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A MOIRÉ PATTERN ON SYMMETRIC SPACES OF THE NONCOMPACT TYPE

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Abstract. We prove that if X is a symmetric space of the noncompact type, just as adding Helgason waves which propagate in all directions will yield an elementary spherical function for X, a Helgason wave can be produced by adding elementary spherical functions whose centers describe a horocycle in X.

1. Introduction

A *moiré pattern* is a visual effect obtained by superimposing plane motifs which are obtained from one another through small Euclidean motions. Moiré patterns often occur in image processing (see [1, 7]), but they also appear in other contexts. Let us start by describing a possible use in neuroscience [11–13] which is the motivation for this short paper.

1.1. On the way from the retina to the primary visual cortex, the visual information is conveyed by the Lateral Geniculate Nucleus (hereafter abridged as LGN). The specialization of neurons in either of these two areas can be described with the help of a *receptive profile*: discarding many important details, one can roughly attach to each neuron a function $\mathcal{R}$ on the visual plane $\mathbf{V}$, with the property that if the input image at time t is described as a function $\mathcal{I}_t$, then the $\mathbf{L}^2$ scalar product between $\mathcal{R}$ and $\mathcal{I}$ is a reasonable description of the electrical activity of the neuron shortly after t.

It is well acknowledged that the receptive profiles of LGN cells have *spherical* symmetry: to each cell is attached a point x_0 of the visual plane, and the receptive profile $\mathcal{R}_{LGN}$ of that cell is a function of the distance to x_0. In addition, as the distance to x_0 grows, $\mathcal{R}_{LGN}$ decreases to zero, then becomes negative in a region in which the presence of light therefore has an inhibitory effect on the cell, then grows again, and tends to zero as the distance to x_0 grows again. A famous suggestion for $\mathcal{R}_{LGN}$ is a mexican hat function, that is, the Laplacian of a Gaussian function.

In the primary visual cortex, however, the receptive profiles do not have spherical symmetry: they have a preferred direction, and the natural candidates for the receptive profiles are products of a *plane wave* with a function which decreases with the distance to a preferred position (popular models include Gabor wavelets,

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products of a plane wave with a Gaussian envelope, or Marr wavelets, products of a plane wave with a mexican hat function).

The transition from the LGN to V1 then involves a change in the symmetry of the receptive profiles. What is the biological basis for this transition?

Hubel and Wiesel famously proposed that the answer lies in the wiring of neurons: if a given V1 neuron receives inputs from LGN cells which have their centers of symmetry *aligned and close to one another*, and if the combination of LGN inputs is a simple summation, then a directional preference can emerge through a Moiré-like pattern (see Figure 1).

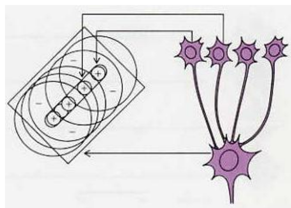

FIGURE 1. Hubel and Wiesel's scenario for the transition between the receptive profiles in the LGN and those in V1: if the electrical activity of a neuron in V1 is close to the sum of activities exhibited by LGN cells which have the centers of their receptive profiles distributed on a small segment of the visual plane, the cortical neuron will have a clear orientation preference.

1.2. It is possible to emphasize the role of symmetries in Hubel and Wiesel's argument, by changing the models for the receptive profiles and relaxing the condition that the receptive profiles decrease at infinity. In fact, if we drop that realistic but symmetry-independent requirement, a natural mathematical counterpart to Hubel and Wiesel's argument sits inside the structure of the Euclidean group. We shall say what the symmetry-based counterpart is first, and then indicate how it is related to the structure of the Euclidean motion group.

As our translation-invariant receptive profile, choose a plane wave $x \mapsto e^{i\langle Ru, x\rangle}$, where R is a positive number and u a unit vector in $\mathbb{R}^2$. As our rotation-invariant receptive profile, choose the Bessel function $J_R := x \mapsto \int_{\mathbb{S}^1} e^{i\langle Ru, x\rangle} du$ (here the Haar measure on $\mathbb{S}^1$ is normalized so as to have total mass one).

We claim that a plane wave can be reconstructed by the constructive interefer-ence of Bessel functions whose centers of symmetry are distributed on a straight line whose direction is orthogonal to that of the wave's propagation. Let us first observe a picture of the constructive interference (Figure 2).

Now, suppose $u^{\perp}$ is a unit vector orthogonal to u. We shall now indicate how the elementary properties of the Fourier transform, in particular the fact that the Fourier transform of the Dirac distribution on the line $\mathbb{R}u$ is the Dirac distribution on the orthogonal line $\mathbb{R}u^{\perp}$, imply

$$\int_{\mathbb{R}} J_R(x + tu)dt = \frac{2}{R}\cos\left(R\langle x, u^{\perp}\rangle\right), \tag{1}$$

an apparently natural guess in view of Figure 2.

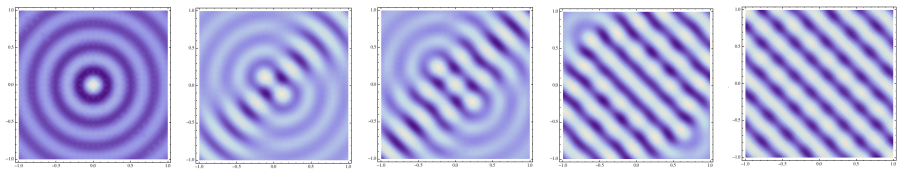

FIGURE 2. On the left is a Bessel function; left to right are superpositions of 3, 5, 11, 21 Bessel functions whose centers of symmetry lie along the obvious line.

The equality cannot hold in the usual, strong sense: J_R is not an integrable function (if it were its Fourier transform would be continuous, whereas it is the Dirac distribution on the circle of radius R), and the left-hand side of (1) is not an absolutely convergent integral. The left-hand side does have a meaning as an improper integral, because when x and u are fixed, we have

$$\int_{-A}^{A} J_R(x+tu)dt = \int_{[-A,A]} dt \int_{\mathbb{S}^1} e^{iR\langle x+tu,v\rangle} dv = \int_{\mathbb{S}^1} e^{iR\langle x,v\rangle} \int_{[-A,A]} dt e^{iR\langle tu,v\rangle}$$
$$= 2\int_{\mathbb{S}^1} e^{iR\langle x,v\rangle} \frac{\sin\left(AR\langle u,v\rangle\right)}{R\langle u,v\rangle} dv,$$

which does have a limit as A goes to infinity; however, because of the stationary-phase lemma (applicable here because $v \mapsto \langle u,v\rangle$ admits only two critical points on $\mathbb{S}^1$, and that these are nondegenerate), there is a constant ℓ such that

$$\int_{-A}^{A} J_R(x+tu)dt \underset{A\to\infty}{\sim} \frac{\ell}{\sqrt{A}}.$$

So the limit is zero.

1.3. If we are to interpret Figure 2 with the help of (1), we have to find a weaker meaning for the left-hand side. We shall now argue that it is best to interpret (1) as an equality of distributions.

When ψ is a Schwartz function on $\mathbb{R}^2$, we can write

$$\int_{x+\mathbb{R}u} \psi J_R = \int_{(x+\mathbb{R}u)\times\mathbb{S}^1} \psi(y) e^{Ri\langle y,v\rangle} dydv = \int_{\mathbb{S}^1} \mathcal{F}\left(\psi\delta_{x+\mathbb{R}u}\right)[Rv]dv, \tag{2}$$

where δ_{x+Ru} is the Dirac distribution on the line $x+\mathbb{R}u$ and $\mathcal{F}$ is the Euclidean-Fourier transform. We shall assume the position x and the direction u to be fixed here.

Now choose a family $(\vartheta_\varepsilon)_{\varepsilon>0}$ of Schwartz functions on $\mathbb{R}^2$ which, as ε goes to zero, goes in the space $\mathcal{S}'(\mathbb{R}^2)$ of tempered distributions on $\mathbb{R}^2$ to the Dirac distribution $\delta_{x+\mathbb{R}u}$ over the line $D = x+\mathbb{R}u$: one can for instance start from a smooth, nonnegative-valued, compactly supported function ϖ on $\mathbb{R}$ which is identically one in a neighborhood of the identity and has integral one, and then set $\vartheta_\varepsilon(y) = \frac{1}{\varepsilon}\varpi\left(\frac{d(y,D)}{\varepsilon}\right)\varpi\left(\varepsilon d(y,0)\right)$, where d is the Euclidean distance in $\mathbb{R}^2$.

Then as ε goes to zero, the Fourier transform $\mathcal{F}(\vartheta_\varepsilon)$ goes to that of $\delta_{x+\mathbb{R}u}$, which is the product $\left(\xi \mapsto e^{i\langle\xi,x\rangle}\right)\delta_{\mathbb{R}u^\perp}$ between a plane wave and the Dirac distribution on $\mathbb{R}u^\perp$.

Assigning to a tempered distribution T on $\mathbb{R}^2$ the distribution on $\mathbb{R}_\star^+$ which sends a smooth and compactly supported function α on $\mathbb{R}_\star^+$ to the number $\langle T, \tilde{\alpha}\rangle$, where $\tilde{\alpha}$ is the radial function on $\mathbb{R}^2$ built on α and the bracket is the duality bracket, we obtain a map I from $\mathcal{S}'(\mathbb{R}^2)$ to the space $\mathcal{D}'(\mathbb{R}_\star^+)$ of distributions on $\mathbb{R}_\star^+$. Noticing by a polar change of coordinates that $R \mapsto R\left(\int_{\mathbb{S}^1} \mathcal{F}(\vartheta_\varepsilon)[Rv]dv\right)$ is the image under I of $\mathcal{F}(\vartheta_\varepsilon)$, and that I is continuous with respect to the natural topologies of $\mathcal{S}'(\mathbb{R}^2)$ and $\mathcal{D}'(\mathbb{R}_\star^+)$, we thus see that $R \mapsto \int_{\mathbb{S}^1} \mathcal{F}(\vartheta_\varepsilon)[Rv]dv$ has a limit in $\mathcal{D}'(\mathbb{R}_\star^+)$ as ε goes to zero. The previous calculation shows that the limit is in fact the continuous map

$$R \mapsto \frac{1}{R}\int_{\mathbb{S}^1} e^{i\langle x,v\rangle}\delta_{\mathbb{R}u^\perp}[Rv]dv = \frac{1}{R}e^{i\langle x,Ru^\perp\rangle} + \frac{1}{R}e^{i\langle x,-Ru^\perp\rangle} = \frac{2}{R}\cos\left(R\langle x,u^\perp\rangle\right).$$

We can thus interpret (1) as identifying the limit, in $\mathcal{D}'(\mathbb{R}_\star^+)$, of $R \mapsto \int_{\mathbb{R}^2} \vartheta_\varepsilon J_R$ as ε goes to zero in $\mathbb{R}$ and thus ϑ_ε goes to $\delta_{x+\mathbb{R}u}$ in $\mathcal{S}'(\mathbb{R}^2)$. This may seem far-fetched, but Figure 2 is there to remind us that the interpretation can be rather convincing; in addition, the fact that the biological receptive profiles R_{LGN} do rapidly decrease at infinity makes it all the more natural in our context to consider (2) before going over to (1).

1.4. We alluded above to the fact that the plane waves and the Bessel function J_R sit inside the structure of the Euclidean motion group: if one starts with the space of smooth (and, say, bounded) functions on $\mathbb{R}^2$, equipped with the natural action of the Euclidean motion group, and if one looks for the invariant subspaces, the space of functions whose Fourier transform is concentrated on a circle of radius R appears as an irreducible invariant subspace (for details, see for instance the Introduction of [9], especially Theorem 2.6). The above special functions are the only elements in that space which are invariant under a one-dimensional Lie subgroup of the Euclidean group: the plane wave $x \mapsto e^{i\langle Ru,x\rangle}$ is, along with its conjugate and the linear combinations of the two, the only element invariant under $\mathbb{R}u$, and the Bessel function $x \mapsto J_R(x-x_0)$ is (up to a scalar multiplication) the only function invariant under the subgroup of rotations around x_0.

In this short paper, we show that the tools of noncommutative harmonic analysis make it possible to exhibit a similar Moiré pattern on a special class of negatively-curved homogeneous spaces–the symmetric spaces of noncompact type.

2. Notation

Suppose G is a real, connected, noncompact semisimple Lie group with finite center.

2.1. We write $\mathfrak{g}$ for the Lie algebra of G, fix a maximal compact subgroup K, and choose Lie subgroups A and N (the Lie algebra of A will read $\mathfrak{a}$) so that $G = KAN$ is an Iwasawa decomposition of G. The choice of N comes with a choice of a positive root system for the pair $(\mathfrak{g}, \mathfrak{a})$; we write ρ for the corresponding half-sum of positive roots, and $\mathfrak{a}_+^\star$ for the corresponding (open) positive Weyl chamber in the dual $\mathfrak{a}^\star$.

We write M for the centralizer of A within K, and B for the compact quotient K/M. We assume the Haar meaure of K to be normalized in such a way that the volume of B is one. We also assume G-invariant measures to have been chosen on

G and the various subgroups and quotients in a coherent manner (see Helgason [9]). The integrations to come will be performed with respect to these invariant measures, lest some precision be given.

2.2. Let X be the Riemannian symmetric space (of the noncompact type) G/K. Let us follow Helgason [10] in calling the orbit in X of any conjugate of N a *horocycle*. If $g = k_0 a_0 n_0$ is in G, the subgroup $gNg^{-1} = k_0 N k_0^{-1}$ depends only on the image $b_0 = k_0 M$ of k_0 in B; let us say that the orbits of gNg^{-1} in X have *direction* b_0.

When x is in X and b is in B, let us write $\xi(b, x)$ for the horocycle through x with direction b; it is the orbit of x under the N-conjugate corresponding to b as above.

Suppose b is in B and $x = naK$ is in X; let us use the Iwasawa projection $\mathcal{A} : G \mapsto \mathfrak{a}$ (defined as $nak \mapsto log_A(a)$) and set $\Delta(b, x) = \mathcal{A}(b^{-1}\tilde{x}) \in \mathfrak{a}$, where $\tilde{x}$ is any lift of x in G. The element $\Delta(b, x)$ of $\mathfrak{a}$ depends only on the horocycle $\xi(b, x)$: when $\xi(b, x)$ and $\xi(b', x')$ coincide, so do $\Delta(b, x)$ and $\Delta(b', x')$.

2.3. When G equals $SU(1, 1)$ and acts through homographies on the open unit disk $\mathbb{D}$ in $\mathbb{C}$, the stabilizer for the origin 0 is a maximal compact subgroup K of G, isomorphic to $SO(2)$; the horocycles in $\mathbb{D}$ (whose above definition depends only on the choice of K) are the circles which are tangent to $\mathbb{D}$ at a point of its boundary (see Figure 3); it is then natural to identify the direction of a horocycle with the tangency point.

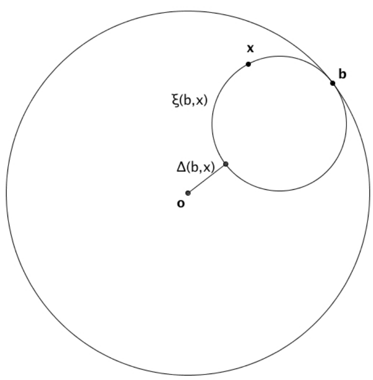

FIGURE 3. A horocycle in $\mathbb{D}$.

3. The Fourier-Helgason transform on X

3.1. Suppose λ is in $\mathfrak{a}^\star$ and b is in B. Set

$$\begin{aligned} e_{\lambda,b} : X &\to \mathbb{R} \\ x &\mapsto e^{\langle i\lambda + \rho \mid \Delta(b,x) \rangle}. \end{aligned}$$

The function $e_{\lambda,b}$ takes a single value on each horocycle with direction b (see Figure 4). It is a building block for G-invariant harmonic analysis on X in much the same way as plane waves are for Fourier analysis on Euclidean space.

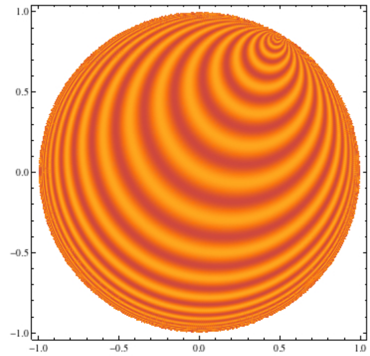

FIGURE 4. This is a plot of the *phase* levels of a Helgason wave (the growth factor has been deleted).

3.2. When f is a map from X to $\mathbb{C}$, the Fourier-Helgason transform of f is the map

$$\hat{f} : (\lambda \in \mathfrak{a}^\star, b \in K/M) \mapsto \int_X e_{-\lambda,b}(x) f(x) dx,$$

defined on the subset of $\mathfrak{a}^\star \times K/M$ where the above integral converges; it is for instance defined on all of $\mathfrak{a}^\star \times K/M$ if f is a smooth function with compact support. When the hat is too short for the notation to be legible, we write $\mathcal{F}(f)$ instead of $\hat{f}$.

When f is an integrable function on X, it is no longer obvious that this integral should converge for $(\lambda, b) \in \mathfrak{a}^\star \times K/M$; yet one can show ([10], p. 209) that it does converge for (λ, b) in $\mathfrak{a}^\star \times B_0$, with B_0 a full-measure subset of B.

3.3. In what follows, we will need an extension of the Fourier-Helgason transform to distributions on X, and a non-Euclidean analogue of Schwartz-class functions and of tempered distributions. In order to get to the point more quickly, the corresponding definitions of $\mathcal{S}(X)$, $\mathcal{S}'(X)$ and their counterparts over $\mathfrak{a} \times K/M$ have been relegated to section 6 below (see [10], around p. 214). But even when f is only integrable, when $\hat{f}$ is integrable with respect to the measure $\left(|\mathbf{c}(\lambda)|^{-2} d\lambda\right) \otimes db$ on $\mathfrak{a}^\star \times K/M$ (the measure features Harish-Chandra's $\mathbf{c}$-function), the following inversion formula will hold for almost every x in X :

$$f(x) = \frac{1}{|W|} \int_{\mathfrak{a}^\star \times B} \hat{f}(\lambda, b)\ e_{\lambda,b}(x)\ |\mathbf{c}(\lambda)|^{-2} d\lambda db.$$

Here $|W|$ is the order of the Weyl group $W(\mathfrak{g}, \mathfrak{a})$; the $\mathbf{c}$-function is real-analytic on the complement in $\mathfrak{a}^\star$ of a finite union of hyperplanes, but we will not need the details of its definition.

3.4. When shifting from Euclidean space to symmetric spaces of the noncompact type, there are often ways to translate interesting questions about the usual Fourier transform (like the Plancherel formula, the Paley-Wiener theorem, etc.) into questions on the Fourier-Helgason transform, and the answers often show some likeness in spite of some important differences due to the curvature of X and the ensuing growth at infinity of Helgason's waves.

4. Elementary spherical functions

4.1. Helgason's waves $e_{\lambda,b}$ make G-invariant harmonic analysis on G/K, a subject depicted in detail in his work, look familiar; before Helgason made it look so, the fact that the function obtained by constructive interference from all Helgason waves with frequency λ is the spherical function φ_λ had already proved to be a key point in Harish-Chandra's program for studying the representation theory of G (see [6]).

4.2. For each $\lambda \in \mathfrak{a}^\star$, the map

$$\begin{aligned} \varphi_\lambda : G &\to \mathbb{C} \\ g &\mapsto \int_B e_{\lambda,b}(gK)\, db \end{aligned}$$

takes the value 1 at zero, is left-and-right K-invariant and is an eigenfunction for all G-invariant differential operators on G: it is an elementary spherical function of G. The only functions with the three properties in the previous sentence are the φ_λ, $\lambda \in \mathfrak{a}^\star$, and two functions of this type are equal if and only if the elements of $\mathfrak{a}^\star$ defining them are on the same orbit for the action of the Weyl group on $\mathfrak{a}^\star$.

4.3. Let us recall that the K-invariant version of Fourier-Helgason analysis led Harish-Chandra to define the **c**-function and opened the way towards the Plancherel formula for G: if f is a smooth function on G which has compact support and is K-bi-invariant, let us write $\tilde{f}(\lambda) = \int_G f(g)\varphi_{-\lambda}(g)dg$ for $\lambda \in \mathfrak{a}^\star$; then

$$f \mapsto \tilde{f} \text{ extends to an isometry between } \mathbb{L}^2(K\backslash G/K) \text{ and } \mathbb{L}^2\left(\mathfrak{a}^\star_+, |\mathbf{c}(\lambda)|^{-2} d\lambda\right).$$

5. A moiré pattern

5.1. In the next few paragraphs, we shall prove that a synthesis formula holds in the opposite direction and that Helgason's waves can be recovered by constructive interference from spherical functions whose centers of symmetry cluster along a horocycle.

To be precise, let us choose a "frequency" λ in $\mathfrak{a}^\star$ and a point in the boundary–say the identity coset $b_0 = 1_K M$ in B. When y is in X, let us write $\varphi_\lambda^{[y]}$ for the only element in the eigenspace $\mathcal{E}_\lambda(X)$ from [10, chapter 6],[1] which takes the value 1 at y and is insensitive to left- and right-translations of the variable along an element of the stabilizer of y in G. We are going to argue that for every x in X, the equality[2]

$$\int_{\xi(b_0,0)} \varphi_\lambda^{[y]}(x)\, dy = \frac{|\mathbf{c}(\lambda)|^2}{|W|} \sum_{w\in W} e_{w\lambda,b_0}(x) \tag{3}$$

holds in a weak sense analogous to (1): here is the statement of what we are actually going to prove.

Choose a family $(\vartheta_\varepsilon)_{\varepsilon>0}$ of Schwartz functions on X which, as ε goes to zero, goes in $\mathcal{S}'(X)$ to the Dirac distribution over the horocycle $\xi(b_0, x)$–an example is

[1]This is the common eigenspace for G-invariant differential operators on X which contains φ_λ.

[2]In the right-hand side of (3), the averaging over W mirrors the appearance of an even, real-valued cosine wave (rather than a complex-valued plane wave) in (1), and the presence of the **c**-function mirrors the normalization coming from the polar change of coordinates used for proving (1) (see section 4).

$\vartheta_\varepsilon(y) = \frac{1}{\varepsilon^{\dim(A)}} \varpi\left(\frac{d(y,\xi(b_0,x))}{\varepsilon}\right) \varpi\left(\varepsilon d(y,o)^2\right)$, where d is the Riemannian distance in X and ϖ is the bump function introduced above.

Proposition. *When x is a point in X, the continuous function*

$$\lambda \in \mathfrak{a}^\star \mapsto \frac{|\mathbf{c}(\lambda)|^2}{|W|} \sum_{w\in W} e_{w\lambda,b_0}(x)$$

is the limit, in $\mathcal{D}'(\mathfrak{a}^\star/W)$, of $\lambda \mapsto \int_{\xi(b_0,0)} \vartheta_\varepsilon(y)\varphi_\lambda^{[y]}(x)\,dy$ as ε goes to zero.

Note that for every $\varepsilon > 0$, $\lambda \mapsto \int_{\xi(b_0,0)} \vartheta_\varepsilon(y)\varphi_\lambda^{[y]}(x)\,dy$ is a well defined, continuous, W-invariant function of λ.

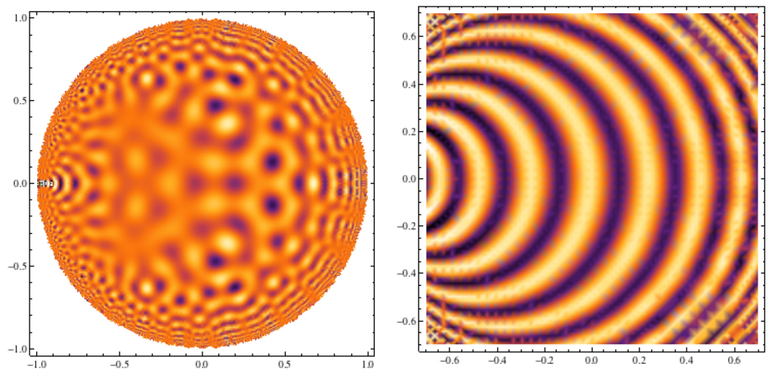

FIGURE 5. The left picture shows a sum of five, and the right picture the detail of a sum of sixty, spherical functions whose centers of symmetry lie on the horocycle $\xi(-1,0)$.

5.2. If $y \in X$ is where the origin $o = eK$ in G/K is sent by $g_y \in G$, then the function $\varphi_\lambda^{[y]}$ is none other than $z \mapsto \varphi_\lambda(g_y^{-1}\cdot z)$. When y is on the horocycle $\xi(b_0,o)$, the element g_y can be assumed to belong to N and $g_y^{-1}\cdot x$ is on the horocycle $\xi(b_0,x)$. So (3) is a weak version of

$$\int_{\xi(b_0,x)} \varphi_\lambda = \frac{|\mathbf{c}(\lambda)|^2}{|W|} \sum_{w\in W} e_{w\lambda,b_0}(x). \tag{4}$$

But disregarding the convergence questions for a few lines, the reconstruction formula by Harish-Chandra cited above (see also [5]) formally yields :

$$\int_{\xi(b_0,x)} \varphi_\lambda = \int_{\xi(b_0,x)} dy \int_B e_{\lambda,b}(y)db. \tag{5}$$

5.3. If we swap the integrals in (5), we will end up with the integral over B of the Fourier-Helgason transform of the characteristic function of the horocycle $\xi(b_0,x)$, and this will bring us very close to the desired conclusion. But we will also end up with divergent integrals. Before we address this, we record the following lemma:

Lemma. *The Helgason-Fourier transform of the Dirac distribution on the horocycle* $\xi(b_0, x)$ *(viewed as a tempered distribution on* $\mathfrak{a}^\star \times B$*) is:*

$$[\lambda \mapsto e_{\lambda,b_0}(x)] \otimes \delta_{b=b_0}. \tag{6}$$

This is an analogue of the projective property of the Fourier transform on Euclidean space used in section 1 above, but it should be noted that the privileged direction in the Helgason-Fourier transform is that of the horocycle itself rather than an "orthogonal" one.

Proof. Let us write $T_{b_0,x} \in \mathcal{S}'(\mathfrak{a}^\star \times B)$ for the distribution (6). What we have to check is that for every ψ in $\mathcal{S}(X)$,

$$\langle T_{b_0,x} | \hat{\psi} \rangle = \int_{\xi(b_0,x)} \psi,$$

in other words

$$\int_{\mathfrak{a}^\star} e_{\lambda,b_0}(x) \widehat{\psi}(\lambda, b_0) d\lambda = \int_{\xi(b_0,x)} \psi. \tag{7}$$

Let us first assume ψ to have compact support and set out from the fact that $\widehat{\psi}(\lambda, b_0) = \int_X e^{\langle -i\lambda+\rho, \mathcal{A}(y)\rangle} \psi(y) dy$. We now use the integration formula on p. 266 of [9]: if f lies in the space $\mathcal{D}(X)$ of smooth and compactly supported functions, then

$$\int_X f(y) dy = \int_{\mathfrak{a}} e^{-2\langle \rho, H\rangle} dH \int_N f(ne^H \cdot o) dn.$$

Setting $F(H) = \int_N \psi(ne^H \cdot o) dn$ for H in $\mathfrak{a}$, we obtain

$$\widehat{\psi}(\lambda, b_0) = \int_{\mathfrak{a}} e^{\langle -i\lambda+\rho, H\rangle} F(H) dH$$

(in other words, the Fourier transform of ψ is the Euclidean-Fourier transform on $\mathfrak{a}$ of its Radon transform over the family of horocycles with direction b). As a consequence, we obtain

$$e_{\lambda,b_0}(x) \widehat{\psi}(\lambda, b_0) = \int_{\mathfrak{a}} e^{\langle i\lambda+\rho, \mathcal{A}(x)-H\rangle} F(\mathcal{A}(x) - H) dH.$$

But this is the Euclidean-Fourier transform of the function

$$\gamma \ : \ H \ \mapsto \ e^{\langle \rho, \ H\rangle} \ F(\mathcal{A}(x) \ - \ H).$$

When we integrate $e_{\lambda,b_0}(x) \widehat{\psi}(\lambda, b_0)$ over $\mathfrak{a}^\star$ as we must in order to get (7), we can use the ordinary Fourier inversion formula (applicable here because γ has compact support and is smooth) and we obtain

$$\int_{\mathfrak{a}^\star} e_{\lambda,b_0}(x) \widehat{\psi}(\lambda, b_0) d\lambda = \gamma(0) = F(\mathcal{A}(x)) = \int_N \psi(ne^{\mathcal{A}(x)} \cdot o) dn,$$

but this is none other than $\int_{\xi(b_0,x)} \psi$. Thus $\delta_{\xi(b,x)}$ and $\mathcal{F}^{-1}\left([\lambda \mapsto e_{\lambda,b_0}(x)] \otimes \delta_{b=b_0}\right)$, both tempered distributions on X (see (A1) below), coincide over $\mathcal{D}(X)$; of course then they do coincide as tempered distributions on X. Taking Fourier transforms proves the lemma. ☐

5.4. Let us come back to the moiré phenomenon (3). Remembering our initial wish to swap the integrals in (5), let us use the fact that ϑ_ε is rapidly decreasing for every $\varepsilon > 0$ and write

$$\int_X \vartheta_\varepsilon \varphi_\lambda = \int_B \left(\int_X \vartheta_\varepsilon e_{\lambda,b} db \right) = \int_B \widehat{\vartheta_\varepsilon}(\lambda, b) db.$$

The right-hand side is W-invariant in λ (this is obvious from the fact that the left-hand side is, but see also [4, Theorem 2], and [10, Lemma 1.2 p. 200]). We can then rewrite the equality as

$$\int_X \vartheta_\varepsilon \varphi_\lambda = \frac{1}{|W|} \sum_{w \in W} \int_B \widehat{\vartheta_\varepsilon}(w\lambda, b) db.$$

Let us now have ε go to zero, so that in the space of tempered distributions on X, ϑ_ε goes to the Dirac distribution over $\xi(b_0, x)$. Because of the continuity properties recalled below in (A1) and (A2), as ε goes to zero, there is a limit in $\mathcal{D}'(\mathfrak{a}^\star/W)$ to the family of distributions given by integration against

$$\lambda \mapsto |\mathbf{c}(\lambda)|^{-2} \frac{1}{|W|} \int_B \left(\sum_{w \in W} \widehat{\vartheta_\varepsilon}(w\lambda, b) \right) db$$

with respect to the measure on $\mathfrak{a}^\star$ inherited from Lebesgue measure. We then see that the distribution associated with integrating against $\lambda \mapsto |\mathbf{c}(\lambda)|^{-2} \int_X \vartheta_\varepsilon \varphi_\lambda$ goes, in $\mathcal{D}'(\mathfrak{a}^\star/W)$, to the distribution $\int_B \mathcal{F}\left(\delta_{\xi(b_0,x)}\right)$ (precisely defined in section 6.2 below), of which the above Lemma says that it is associated with integrating against the almost-everywhere-defined function $\lambda W \mapsto \frac{1}{|W|} \sum\limits_{w \in W} e_{w\lambda, b_0}(x)$ with respect to the Lebesgue-inherited measure.

After a very slight change notation for ϑ in order to revert back from (4) to (3), we conclude that $\lambda W \mapsto \int_{\xi(b_0,0)} \vartheta(y) \varphi_\lambda^{[y]}(x)\, dy$ goes, in $\mathcal{D}'(\mathfrak{a}^\star/W)$, to the distribution given by integration against $\lambda W \mapsto |\mathbf{c}(\lambda)|^2 \frac{1}{|W|} \sum_{w \in W} e_{w\lambda, b_0}(x)$ with respect to the usual Lebesgue-inherited measure on $\mathfrak{a}^\star/W$: that was the weaker form of (3) aimed at in this short paper, and the Proposition is now proven.

6. Appendix: Tempered distributions on X and their Fourier transforms

6.1. **Schwartz functions on X.** Let us write $\mathbf{D}(G)$ for the algebra of left-invariant differential operators on G, and $\bar{\mathbf{D}}(G)$ for the algebra of right-invariant differential operators.

Recall that every element of G can be written as a product $k_1 a k_2$ with k_1, k_2 in K and a in A, and that two such decompositions have their a-part related by the action of an element in the Weyl group. If we set $|g| = |\log(a)|$ (the right-hand side refers to a Euclidean norm on $\mathfrak{a}$), we can make the following definition: a smooth function f on G is rapidly decreasing (or *Schwartz*) if for every $\ell \in \mathbb{N}$, $L \in \mathbf{D}(G)$ and $R \in \bar{\mathbf{D}}(G)$,

$$\sup_{g \in G} \left| (1 + |g|)^\ell \Xi(g)^{-1} (LRf)(g) \right| < \infty, \tag{8}$$

where the map Ξ is the spherical function φ_0. In [6, Theorem 3], we find the following estimate:

$$\Xi(g) \leq c(1 + |g|)^d e^{-\langle \rho | \log a \rangle},$$

where c is a positive real number and d a nonnegative integer.

The rapidly decreasing functions on G gather in the Schwartz space $\mathcal{S}(G)$; those which are right-invariant under K gather in the Schwartz space $\mathcal{S}(X)$. The quantities (8) provide natural seminorms turning $\mathcal{S}(X)$ into a Fréchet space; we write $\mathcal{S}'(X)$ for the topological dual $\mathcal{S}(X)$, the space of *tempered distributions* on X.

6.2. **Schwartz functions on $\mathfrak{a}^\star \times K/M$ and continuity of the Fourier transform.** When taking the Helgason-Fourier transform of a function in $\mathcal{S}(X)$, we get a smooth function on $\mathfrak{a}^\star \times K/M$ which satisfies ([10, chap. 3, thm 1.10]):

(9) For each $P \in \mathbb{R}[X,Y]$ and every $\ell \in \mathbb{N}$,

$$\sup_{\lambda,b} \left|(1+|\lambda|)^\ell \left(P(\Delta_{K/M}, \Delta_{\mathfrak{a}^\star}) \cdot g\right)(\lambda,b)\right| < \infty$$

where $\Delta_{K/M}$, $\Delta_{\mathfrak{a}^\star}$ are the Laplace-Beltrami operators on B and $\mathfrak{a}^\star$.

Let us write $\mathcal{S}(\mathfrak{a}^\star \times K/M)$ for the space of smooth functions on $\mathfrak{a}^\star \times B$ for which (9) holds; as before it comes with natural seminorms which make it a Fréchet space; taking Fourier-Helgason transforms of course defines a continuous, injective map from $\mathcal{S}(X)$ into $\mathcal{S}(\mathfrak{a}^\star \times K/M)$.

Harish-Chandra and Helgason proved that this map defines a homeomorphism between the subspaces gathering the K-invariants in both spaces ([10, th. 1.17]; see also Anker [2]). Eguchi [3,4] proved (together with Okamoto) that the Fourier transform of an element f of $\mathcal{S}(\mathfrak{a}^\star \times K/M)$ satisfies some form of Weyl-group invariance (see [4, Theorem 2]): the averages over B of $b \mapsto \widehat{f}(\lambda,b)$ $b \mapsto \widehat{f}(w\lambda,b)$ coincide for every w in W. Writing $\mathcal{S}(\mathfrak{a}^\star \times K/M)_W$ for the space of Schwartz functions on $\mathfrak{a}^\star \times K/M$ satisfying that Weyl-group invariance condition, Eguchi and Okamoto proved that $\mathcal{F}$ induces a homeomorphism between $\mathcal{S}(X)$ and $\mathcal{S}(\mathfrak{a}^\star \times K/M)_W$.

Now, define the space $\mathcal{S}'(\mathfrak{a}^\star \times K/M)$ of tempered distributions on $\mathfrak{a}^\star \times K/M$ as the topological dual of $\mathcal{S}(\mathfrak{a} \times K/M)_W$, the space $\mathcal{S}'(X)$ of tempered distributions on X as the topological dual of $\mathcal{S}(X)$, and if T is a tempered distribution on X, define $\hat{T}$ as the distribution $\psi = \hat{\varphi} \in \mathcal{S}(\mathfrak{a}^\star \times B)_W \mapsto \langle T \mid \varphi\rangle$ on $\mathfrak{a}^\star \times B$. In section 5.3, we used the following fact:

(A1) $T \mapsto \hat{T}$ defines a homeomorphism between $\mathcal{S}'(X)$ and $\mathcal{S}'(\mathfrak{a}^\star \times K/M)$.

To complete our list of disribution-theory-based ingredients for section 5, let us record the following remark: if T is an element of $\mathcal{S}'(\mathfrak{a} \times K/M)$, one can define a distribution U on $\mathfrak{a}^\star$ ("the integral of T over K/M") by setting, for ζ in $\mathcal{D}(\mathfrak{a}^\star/W)$, $\langle U \mid \zeta\rangle = \langle T \mid \tilde{\zeta} \otimes 1_B\rangle$ (where $\tilde{\zeta}$ is the inflation of ζ to $\mathfrak{a}^\star$). Writing $\int_B T$ for the distribution U, we then of course have:

(A2) $T \mapsto \displaystyle\int_B T$ is continuous as a map from $\mathcal{S}'(\mathfrak{a}^\star \times K/M)$ to $\mathcal{D}'(\mathfrak{a}^\star/W)$.

We should point out here that an almost-everywhere defined and bounded function u on $\mathfrak{a}^\star$ defines an element of $\mathcal{S}'(\mathfrak{a}^\star \times K/M)$, but that given the form of the Plancherel formula for the Helgason-Fourier transform (paragraph 4.3 above; see [10, p. 203]), if the above definition of the Fourier transform of tempered distributions is to extend the Helgason-Fourier transform of smooth and rapidly decreasing functions, the distribution should be given by integration against

$$|\mathbf{c}(\lambda)|^{-2}\ u(\lambda, b)\ d\lambda \otimes db$$

rather than against $u(\lambda, b)d\lambda \otimes db$.

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References

[1] Isaac Amidror, *The theory of the Moiré phenomenon*, Computational Imaging and Vision, vol. 15, Kluwer Academic Publishers, Dordrecht, 2000. MR1751208

[2] J-Ph. Anker, *The spherical Fourier transform of rapidly decreasing functions*, J. Funct. Anal. **96-2**, pp. 331–349.

[3] Masaaki Eguchi, *Asymptotic expansions of Eisenstein integrals and Fourier transform on symmetric spaces*, J. Funct. Anal. **34** (1979), no. 2, 167–216. MR552702

[4] Masaaki Eguchi and Kiyosato Okamoto, *The Fourier transform of the Schwartz space on a symmetric space*, Proc. Japan Acad. Ser. A Math. Sci. **53** (1977), no. 7, 237–241. MR0499949

[5] Harish-Chandra, *Spherical functions on a semisimple Lie group*, Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 408–409. MR0087043

[6] Harish-Chandra, *Spherical functions on a semisimple Lie group. I*, Amer. J. Math. **80** (1958), 241–310. MR0094407

[7] J. Harthong, *Le moiré*, Publications de l'IRMA, Strasbourg ; vol. 93.

[8] Sigurdur Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Pure and Applied Mathematics, vol. 80, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1978. MR514561

[9] Sigurdur Helgason, *Groups and geometric analysis*, Pure and Applied Mathematics, vol. 113, Academic Press, Inc., Orlando, FL, 1984. Integral geometry, invariant differential operators, and spherical functions. MR754767

[10] Sigurdur Helgason, *Geometric analysis on symmetric spaces*, 2nd ed., Mathematical Surveys and Monographs, vol. 39, American Mathematical Society, Providence, RI, 2008. MR2463854

[11] D. Hubel, *Eye, Brain and Vision*, Scientific American Library.

[12] D. H. Hubel and T. N. Wiesel, *Receptive fields, binocular interaction and functional architecture in the cat's visual cortex*, Journal of Physiology **160-1** ; pp. 106-154.

[13] D. Hubel, *Transformation of information in the cat's visual system*, Proceedings of the International Union of Physiological Sciences, XXII International Congress, Leiden.

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