ such that

$$h|_{\mathcal{O}(p)\cup(M\setminus\mathcal{U})} = g|_{\mathcal{O}(p)\cup(M\setminus\mathcal{U})}.$$

Furthermore, for all $y \in \mathcal{U}$ it holds that

$$h(y) = \exp_{g^{i+1}(p)} \circ Dg(g^i(p)) \circ \exp^{-1}_{g^i(p)}(y), \tag{9}$$

where exp is the exponential map.

As a consequence of (9) we have

$$d(h(y), g^{i+1}(p)) = ||Dg(g^i(p)) \exp^{-1}_{g^i(p)}(y)||.$$

Hence, if $N := \max_{y\in M} ||Dg(y)||$, then

$$d(h(y), g^{i+1}(p)) \leq N|| \exp^{-1}_{g^i(p)}(y)|| = Nd(y, g^i(p)). \tag{10}$$

We define $\epsilon_i := \dfrac{\epsilon}{(1+N)^{\tau(p)-i}}$, $i = 0, \ldots, \tau(p)-1$. Note that $N\epsilon_{i-1} < \epsilon_i < \epsilon_{i+1}$. Let $V := B_{\epsilon_0}(p)$ and pick an arbitrary point $x \in V$.

Claim 2. *Let $d(h^i(x), g^i(p)) < \epsilon_i$ for all $i = 0, \ldots, \tau(p)-1$.*

The proof goes by induction. For $i = 0$, by definition of V, it holds that $d(x,p) < \epsilon_0$. Suppose it is true for $1 < i < \tau(p) - 1$. Then

$$d(h^i(x), g^i(p)) < \epsilon_i, \quad \text{with } \epsilon_i < \epsilon \quad \text{for} \quad 1 < i < \tau(p) - 1\,.$$

By (10) we get

$$d(h^{i+1}(x), g^{i+1}(p)) \leq Nd(h^i(x), g^i(p)) < N\epsilon_i < \epsilon_{i+1},$$

completing the induction. This proves the claim. □

Since for any $i = 0, \ldots, \tau(p)-1$ we have that $d(h^i(x), g^i(p)) < \epsilon$, we can apply (9) and get

$$h(x) = \exp_{g(p)} \circ Dg(p) \circ \exp_p^{-1}(x),$$
$$h^2(x) = \exp_{g^2(p)} \circ Dg(g(p)) \circ \exp^{-1}_{g(p)}(h(x)),$$
$$\vdots$$

and

$$h^{\tau(p)}(x) = \exp_{g^{\tau(p)}(p)} \circ Dg(g^{\tau(p)-1}(p)) \circ \exp^{-1}_{g^{\tau(p)-1}(p)}(h^{\tau(p)-1}(x)).$$

Therefore, by the chain rule,

$$h^i(x) = \exp_{g^i(p)} \circ Dg^i(p) \circ \exp_p^{-1}(x), \qquad \forall\, x \in V, \qquad i \in \{0, \dots, \tau(p) - 1\}.$$

In particular,

$$h^{\tau(p)}(x) = \exp_p \circ Dg^{\tau(p)}(p) \circ \exp_p^{-1}(x), \qquad \forall\, x \in V. \tag{11}$$

Now, let E_p be the direct sum of all eigenspaces of the eigenvalues λ, $|\lambda| \leq 1$ of $Dg^{\tau(p)}(p)$ and define $\mathcal{I}_p$ as

$$\mathcal{I}_p := \exp_p(E_p) \cap V.$$

Claim 3. *$\mathcal{I}_p$ defined as above is $h^{\tau(p)}$-invariant.*

For this, given $x \in \mathcal{I}_p$ we have $x = \exp_p(v)$ with $v \in E_p$ and $\|v\| < \epsilon_0$, and so

$$h^{\tau(p)}(x) = h^{\tau(p)}(\exp_p(v)) = \exp_p \circ Dg^{\tau(p)}(p) \circ \exp_p^{-1}(\exp_p(v)) = \exp_p \circ Dg^{\tau(p)}(p)v.$$

Modifying the Riemannian metric, if necessary, we can suppose that all the eigenspaces of the eigenvalues λ, with $|\lambda| \leq 1$, are two by two orthogonal. In particular, $\|Dg^{\tau(p)}(p)v\| \leq \|v\|$, for all $v \in E_p$. Hence, we have that

$$d(h^{\tau(p)}(x), p) = \|Dg^{\tau(p)}(p)v\| \leq \|v\| < \epsilon_0, \quad \text{and so} \quad h^{\tau(p)}(x) \in V.$$

Therefore $h^{\tau(p)}(x) \in \mathcal{I}_p$ because E_p is invariant by $Dg^{\tau(p)}(p)$. This concludes the proof that $\mathcal{I}_p$ is $h^{\tau(p)}$-invariant. □

Now we are ready to proof the lemma:

Proof of item (1). Given $\delta > 0$ and taking $\epsilon < \delta$ in the definition of $\mathcal{I}_p$ we get that

$$\begin{aligned}
\Gamma_\delta^+(h,p) &= \bigcap_{i=0}^{\infty} (h^i)^{-1}\left(B_\delta\left(h^i(p)\right)\right) \\
&= \bigcap_{k=0}^{\infty}\left(\bigcap_{j=0}^{\tau(p)-1}\left(h^{k\tau(p)+j}\right)^{-1}\left(B_\delta\left(h^{k\tau(p)+j}(p)\right)\right)\right) \\
&= \bigcap_{k=0}^{\infty}\left(\bigcap_{j=0}^{\tau(p)-1}\left(h^{k\tau(p)}\right)^{-1}\left(\left(h^j\right)^{-1}\left(B_\delta\left(h^j(p)\right)\right)\right)\right).
\end{aligned}$$

By Claim 2, $V \subset \left(h^j\right)^{-1}\left(B_{\epsilon_j}\left(h^j(p)\right)\right) \subset \left(h^j\right)^{-1}\left(B_\delta\left(h^j(p)\right)\right)$, and so

$$\begin{aligned}
\Gamma_\delta^+(h,p) &= \bigcap_{k=0}^{\infty}\left(\bigcap_{j=0}^{\tau(p)-1}\left(h^{k\tau(p)}\right)^{-1}\underbrace{\left(\left(h^j\right)^{-1}\left(B_\delta\left(h^j(p)\right)\right)\right)}_{\supset V \supset \mathcal{I}_p}\right) \\
&\supset \bigcap_{k=0}^{\infty}\left(\left(h^{k\tau(p)}\right)^{-1}(\mathcal{I}_p)\right).
\end{aligned}$$

But, since

$$h^{\tau(p)}(\mathcal{I}_p) \subset \mathcal{I}_p$$

we get

$$\left(h^{\tau(p)}\right)^{-1}(\mathcal{I}_p) \supset \mathcal{I}_p.$$

Thus, $\Gamma_\delta^+(h,p) \supset \mathcal{I}_p$, concluding the proof of item (1). □

Proof of item (2). Observe that $\mathcal{I}_p = V$ because $E_p = T_pM$, and so (11) implies that

$$h^{\tau(p)}|_V = id_V, \tag{12}$$

and hence $\left(h^{k\tau(p)}\right)^{-1}|_V = id_V$. Thus, for $\delta' < \epsilon_0$, $\Gamma^+_{\delta'}(h,p) \subset V$. In fact, we have

$$\begin{aligned}
\Gamma^+_{\delta'}(h,p) &= \bigcap_{k=0}^{\infty} \left(h^{k\tau(p)}\right)^{-1} \underbrace{\bigcap_{j=0}^{\tau(p)-1} \left(h^j\right)^{-1} \left(B_{\delta'}\left(h^j(p)\right)\right)}_{\subset B_{\epsilon_0}(p)=V} \\
&= \bigcap_{k=0}^{\infty} \left(\bigcap_{j=0}^{\tau(p)-1} id_V \left(\left(h^j\right)^{-1} \left(B_{\delta'}\left(h^j(p)\right)\right)\right)\right) \\
&= \bigcap_{k=0}^{\infty} \bigcap_{j=0}^{\tau(p)-1} \left(h^j\right)^{-1} \left(B_{\delta'}\left(h^j(p)\right)\right) \\
&= \bigcap_{j=0}^{\tau(p)-1} \left(h^j\right)^{-1} \left(B_{\delta'}\left(h^j(p)\right)\right).
\end{aligned}$$

Then $\Gamma^+_{\delta'}(h,p) = \bigcap_{i=0}^{\infty} \left(h^i\right)^{-1} \left(B_{\delta'}\left(h^i(p)\right)\right) = \bigcap_{j=0}^{\tau(p)-1} \left(h^j\right)^{-1} \left(B_{\delta'}\left(h^j(p)\right)\right)$, which is a finite intersection of open sets and so it is also an open set. Set $V' := \Gamma^+_{\delta'}(p)$.

Now, note that (12) implies that $h(A) = A$ for all invariant sets $A \subset V$. In particular, $h(V') = V'$.

Next we construct a non-atomic f-invariant probability μ_h on M. For this, we consider μ a non-atomic probability measure supported on $\mathcal{I}_p$. To this end, given a Borel set $C \subset M$, define the measure μ_h by

$$\mu_h(C) = \frac{1}{\tau(p)} \sum_{i=0}^{\tau(p)-1} \frac{\mu\left(h^i(C \cap V')\right)}{\mu(V')}.$$

Since $h(V') = V'$, we obtain

$$\begin{aligned}
\mu_h(M) &= \frac{1}{\tau(p)} \sum_{i=0}^{\tau(p)-1} \frac{\mu\left(h^i(M \cap V')\right)}{\mu(V')} = \frac{1}{\tau(p)} \sum_{i=0}^{\tau(p)-1} \frac{\mu\left(h^i(V')\right)}{\mu(V')} \\
&= \frac{1}{\tau(p)} \sum_{i=0}^{\tau(p)-1} \frac{\mu(V')}{\mu(V')} = \frac{\tau(p)}{\tau(p)} \frac{\mu(V')}{\mu(V')} = 1.
\end{aligned}$$

Hence, μ_h is a probability measure. Since

$$\begin{aligned}
\mu_h\left(h^{-1}(C)\right) &= \frac{1}{\tau(p)} \sum_{i=0}^{\tau(p)-1} \frac{\mu\left(h^i\left(h^{-1}(C) \cap V'\right)\right)}{\mu(V')} \\
&= \frac{1}{\tau(p)} \sum_{i=0}^{\tau(p)-1} \frac{\mu\left(h^i(C \cap V')\right)}{\mu(V')} = \mu_h(C),
\end{aligned}$$

we conclude that μ_h is h-invariant.

For $\mathcal{I}_p$, we obtain

$$\begin{aligned}\mu_h(\mathcal{I}_p) &= \frac{1}{\tau(p)}\sum_{i=0}^{\tau(p)-1}\frac{\mu\left(h^i(\mathcal{I}_p\cap V')\right)}{\mu(V')}\\ &= \frac{1}{\tau(p)}\sum_{i=0}^{\tau(p)-1}\frac{\mu\left(h^i(\mathcal{I}_p\cap V')\right)}{\mu(V')} \geq \frac{1}{\tau(p)}\frac{\mu(\mathcal{I}_p\cap V')}{\mu(V')}\end{aligned}$$

because $\mathcal{I}_p\cap V'$ is in this case an open subset of $V'\subset \operatorname{supp}\mu$. Therefore, we have $\mu_h(\mathcal{I}_p)>0$, finishing the proof of item (2). All together this concludes the proof of Lemma A. □

Corollary 3. *Under the same hypotheses of the lemma, if p is a periodic hyperbolic saddle, then the statement of item* (1) *can be obtained for g itself without perturbations.*

Proof. Since p is a periodic hyperbolic saddle, the local stable manifold $W^s_\varepsilon(p,g)$ is a C^1-submanifold. We will denote it by I_g. Hence we can define $\mu = m_{I_g}$ as the induced Lebesgue measure on it, and as before we get that $\mu = m_{I_g}$ is not measure-expansive, leading to a contradiction. □

4.2. **Proof of Theorem B.** Let f be a local C^1-diffeomorphism robustly positive measure-expansive map. Then there exists a neighborhood $\mathcal{U}(f)$ such that all of its elements are positive measure-expansive. Assume, by contradiction, that f is not expanding. By [2, Theorem 1.3], there exists $g\in\mathcal{U}(f)$, with $p\in\operatorname{Per}(g)$ of period $\tau(p)$, such that $Dg^{\tau(p)}(p)$ has at least one eigenvalue with modulus less than or equal to one.

By Lemma A(1), there are h near g, $\delta>0$, a submanifold $\mathcal{I}_p\subset\Gamma^+_\delta(h,p)$, and a measure $\mu\in\mathcal{M}(M)\setminus\mathcal{A}(M)$ such that $\mu(\mathcal{I}_p)>0$. This implies that h is not measure-expansive. As h is near f, this contradicts that f is robustly positive measure-expansive. This finishes the proof of Theorem B. □

4.3. **Proof of Theorem 1.** We use the same ideas as in [1, Theorem 1.1], but here we consider a dense and open set $\mathcal{R}$ defined by $\mathcal{R}=\mathcal{H}\cup\overline{\mathcal{H}}^c$, where $\mathcal{H}$ is the set

$$\mathcal{H}=\{g\in\operatorname{Diff}^1_{loc}(M);\ g \text{ has a saddle hyperbolic periodic point } q\}.$$

First, observe that $\mathcal{H}$ is open: given $g\in\mathcal{H}$ there exists a hyperbolic saddle point $q\in\operatorname{Per}_h(g)$. Then there is a neighborhood $\mathcal{U}$ of g in the C^1-topology and a continuous map $p:\mathcal{U}\to M$ with $p(g)=q$ such that for $\varphi\in\mathcal{U}$, $p(\varphi)$ is a hyperbolic saddle point of φ. Since the map $\varphi\mapsto D\varphi^{\tau(p(\varphi))}(p(\varphi))$ is continuous there is a neighborhood $\mathcal{V}\subset\mathcal{U}$ of g such that all eigenvalues of $D\varphi^{\tau(p(\varphi))}(p(\varphi))$ do not belong to the unit circle. Hence, for each $\varphi\in\mathcal{V}$, $p(\varphi)$ is a saddle hyperbolic point of φ. Therefore, $\mathcal{V}\subset\mathcal{H}$, concluding that $\mathcal{H}$ is open.

Since $\mathcal{H}$ is open, we get that $\mathcal{R}$ defined above is open and dense on $\operatorname{Diff}^1_{loc}(M)$.

Now we claim that if $f\in\mathcal{R}$ and f is μ-expansive for all $\mu\in\mathcal{M}(M)\setminus\mathcal{A}(M)$, then f is expanding.

The proof goes by contradiction. Assume that $f\in\mathcal{R}$, f is μ-expansive, and f is not expanding. Then [2, Theorem 1.3] implies that there is a sequence $\{f_n\}$ converging to f with periodic points p_n for f_n which have at least one eigenvalue with modulus less than or equal to one. Then, by [7, Lemma (1.1)], we can find

a sequence $\{g_n\}$ converging to f such that p_n is a hyperbolic saddle for every n. Hence $f \notin \overline{\mathcal{H}}^c$. Thus $f \in \mathcal{H}$ and so has a hyperbolic saddle periodic point p, and by Corollary 3, there exists a measure supported in $\mathcal{I}_p$ which is not positively expansive, leading to a contradiction. □

5. Proof of Theorem C

To prove Theorem C, recall that $\mathcal{PIM}$ is the set of C^1-diffeomorphisms f on M that are positively μ-expansive for every $\mu \in \mathcal{M}_f(M) \setminus \mathcal{A}(M)$, $\mathcal{M}_f(M)$ is the set of f-invariant measures on M, and $\mathcal{A}(M)$ is the set of atomic measures on M.

The proof goes by contradiction. Let $f \in \mathcal{D}iff^1_\omega(M)$ and let $\mathcal{U}(f)$ be an open neighborhood of f with $\mathcal{U}(f) \subset \mathcal{PIM} \cup \{f\}$ and assume that f has no dominated splitting. By [4, Theorem 6], since f has no dominated splitting there are a conservative diffeomorphism g in $\mathcal{U}(f)$ and a periodic point p of g such that $Dg^{\tau(p)}(p) = Id$, where $\tau(p)$ is the period of p.

Applying Lemma A we conclude that there exists a local C^1-diffeomorphism $h \in \mathcal{U}(f)$ such that it coincides with g in the orbit of p and for every $\delta > 0$ there exists a submanifold $\mathcal{I}_p \subset \Gamma^+_\delta(h,p)$, $\mathcal{I}_p \ni p$, and there is a probability h-invariant measure μ_h over M such that $\mu_h(\mathcal{I}_p) > 0$. Therefore $\mu_h\left(\Gamma^+_\delta(h,p)\right) > 0$ for δ arbitrarily small, leading to a contradiction (because $h \in \mathcal{PIM}$). This ends the proof of Theorem C. □

Acknowledgments

The authors thank A. Arbieto for helpful conversations on this subject, and the anonymous referee for comments and suggestions that certainly improved this text.

References

[1] Jiweon Ahn, Keonhee Lee, and Manseob Lee, *Positively measure expansive and expanding*, Commun. Korean Math. Soc. **29** (2014), no. 2, 345–349. MR3206420

[2] Alexander Arbieto, *Periodic orbits and expansiveness*, Math. Z. **269** (2011), no. 3-4, 801–807. MR2860265

[3] A. Arbieto and C. A. Morales, *Expansive measures*, Publ. Mat. Urug. **14** (2013), 61–71. MR3235344

[4] C. Bonatti, L. J. Díaz, and E. R. Pujals, *A C^1-generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sinks or sources* (English, with English and French summaries), Ann. of Math. (2) **158** (2003), no. 2, 355–418. MR2018925

[5] Jérôme Buzzi and Todd Fisher, *Entropic stability beyond partial hyperbolicity*, J. Mod. Dyn. **7** (2013), no. 4, 527–552. MR3177771

[6] Albert Fathi, *Expansiveness, hyperbolicity and Hausdorff dimension*, Comm. Math. Phys. **126** (1989), no. 2, 249–262. MR1027497

[7] John Franks, *Necessary conditions for stability of diffeomorphisms*, Trans. Amer. Math. Soc. **158** (1971), 301–308. MR0283812

[8] Koichi Hiraide, *Expansive homeomorphisms with the pseudo-orbit tracing property of n-tori*, J. Math. Soc. Japan **41** (1989), no. 3, 357–389. MR999503

[9] Keonhee Lee, Manseob Lee, Kazumine Moriyasu, and Kazuhiro Sakai, *Positively measure-expansive differentiable maps*, J. Math. Anal. Appl. **435** (2016), no. 1, 492–507. MR3423410

[10] Jorge Lewowicz, *Expansive homeomorphisms of surfaces*, Bol. Soc. Brasil. Mat. (N.S.) **20** (1989), no. 1, 113–133. MR1129082

[11] Ricardo Mañé, *Ergodic theory and differentiable dynamics*, translated from the Portuguese by Silvio Levy, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 8, Springer-Verlag, Berlin, 1987. MR889254

[12] Carlos A. Morales and Víctor F. Sirvent, *Expansive measures*, Publicações Matemáticas do IMPA. [IMPA Mathematical Publications], 29º Colóquio Brasileiro de Matemática. [29th Brazilian Mathematics Colloquium], Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2013,. MR3100422

[13] Krerley Oliveira, *Every expanding measure has the nonuniform specification property*, Proc. Amer. Math. Soc. **140** (2012), no. 4, 1309–1320. MR2869114

[14] V. I. Oseledec, *A multiplicative ergodic theorem. Characteristic Ljapunov, exponents of dynamical systems* (Russian), Trudy Moskov. Mat. Obšč. **19** (1968), 179–210. MR0240280

[15] M. J. Pacifico and J. L. Vieitez, *On measure expansive diffeomorphisms*, Proc. Amer. Math. Soc. **143** (2015), no. 2, 811–819. MR3283667

[16] Kazuhiro Sakai, *Positively expansive differentiable maps*, Acta Math. Sin. (Engl. Ser.) **26** (2010), no. 10, 1839–1846. MR2718084

[17] Kazuhiro Sakai, Naoya Sumi, and Kenichiro Yamamoto, *Measure-expansive diffeomorphisms*, J. Math. Anal. Appl. **414** (2014), no. 2, 546–552. MR3167979

[18] W. R. Utz, *Unstable homeomorphisms*, Proc. Amer. Math. Soc. **1** (1950), 769–774. MR0038022

[19] José L. Vieitez, *Lyapunov functions and expansive diffeomorphisms on 3D-manifolds*, Ergodic Theory Dynam. Systems **22** (2002), no. 2, 601–632. MR1898808

Departamento de Matemática, Universidad de las Américas and Universidad de Santiago de Chile, Santiago, Chile

Email address: almaarmijo@gmail.com

Instituto de Matemática, Universidade Federal do Rio de Janeiro, C. P. 68.530, CEP 21.945-970, Rio de Janeiro, RJ, Brazil

Email address: pacifico@im.ufrj.br