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ON THE L_r-OPERATORS PENALIZED BY $(r+1)$-MEAN CURVATURE

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Abstract. In this paper, we establish the non-positivity of the second eigenvalue of the Schrödinger operator $-\mathrm{div}\big(P_r\nabla\,\cdot\big) - W_r^2$ on a closed hypersurface Σ^n of $\mathbb{R}^{n+1}$, where W_r is a power of the $(r+1)$-th mean curvature of Σ^n, which we will ask to be positive. If this eigenvalue is null, we will have a characterization of the sphere. This theorem generalizes the result of Harrell and Loss proved to the Laplace-Beltrame operator penalized by the square of the mean curvature.

1. Introduction

This work is based on the ideas presented by Harrell and Loss in [4]. We obtain an elegant and more simplified proof that allowed us to generalize their results to a more general class of operators, L_r, penalized by a power of $(r+1)$-th mean curvature. In 1997, Harrell and Loss obtained the following rigidity result.

Theorem 1.1. *Let Ω be a smooth compact oriented hypersurface of dimension d immersed in $\mathbb{R}^{d+1}$; in particular self-intersections are allowed. The metric on that surface is the standard Euclidean metric inherited from $\mathbb{R}^{d+1}$. Then the second eigenvalue λ_2 of the operator*

$$H = -\Delta - \frac{1}{d}h^2$$

is strictly negative unless Ω is a sphere, in which case λ_2 equals zero.

Here h is the mean curvature of the immersion. In particular, when $d = 2$ the previous result gives a proof for a conjecture of Alikakos and Fusco about hypersurfaces embedded in $\mathbb{R}^3$.

The aim of this paper is to extend this result for a more general class of elliptic geometric operators. To present our main result, we need to introduce some definitions and notation.

Let $\phi\colon M^n \to \overline{M}^{n+1}$ be an isometric immersion, and denote by $\mathbf{A}$ the second fundamental form associated to ϕ. It is known that $\mathbf{A}$ has n-geometric invariants. They are given by the elementary symmetric functions S_r of the principal curvatures $\kappa_1, \ldots, \kappa_n$ as follows:

$$S_r := \sum_{i_1 < \cdots < i_r} \kappa_{i_1} \ldots \kappa_{i_r} \quad (1 \le r \le n).$$

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The r-curvature H_r of ϕ is then defined by

$$H_r := \frac{S_r}{\binom{n}{r}}.$$

Notice that H_1 corresponds to the mean curvature and H_n to the Gauss-Kronecker curvature of ϕ. Newton's transformations of ϕ are the operators P_r defined inductively by

$$P_0 = I,$$

$$P_r = S_r I - \mathbf{A} P_{r-1}.$$

The so-called L_r-operators are defined by $L_r := \operatorname{div}\big(P_r \nabla \cdot\big)$. It is known that if every H_r is positive, then L_r is elliptic by Proposition 3.2 in [3].

Let Σ be a compact hypersurface of $\mathbb{R}^{n+1}$ with the operator L_r being elliptic. We have that $-L_r$ is a positive, unbounded, self-adjoint operator with the spectrum formed only by eigenvalues

$$\sigma(-L_r) = \{0 = \lambda_1(-L_r) < \lambda_2(-L_r) \leq \ldots\}.$$

We consider the following class of Schrödinger operators:

$$\mathcal{L}_r := -L_r - W_r^2,$$

where the potential $W_r = \big(c_r\, H_{r+1}^{\frac{r+2}{r+1}}\big)^{1/2}$ and $c_r = (n-r)\binom{n}{r}$, with $0 \leq r \leq n-1$.

Now we can present the main result of this article.

Theorem 1.2. *Let Σ be an n-dimensional closed hypersurface embedded in $\mathbb{R}^{n+1}$. Assume that $H_{r+1} > 0$. Then the second eigenvalue of $\mathcal{L}_r$, $\lambda_2(\mathcal{L}_r)$, is strictly negative unless Σ is a sphere; in this case $\lambda_2(\mathcal{L}_r)$ equals zero.*

Note that the potential W_r^2 has dimension $(vol.\ \Sigma)^{-(r+2)}$, the same as the differential operator L_r. The implies that the number of negative eigenvalues is independent of the volume of the hypersurface.

The proof is based on the following principle:

Lemma 1.3 (Birman-Schwinger's principle)**.** *Let $L = \operatorname{div}(A(x)\nabla.)$, where $A(x)$ is a matrix uniformly elliptic and $L : H^2(\Omega) \to L^2(\Omega)$, for Ω a bounded domain.*

Consider the self-adjoint operator $-L - W^2(x)$, where W^2 is relatively bounded with respect to $-L$ (i.e., $Dom(-L) \subset Dom(W^2)$ and there exist constants $a, b \geq 0$ such that

$$\|W^2 u\|_2 \leq a\|u\|_2 + b\| - Lu\|_2, \ \ \textit{for all } u \in Dom(-L)).$$

A number $-\mu < 0$ is an eigenvalue of $-L - W^2$ if and only if 1 is an eigenvalue of the bounded positive operator

$$K_\mu := W(-L + \mu)^{-1} W.$$

This result can be obtained as a corollary of a more general principle demonstrated by Klaus in the paper [5].

2. Proof of Theorem 1.2

For the above proof, the following lemma will be used.

Lemma 2.1. *Let Σ be an n-dimensional closed hypersurface embedded in $\mathbb{R}^{n+1}$ with $H_{r+1} > 0$ and consider the operator $\mathcal{L}_r = -L_r - W_r^2$. Suppose there exists $f \in L^2(\Sigma)$ satisfying:*

(1) $\langle f, W_r\rangle = 0$;

(2) $\langle R_0(W_r f), W_r f\rangle > \|f\|_2^2$, where $\langle\ ,\ \rangle$ is the inner product in $L^2(\Sigma)$, $R_0 = (-L_r|_{[1]^\perp})^{-1}$, and

$$[1]^\perp = \{u \in L^2(\Sigma); \langle u, 1\rangle = 0\}.$$

Then the operator $\mathcal{L}_r$ has two negative eigenvalues.

Proof. The proof is herein presented in the three following steps:

Recall that for $\mu > 0$ the resolvent operator $R_\mu = (-L_r + \mu)^{-1}$ is a bounded operator in $L^2(\Sigma)$.

Step 1. For $g \in L^2(\Sigma)$ with $\int_\Sigma g\, d\Sigma = 0$, we have

$$\lim_{\mu\to 0} \|R_0 g - R_\mu g\|_2 = 0.$$

In fact, if

$$(-L_r + \mu)^{-1} g = \varphi, \tag{2.1}$$

then

$$-L_r\varphi + \mu\varphi = g. \tag{2.2}$$

Therefore $\int_\Sigma \varphi d\Sigma = 0$ since $\int_\Sigma L_r\varphi d\Sigma = 0$. The last integral follows from the divergence theorem. Applying $(-L_r)^{-1}$ in equation (2.2), we obtain that

$$\varphi + \mu(-L_r)^{-1}\varphi = (-L_r)^{-1} g$$

and therefore

$$\begin{aligned} \|R_0 g - R_\mu g\|_2 = \|(-L_r)^{-1} g - (-L_r+\mu)^{-1} g\|_2 &= \|\mu(-L_r)^{-1}\varphi\|_2 \\ &\le \mu\|{-L_r}^{-1}\|\|\varphi\|_2 \\ &= \mu\|{-L_r}^{-1}\|\|(-L_r+\mu)^{-1} g\|_2. \end{aligned} \tag{2.3}$$

Now in order to estimate the norm $\|(-L_r+\mu)^{-1} g\|_2$, we will multiply the two sides of (2.2) by φ and apply the divergence theorem to obtain

$$\int_\Sigma \langle P_r\nabla\varphi, \nabla\varphi\rangle d\Sigma + \mu\int_\Sigma \varphi^2 d\Sigma = \int_\Sigma \varphi g d\Sigma. \tag{2.4}$$

Since φ has zero mean using the characterization of the Min-Max Principle $\lambda_2(-L_r)$, we obtain

$$\lambda_2(-L_r)\int_\Sigma \varphi^2 d\Sigma \le \int_\Sigma \langle P_r\nabla\varphi, \nabla\varphi\rangle d\Sigma.$$

On the other hand, the arithmetic-geometric mean inequality gives us

$$\int_\Sigma \varphi g d\Sigma \le \left(\frac{1}{4\varepsilon}\int_\Sigma g^2 d\Sigma + \varepsilon\int_\Sigma \varphi^2 d\Sigma\right).$$

Thus for $\varepsilon = (\lambda_2(-L_r)+\mu)/2$ we have

$$\int_\Sigma \varphi^2 d\Sigma \le \frac{1}{(\lambda_2(-L_r)+\mu)^2}\int_\Sigma g^2 d\Sigma.$$

Therefore,

$$\|R_\mu g\|_2 \le \frac{1}{(\lambda_2(-L_r)+\mu)}\|g\|_2.$$

From the estimate above, it follows that

$$\|R_\mu g - R_\gamma g\|_2 \le \frac{|\mu-\gamma|\|g\|_2}{(\lambda_2(-L_r)+\mu)(\lambda_2(-L_r)+\gamma)}.$$

Set $K_\mu := W_r(-L_r+\mu)^{-1}W_r$. Now for μ positive and sufficiently small, K_μ has an eigenvalue greater than 1.

Step 2. There exists $-\mu_1 < 0$ such that $\|K_{\mu_1}|_{[W_r]^\perp}\| > 1$.

The operator $K_\mu|_{[W_r]^\perp}$ is compact, symmetric, and positive; therefore $\|K_\mu|_{[W_r]^\perp}\|$ is an eigenvalue of $K_\mu|_{[W_r]^\perp}$.

We know from (2) that

$$\langle R_0(W_r f), W_r f\rangle > \|f\|_2^2$$

and $K_\mu \to K_0$ in $\mathcal{B}([W_r]^\perp)$ when $\mu \to 0$ with $K_0 = W_r R_0 W_r$. Then we have $\|K_0|_{[W_r]^\perp}\| > 1$, so there exists $-\mu_1 < 0$ such that $\|K_{\mu_1}|_{[W_r]^\perp}\| > 1$.

Step 3. Step 2 implies the lemma.

Since K_μ is positive, we have $\|K_\mu\|$ is the largest eigenvalue of K_μ. Furthermore,

$$\|K_\mu\| \le \frac{1}{\lambda_2(-L_r|_{H^2(\Sigma)\cap[1]^\perp})+\mu}\|W_r\|_\infty^2. \tag{2.5}$$

Thus, the eigenvalue $\|K_\mu\| \to 0$ when $\mu \to \infty$. Therefore, there is $-\mu_2 < 0$ such that $\|K_{\mu_2}\| < 1$.

Hence, we show that there exist μ_2 and μ_1 constants such that

$$\|K_{\mu_2}\| < 1 < \|K_{\mu_1}\|,$$

and $\mu \mapsto \|K_\mu\|$ is continuous. By the Intermediate Value Theorem, there exists $-\mu_0$ such that $\|K_{\mu_0}\| = 1$.

Thus by Birman-Schwinger's principle, we have that $-\mu_0 < 0$ is an eigenvalue of $\overline{\mathcal{L}_r} = \mathcal{L}_r|_{[1]^\perp}$; i.e., there exists a non-zero function $f \in H^2(\Sigma)\cap[1]^\perp$ such that $\overline{\mathcal{L}_r} f = -\mu_0 f$. Thus $-\mu_0$ is also an eigenvalue of the operator $\mathcal{L}_r$.

Suppose by contradiction that $-\mu_0$ is the only negative eigenvalue of $\mathcal{L}_r$.

In this case, $-\mu_0$ would be the first eigenvalue with a first self-space given by $[f] = \{c\,f;\ c \in \mathbb{R}\}$, and $\mathcal{L}_r$ restricted to the subspace $[f]^\perp$ should be a positive operator. On the other hand, we have $f \in [1]^\perp$. Thus, the constant function $1 \in [f]^\perp$, implying that $\langle \mathcal{L}_r 1, 1\rangle_2 \ge 0$, a contradiction.

Hence the operator $\mathcal{L}_r$ has more than one negative eigenvalue if there is $f \in L^2(\Sigma)$ satisfying (1) and (2). □

Now we present the proof of Theorem 1.2.

Proof. Let $\phi : \Sigma^n \to \mathbb{R}^{n+1}$ be an isometric immersion. By [1], we have the following equation satisfied:

$$-L_r\phi = c_r H_{r+1} N, \tag{2.6}$$

where N is the normal vector of the surface. Thus each coordinate satisfies

$$-L_r\phi_i = c_r H_{r+1} N_i, \tag{2.7}$$

with $i \in \{1, \dots, n+1\}$.

Denote

$$(\phi_i)_\Sigma := \frac{1}{vol.\ \Sigma} \int_\Sigma \phi_i d\Sigma$$

and $(\phi)_\Sigma := ((\phi_1)_\Sigma, \dots, (\phi_{n+1})_\Sigma)$.

Choosing f_i so that

$$f_i W_r = c_r H_{r+1} N_i, \tag{2.8}$$

we have

$$f_i = (c_r H_{r+1}^{\frac{r}{r+1}})^{\frac{1}{2}} N_i \tag{2.9}$$

and $\langle f_i, W_r \rangle = 0$, by (2.7).

Observe that

$$R_0(W_r f_i) = R_0(c_r H_{r+1} N_i) = R_0(-L_r(\phi_i - (\phi_i)_\Sigma)) = \phi_i - (\phi_i)_\Sigma.$$

By multiplying both sides equal to $W_r f_i$ and using the divergence theorem, we conclude that

$$\langle R_0(W_r f_i), W_r f_i \rangle_2 = \langle P_r \nabla \phi_i, \nabla \phi_i \rangle_2 = \int_\Sigma c_r H_{r+1} (\phi_i - (\phi_i)_\Sigma) N_i d\Sigma.$$

Summing up both sides with i varying from 1 to $n+1$, we have

$$\sum_{i=1}^{n+1} \langle R_0(W_r f_i), W_r f_i \rangle_2 = \sum_{i=1}^{n+1} \langle P_r \nabla \phi_i, \nabla \phi_i \rangle_2 = \int_\Sigma c_r H_{r+1} \langle \phi - (\phi)_\Sigma, N \rangle d\Sigma.$$

We know from Minkowski's integral formula that

$$\int_\Sigma H_r d\Sigma - \int_\Sigma H_{r+1} \langle \phi - (\phi)_\Sigma, N \rangle d\Sigma = 0.$$

Thus, replacing the previous expression, we have

$$\sum_{i=1}^{n+1} \langle R_0(W_r f_i), W_r f_i \rangle_2 = \sum_{i=1}^{n+1} \langle P_r \nabla \phi_i, \nabla \phi_i \rangle_2 = \int_\Sigma c_r H_r d\Sigma.$$

By [1], using the classical inequality $H_r^{\frac{1}{r}} \geq H_{r+1}^{\frac{1}{r+1}}$, for $r \geq 1$, we have

$$\begin{aligned} \sum_{i=1}^{n+1} \langle R_0(W_r f_i), W_r f_i \rangle_2 = \int_\Sigma c_r H_r d\Sigma \geq \int_\Sigma c_r H_{r+1}^{\frac{r}{r+1}} d\Sigma = \sum_{i=1}^{n+1} \int_\Sigma c_r H_{r+1}^{\frac{r}{r+1}} N_i^2 d\Sigma \\ = \sum_{i=1}^{n+1} \|f_i\|_2^2. \end{aligned}$$

Remark 2.2. If $r = 0$, we have written the sums above as being identical, and the only step that does not appear is the gap between the bends. However it is easy to see that the rest of the argument follows analogously to other cases.

Define $d_i = \langle R_0(W_r f_i), W_r f_i \rangle_2 - \|f_i\|_2^2$. Thus $\sum_{i=1}^{n+1} d_i \geq 0$, and then two possibilities may occur:

(i) There is $i \in \{1, \dots, n+1\}$ such that $d_i > 0$;

(ii) $d_i = 0$, for all $i \in \{1, \dots, n+1\}$.

If (i) occurs, we have that f_i satisfies the hypotheses (1) and (2) of Lemma 2.1 and therefore

$$\lambda_2(\mathcal{L}_r) < 0.$$

If (ii) occurs, we have all the d_i void. In this case we use Lagrange multipliers.

Now consider the functionals $\Psi, \Phi : L^2(\Sigma) \to \mathbb{R}$ given by

$$\Psi(f) = \langle R_0(W_r f), W_r f\rangle - \|f\|_2^2, \qquad \Phi(f) = \langle W_r, f\rangle_2,$$

and the set of constraints

$$S = \{f \in L^2(\Sigma);\ \Phi(f) = \langle W_r, f\rangle_2 = 0\}.$$

We have to study two possibilities:

(a) $\inf\{\Psi(f);\ f \in S\} < 0$ or

(b) $\inf\{\Psi(f);\ f \in S\} = 0$.

In the first case, there is a function $f \in S$ such that $\Psi(f) < 0$ and f is a critical function for Ψ on S. Then the method of Lagrange multipliers exists for $\Gamma \in \mathbb{R}$, such that

$$\Psi'(f) = \Gamma\Phi'(f),$$

which results in the Euler-Lagrange equation

$$W_r R_0(W_r f) - f = \Gamma W_r.$$

Multiplying both sides of the above equation by f and integrating, we have

$$0 = \Gamma\langle W_r, f\rangle = \langle R_0(W_r f), W_r f\rangle - \|f\|_2^2 < 0.$$

This is a contradiction, and the case (a) does not occur. In the second case, we have seen that each $f_i \in S$ and $\Psi(f_i) = \inf\{\Psi(f);\ f \in S\} = 0$. By the Method of Lagrange Multipliers, there exists $\Gamma \in \mathbb{R}$ such that $\Psi'(f_i) = \Gamma\Phi'(f_i)$. Hence, we obtain that each f_i satisfies the following Euler-Lagrange equation:

$$W_r R_0(W_r f_i) = f_i + \Gamma W_r.$$

Therefore we conclude that

$$W_r(R_0(W_r f_i) - \Gamma) = f_i,$$
$$W_r(\phi_i - (\phi_i)_\Sigma - \Gamma) = f_i;$$

then

$$\phi_i - (\phi_i)_\Sigma - \Gamma = \frac{f_i}{W_r} = H_{r+1}^{\frac{1}{r+1}} N_i.$$

Thus, we have its version vector

$$\phi - (\phi)_\Sigma - \Gamma = H_{r+1}^{\frac{1}{r+1}} N.$$

Differentiating the above expression along any curve Σ, we conclude that the derivative of $H_{r+1}^{\frac{1}{r+1}}$ is zero, so H_{r+1} is constant. Then Σ is a sphere by Alexandrov's Theorem in [6].

In fact in this case we have $\lambda_2(\mathcal{L}_r) = 0$, as we have

$$W_r(\phi_i - (\phi_i)_\Sigma - \Gamma) = f_i,$$

and multiplying both sides by the expression W_r, we obtain

$$W_r^2(\phi_i - (\phi_i)_\Sigma - \Gamma) = W_r f_i = -L_r(\phi_i - (\phi_i)_\Sigma - \Gamma).$$

Thus $\psi = \phi_i - (\phi_i)_\Sigma - \Gamma$ is the second eigenfunction of $\mathcal{L}_r = -L_r - W_r^2$, and $\mathcal{L}_r\psi = 0$. $\square$

Finally, we observe that the result obtained is also valid for the operator

$$T_r = -L_r - c_r \|\mathbf{A}\|^{r+2}.$$

Corollary 2.3. *Under the same conditions of Theorem* 1.2, $\lambda_2(T_r) \leq 0$ *with equality if and only if* Σ *is a sphere.*

The proof of the corollary follows immediately from Jensen's inequality and the min-max principle. This finishes the proof. □

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