PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 9, September 2018, Pages 3955–3959
http://dx.doi.org/10.1090/proc/14096
Article electronically published on May 2, 2018

A NOTE ON ANOSOV FLOWS OF NON-COMPACT RIEMANNIAN MANIFOLDS

GERHARD KNIEPER

(Communicated by Nimish Shah)

ABSTRACT. In this note we formulate a condition for complete non-compact Riemannian manifolds, which implies no conjugate points in case that the geodesic flow is Anosov with respect to the Sasaki metric.

In 1974 Klingenberg [6] proved, using Morse theory, that geodesic Anosov flows on compact Riemannian manifolds do not have conjugate points. In 1987 Mané [8] showed with the help of the Maslov index that geodesic flows on complete Riemannian manifolds with finite volume have no conjugate points provided there exists a continuous flow invariant Lagrangian section. Since the stable and unstable bundles provide such invariant sections, geodesic Anosov flows on manifolds with finite volume do not have conjugate points as well. In the same paper, Mané claimed that any complete non-compact Riemannian manifold with lower curvature bound has no conjugate points in case the geodesic flow is Anosov with respect to the Sasaki metric. However, as we noticed in [7], the proof contains a mistake which occurs in Proposition II.2 of his article.

This note was inspired by a question of the authors of [4], whether one could prove Mané's claim under certain extra geometric conditions. The positive answer to this question would allow them to remove an assumption in their work. To formulate this result, we start by introducing some notation and definitions.

In the following, (M, g) will denote a complete Riemannian manifold, $\pi : TM \to M$ the tangent bundle with the canonical projection, and $SM = \{v \in TM \mid \|v\| = 1\}$ the unit tangent bundle with respect to the Riemannian metric g. The tangent space T_vSM of SM at $v \in SM$ is given by

$$\{(x, y) \mid x, y \in T_{\pi(v)}M, y \perp v\},$$

where we use the splitting of T_vTM into horizontal and vertical spaces. Using this decomposition, the linearization

$$D\phi^t(v) : T_{\pi v}M \times T_{\pi v}M \to T_{\pi\phi^t(v)}M \times T_{\pi\phi^t(v)}M$$

of the geodesic flow $\phi^t : SM \to SM$ is given by $D\phi^t(v)(x, y) = (J(t), J'(t))$, where $J(t)$ is the Jacobi field which is the solution of the Jacobi equation

$$J''(t) + R(J(t), \phi^t(v))\phi^t(v) = 0,$$

Received by the editors September 22, 2017, and, in revised form, December 9, 2017.
2010 *Mathematics Subject Classification.* Primary 37D40, 53C22.
Key words and phrases. Geodesic Anosov flows, conjugate points.
This work was partially supported by the German Research Foundation (DFG), CRC TRR 191, Symplectic structures in geometry, algebra and dynamics.

along the geodesic $c_v(t) = \pi(\phi^t(v))$ with initial conditions $J(0) = x, J'(0) = y \perp v$. Here $J' = \frac{D}{dt}J$ denotes the covariant derivative along c_v and $R(X,Y)Z$ the Riemannian curvature tensor. There is a natural 1-form Θ on TM defined by

$$\Theta_v(\xi) = \langle v, d\pi_v(\xi)\rangle.$$

Using the decomposition of $\xi = (x, y)$ into horizontal and vertical parts, introduced above, we obtain

$$\Theta_v(x, y) = \langle v, x\rangle.$$

The differential $d\Theta$ is the canonical symplectic form on TM and is given by

$$d\Theta_v((x_1, y_1), (x_2, y_2)) = \langle y_1, x_2\rangle - \langle y_2, x_1\rangle.$$

The canonical metric g_S on TM given by

$$g_S((x_1, y_1), (x_2, y_2)) = \langle x_1, x_2\rangle + \langle y_1, y_2\rangle$$

for $(x_1, y_1), (x_2, y_2) \in T_vTM$ is called the Sasaki metric on TM. Denote by

$$E^\phi(v) = \{(\lambda v, 0) \mid \lambda \in \mathbb{R}\}$$

the 1-dimensional space tangent to the geodesic flow at $v \in SM$. The orthogonal complement

$$N(v) := E^\phi(v)^\perp$$

in T_vSM with respect to the Sasaki metric defines a bundle N invariant under the linearization of the geodesic flow. Furthermore, N is a symplectic bundle; i.e., the symplectic form restricted to N is non-degenerate. A Lagrangian subspace $L(v) \subset N(v)$ is called a Lagrangian graph if $V(v) \cap L(v) = \emptyset$.

Lemma 1. *If $L(\phi^t(v)) = D\phi^t(v)L(v)$ is a Lagrangian graph for all $t \in [a, b]$, then $c_v : [a, b] \to M$ has no conjugate points.*

Proof. For a proof see Lemma 2.7 in section 1.2 of [7]. □

Definition 2. Let (M, g) be a complete Riemannian manifold and let $\|\cdot\|$ be the norm on TSM induced by the Sasaki metric. The geodesic flow $\phi^t : SM \to SM$ is called Anosov flow if there exist constants $k, C > 0$ and a splitting

$$T_vSM = E^s(v) \oplus E^u(v) \oplus E^\phi(v)$$

such that $E^\phi(v) = \text{span}\{X_G(v)\}$ and

$$\|D\phi^t(v)\xi\| \le C \cdot e^{-kt}\|\xi\|$$

for all $\xi \in E^s(v), t \ge 0$, as well as

$$\|D\phi^{-t}(v)\xi\| \le Ce^{-kt}\|\xi\|$$

for all $\xi \in E^u(v)$, $t \ge 0$.

Lemma 3. *Let (M, g) be a complete Riemannian manifold with lower sectional curvature bound $-\beta^2$. If the geodesic flow is Anosov with constants $k, C > 0$ as in Definition 2, there exists a constant $\sigma = \sigma(\beta, k, C)$ with the following property. If*

$$E^s(v) \cap V(v) \ne \{0\},$$

then the geodesic c_v has conjugate points on the interval $[-1, \sigma]$.

Proof. For a proof see Lemma 3.5 in section 1.3 of [7]. □

Remark. The lemma above is related to Proposition II.2 in Mané's article [8]. In this proposition he even claimed that $c_v(0)$ should be conjugated to $c_v(t_0)$ for some $t_0 > 0$. However, the proof which is based on the index form needs information about the geodesic on an interval containing 0 as an interior point.

The following theorem contains the main result of this paper.

Theorem 4. *Let (M, g) be a complete, connected, and non-compact Riemannian manifold with sectional curvature bounded from below by $-\beta^2$ such that the geodesic flow is Anosov with respect to the Sasaki metric with constants k, C as in Definition 2. Assume that the following three conditions are satisfied:*

(1) *For all $v \in SM$ there exists an open neighborhood $U(v) \subset SM$ such that $\lim_{t\to\infty} d(c(0), c(t)) = \infty$ uniformly for all geodesic $c : \mathbb{R} \to M$ with $\dot{c}(0) \in U(v)$.*
(2) *There exists a compact set $K \subset M$ such that for all $p \in M \setminus K$ and all geodesics c with $c(0) = p$ the segment $c : [-1, \sigma] \to M$ has no conjugate points with $\sigma = \sigma(\beta, k, C)$ as in Lemma 3.*
(3) *There exists at least one geodesic without conjugate points.*

Then (M, g) has no conjugate points.

Remark. Note that the existence of a geodesic without conjugate points is guaranteed if their exists a geodesic c_v which does not intersect K. Otherwise, by Lemma 1 there would exist $t \in \mathbb{R}$ such that $E^s(\phi^t(v)) \cap V(\phi^t(v)) \neq \emptyset$ and from Lemma 3 follows that $c_v : [-1 + t, \sigma + t] \to M$ has conjugate points contradicting the assumption that c_v is contained in the complement of K.

Proof. Consider the set

$$C(SM) = \{v \in SM \mid c_v : \mathbb{R} \to M \text{ has no conjugate points}\}.$$

It is known that $C(SM)$ is closed. For a proof see [8]. Now we show that $C(SM)$ is open. To prove this, consider a sequence $v_n \in SM \setminus C(SM)$ converging to v. Then by Lemma 1 there is $t_n \in \mathbb{R}$ such that $E^s(\phi^{t_n}(v_n)) \cap V(\phi^{t_n}(v_n)) \neq \emptyset$. Lemma 3 implies that $c_{v_n} : [-1 + t_n, \sigma + t_n] \to M$ has conjugate points, and from condition (2) we conclude that $c_{v_n}(t_n) \in K$. By condition (1) there exist open neighborhoods $U(v), U(-v) \subset SM$ of v such that

$$\lim_{t\to\infty} d(c_w(t), c_w(0)) = \infty$$

uniformly for all $w \in U(v)$ and $w \in U(-v)$. This implies the existence of some $T > 0$ such that for all $n \in \mathbb{N}$ with $v_n \in U(v)$ and $-v_n \in U(-v)$ we have $|t_n| \leq T$, and, therefore, $c_v : [-1 - T, \sigma + T] \to M$ has conjugate points as well. This shows that $C(SM)$ is open, and since by condition (3) the set $C(SM)$ is non-empty we have $C(SM) = SM$. Hence (M, g) has no conjugate points. □

Corollary 5. *Let (M, g) be a complete, connected, and non-compact Riemannian manifold such that the sectional curvature is bounded from below by $-\beta^2$ and such that the geodesic flow is Anosov with respect to the Sasaki metric with constants k, C as in Definition 2. Assume that the following three conditions are satisfied.*

(1) *For all* $v \in SM$ *there exists an open neighborhood* $U(v) \subset SM$ *such that* $\lim_{t\to\infty} d(c(0), c(t)) = \infty$ *uniformly for all geodesic* $c : \mathbb{R} \to M$ *with* $\dot{c}(0) \in U(v)$.
(2) *There exists a geodesic ball* $B(p, r)$ *of radius* r *about* p *such that the sectional curvature on* $M \setminus B(p, r)$ *is smaller than* $(\frac{\pi}{\sigma+1})^2$ *with* $\sigma = \sigma(k, C, \beta)$ *as in Lemma* 3.
(3) *There exists a geodesic without conjugate points.*

Then (M, g) *has no conjugate points.*

Proof. Consider $r > 0$ and $B(p, r)$ such that the sectional curvature on $M \setminus B(p, r)$ is smaller than $(\frac{\pi}{\sigma+1})^2$. Then each geodesic $c : [0, \sigma + 1] \to M \setminus B(p, r)$ has no conjugate points and the set $K = B(p, r+1)$ fulfills the assumption in the theorem above. In particular, (M, g) has no conjugate points. □

Remark. With the same argument as in the remark above a geodesic without conjugate points exists, provided there is a geodesic which does not intersect $B(p, r)$.

We would like to close this note by providing a sufficient geometric condition which implies the assumptions made in Theorem 4.

Proposition 6. *Let* (M, g_0) *be a Hadamard manifold of bounded sectional curvature, i.e., a complete simply connected Riemannian manifold with bounded non-positive sectional curvature. Furthermore, assume that there exists a geodesic ball* $B(p, r)$ *of some radius* r *about* $p \in M$ *such that outside* $B(p, r)$ *the sectional curvature is bounded by a negative constant from above, while inside a smaller ball the curvature is allowed to vanish. Then in the set of smooth metrics on* M *there is an open* C^2*-neigborhood of* g_0 *such that the assumptions of Theorem* 4 *are satisfied. In particular, these metrics can have open regions of positive sectional curvature.*

Proof. Let (M, g_0) be a Hadamard manifold fulfilling the assumption stated in the proposition. Consider for $v \in SM$ the stable and unstable subspaces

$$E^s(v) = \{(w, S(v)w) \mid w \perp v\} \text{ and } E^u(v) = \{(w, U(v)w) \mid w \perp v\}$$

of T_vSM, where $S(v)$ and $U(v)$ denote the second fundamental form of the stable and unstable horospheres (see e.g. [7] for more details). Then, $E^s(v)$ and $E^u(v)$ are transversal since otherwise there would exist a parallel orthogonal Jacobi field along the geodesic c_v contradicting our assumptions on (M, g_0) which imply that c_v has to enter regions of negative curvature. From the continuity of $S(v)$ and $U(v)$ and from the assumption that the curvature of M is negative outside $B(p, r)$ we obtain the existence of a constant $a > 0$ such that $U(v) - S(v) \geq a\,\mathrm{id}$. This yields that the geodesic flow is Anosov with respect to the Sasaki metric (see [2], [3]). Then, for a sufficiently small open C^2-neighborhood of g_0 the assumptions in Theorem 4 are satisfied. The reason is that for such a neighborhood the corresponding geodesic flows are C^1 close to the geodesic flow of g_0 and that the Anosov condition is open in the C^1-topology of vector fields. This can be proved using invariant cone families (see e.g. [5] and [1] for the original approach). Hence, for any metric in a sufficiently small C^2-neighborhood of g_0 the geodesic flow is still Anosov. Furthermore, one can choose the neighborhood such that the curvature remains negative outside the ball $B(p, r)$ and that geodesics are escaping. Moreover, there are geodesics which do not intersect $B(p, r)$ and therefore do not have conjugate points. Nevertheless, inside $B(p, r)$ the curvature can be positive. □

References

[1] D. V. Anosov, *Geodesic flows on closed Riemannian manifolds of negative curvature* (Russian), Trudy Mat. Inst. Steklov. **90** (1967), 209. MR0224110

[2] J. Bolton, *Conditions under which a geodesic flow is Anosov*, Math. Ann. **240** (1979), no. 2, 103–113. MR524660

[3] Patrick Eberlein, *When is a geodesic flow of Anosov type? I,II*, J. Differential Geometry **8** (1973), 437–463; ibid. **8** (1973), 565–577. MR0380891

[4] C. R. Graham, C. Guillarmou, G. Uhlmann and P. Stefanov: *X-ray transform and boundary rigidity for asymptotically hyperbolic manifolds*, arXiv:1709.05053v1, math.DG, 15 Sep. 2017, 1–54.

[5] Anatole Katok and Boris Hasselblatt, *Introduction to the modern theory of dynamical systems*, with a supplementary chapter by Katok and Leonardo Mendoza, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, Cambridge, 1995. MR1326374

[6] Wilhelm Klingenberg, *Riemannian manifolds with geodesic flow of Anosov type*, Ann. of Math. (2) **99** (1974), 1–13. MR0377980

[7] Gerhard Knieper, *Hyperbolic dynamics and Riemannian geometry*, Handbook of dynamical systems, Vol. 1A, North-Holland, Amsterdam, 2002, pp. 453–545. MR1928523

[8] R. Mañé, *On a theorem of Klingenberg*, Dynamical systems and bifurcation theory (Rio de Janeiro, 1985), Pitman Res. Notes Math. Ser., vol. 160, Longman Sci. Tech., Harlow, 1987, pp. 319–345. MR907897

Faculty of Mathematics, Ruhr University Bochum, 44780 Bochum, Germany

Email address: `gerhard.knieper@rub.de`