PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 10, October 2018, Pages 4275–4282
http://dx.doi.org/10.1090/proc/14076
Article electronically published on May 2, 2018

FAMILIES OF MONGE-AMPÈRE MEASURES WITH HÖLDER CONTINUOUS POTENTIALS

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(Communicated by Filippo Bracci)

Abstract. Let X be a compact Kähler manifold of dimension n. Let $\mathcal{F}$ be a family of probability measures on X whose superpotentials are of uniformly bounded $\mathcal{C}^\alpha$ norms for some fixed constant $\alpha \in (0, 1]$. We prove that the corresponding family of solutions of the complex Monge-Ampère equations $(dd^c\varphi + \omega)^n = \mu$ with $\mu \in \mathcal{F}$ is Hölder continuous.

1. Introduction

Let X be a compact Kähler manifold of dimension n with a fixed Kähler form ω so normalized that $\int_X \omega^n = 1$. Let μ be a probability measure on X. For every bounded ω-psh function φ on X, we put $\omega_\varphi := dd^c\varphi + \omega$. By [1, 10], the currents ω_φ^j are well defined for $1 \le j \le n$. Consider the complex Monge-Ampère equation

$$\omega_\varphi^n = \mu, \tag{1.1}$$

where φ is a bounded ω-psh function on X and $\int_X \varphi\, \omega^n = 0$. The equation (1.1) and its variants have been extensively studied and have a wide range of applications. Instead of giving details on the development of the research on (1.1), in this short paper, we refer the readers to [2–5, 9, 11, 13–15, 18] and the references therein for detailed information.

In this work, we study the Hölder continuity of solutions of (1.1). Recently, based on [3], Dinh and Nguyên proved in [6] that (1.1) has a unique Hölder continuous solution φ_μ if and only if μ has a Hölder continuous superpotential $\mathcal{U}_\mu$; see Definition 2.1 below. Precisely, they proved that if $\mathcal{U}_\mu$ is Hölder continuous with Hölder exponent $\alpha \in (0, 1]$, then $\varphi_\mu \in \mathcal{C}^\beta(X)$ for any $\beta \in (0, \frac{2\alpha}{n+1})$, where $\mathcal{C}^\beta(X)$ denotes the set of Hölder continuous functions with Hölder exponent β on X. In this case, we call φ_μ *the (Monge-Ampère) potential of* μ. In view of the last result, we would like to address the question of the stability of the Hölder continuity of the solution of (1.1) with respect to μ: given a family of probability measures with Hölder continuous superpotentials, does φ_μ depend Hölder continuously on μ in that family? Let us be more clear in the next paragraph.

Let $\alpha \in (0, 1]$. By [6, Pro. 4.1], if φ varies in a bounded subset of $\mathcal{C}^\alpha(X)$, then ω_φ^n has a Hölder continuous superpotential with uniformly bounded Hölder exponent and Hölder constant. Hence, in order to study the above stability problem, it is necessary to consider sets of probability measures having the last property.

Received by the editors September 9, 2017, and, in revised form, December 18, 2017.
2010 *Mathematics Subject Classification.* Primary 32Uxx, 32Qxx.
Key words and phrases. Monge-Ampère measure, Monge-Ampère equation, superpotential.

Let C_0 be a positive constant and denote by $\mathcal{P}_{\alpha,C_0}$ the set of probability measures μ on X whose superpotentials $\mathcal{U}_\mu$ are Hölder continuous with Hölder exponent α and Hölder constant C_0. Since the specific value of C_0 is not important in our setting below, from now on, we will write $\mathcal{P}_\alpha$ instead of $\mathcal{P}_{\alpha,C_0}$ for simplicity.

Let $\beta \in (0, \frac{2\alpha}{n+1})$. Define $\Phi : \mathcal{P}_\alpha \to \mathcal{C}^\beta(X)$ by sending $\mu \in \mathcal{P}_\alpha$ to the unique solution φ_μ of (1.1). Recall that the set of probability measures on X endowed with the weak topology is a metric space with the distance dist defined as follows: for measures μ, μ',

$$\operatorname{dist}(\mu, \mu') := \sup_{\|v\|_{\mathcal{C}^1} \le 1} \big|\langle \mu - \mu', v\rangle\big|,$$

where v is a smooth real-valued function on X. The following is our main result.

Theorem 1.1. *The map Φ is Hölder continuous with Hölder exponent α' for any $0 < \alpha' < \beta(\frac{2\alpha}{n+1} - \beta)2^{-n-1}$.*

Equivalently, the last theorem says that there is a constant C (depending on β, α') such that

$$\|\varphi_{\mu_1} - \varphi_{\mu_2}\|_{\mathcal{C}^\beta} \le C[\operatorname{dist}(\mu_1, \mu_2)]^{\alpha'} \tag{1.2}$$

for every $\mu_1, \mu_2 \in \mathcal{P}_\alpha$. Consequently, if $\{\mu_k\}_{k\in\mathbb{N}} \in \mathcal{P}_\alpha$ converges weakly to $\mu \in \mathcal{P}_\alpha$, then the associated solution φ_{μ_k} converges to φ_μ in $\mathcal{C}^\beta(X)$. An interesting feature in the last assertion is that μ_k and μ can be singular to each other for every k. An imitation of Kołodziej's arguments in [12] only gives an estimate of type (1.2) but with $\operatorname{dist}(\mu_1, \mu_2)$ replaced by the mass norm $\|\mu_1 - \mu_2\|$ of $(\mu_1 - \mu_2)$. Such an estimate is not useful when μ_1, μ_2 are singular to each other. For example, as in the situation described in Corollary 1.2 below, the supports of measures μ_1, μ_2 in question are disjoint, hence $\|\mu_1 - \mu_2\| = 2$ in this case.

We give now an application of our main result. Recall that a real submanifold of X is said to be *Cauchy-Riemann generic* if the real tangent space at any point of it isn't contained in a complex hypersurface of the real tangent space at that point of X. By [17], the restriction of a smooth volume form of an immersed (Cauchy-Riemann) generic submanifold Y of X to a compact subset K of Y has a Hölder continuous superpotential. It is also clear from the arguments there that if the compact K depends smoothly on a parameter τ, then the Hölder exponent and Hölder constant of the superpotential can be chosen to be fixed numbers for every τ; see Proposition 2.7 below. More precisely, let M be a compact real manifold and let Y be a real Riemannian manifold. Assume that there is a smooth map $\Psi : Y \times M \to X$ such that $\Psi_\tau := \Psi|_{Y\times\{\tau\}} : Y \to X$ is an embedding into X such that $Y_\tau := \Psi_\tau(Y)$ is a generic submanifold Y_τ for every $\tau \in M$. Then $\{Y_\tau\}_{\tau\in M}$ is a smooth family of generic submanifolds of X. Note that using local charts of X, we see that such a family exists abundantly. With this setting, we get the following nice geometric result.

Corollary 1.2. *Let K be a compact subset of Y. For $\tau \in M$, define μ_τ to be the pushforward measure of the volume form of Y on K under Ψ_τ. Then the family of the Monge-Ampère potential φ_{μ_τ} of μ_τ is Hölder continuous in τ.*

Note that as in Theorem 1.1, we can give an explicit Hölder exponent in Corollary 1.2. In the next section, we will give a proof of Theorem 1.1.

2. Proof of Theorem 1.1

Let $\mathcal{P}_0$ be the set of ω-psh functions φ on X such that $\int_X \varphi\, \omega^n = 0$. We define the distance dist_{L^1} on $\mathcal{P}_0$ by putting

$$\mathrm{dist}_{L^1}(\varphi_1, \varphi_2) := \int_X |\varphi_1 - \varphi_2|\, \omega^n$$

for every $\varphi_1, \varphi_2 \in \mathcal{P}_0$.

Definition 2.1. The superpotential of a probability measure μ (of mean 0) is the function $\mathcal{U} : \mathcal{P}_0 \to \mathbb{R}$ given by $\mathcal{U}(\varphi) := \int_X \varphi d\mu$. We say that $\mathcal{U}$ is Hölder continuous with Hölder exponent $\alpha \in (0, 1]$ and Hölder constant C if it is so with respect to the distance dist_{L^1}, i.e.,

$$|\mathcal{U}(\varphi_1) - \mathcal{U}(\varphi_2)| \leq C[\mathrm{dist}_{L^1}(\varphi_1, \varphi_2)]^\alpha$$

for every $\varphi_1, \varphi_2 \in \mathcal{P}_0$. The $\mathcal{C}^\alpha$-norm of $\mathcal{U}$ is defined as usual.

By [6, Le. 3.3], that $\mathcal{U}$ is Hölder continuous with Hölder exponent $\alpha \in (0, 1]$ is equivalent to having

$$\int_X |\varphi_1 - \varphi_2| d\mu \leq C \max\big\{\|\varphi_1 - \varphi_2\|^\alpha_{L^1(X)}, \|\varphi_1 - \varphi_2\|_{L^1(X)}\big\} \tag{2.1}$$

for some constant C independent of φ_1, φ_2. By the arguments in [6], we immediately get the following.

Lemma 2.2. *Assume that the superpotential $\mathcal{U}$ of a probability measure μ on X is Hölder continuous with Hölder exponent α and Hölder constant C. Let $\beta \in (0, \frac{2\alpha}{n+1})$. Then the unique solution φ_μ of (1.1) with $\int_X \varphi_\mu\, \omega^n = 0$ is Hölder continuous with Hölder exponent β and Hölder constant $\tilde{C}$ depending only on α, β, C, and X. In particular, φ_μ is bounded by $\tilde{C}$ independent of $\mu \in \mathcal{P}_\alpha$.*

Let K be a Borel subset of X. The *capacity* of K is given by

$$\mathrm{cap}_\omega(\mathrm{K}) := \sup\Big\{\int_\mathrm{K} \omega^\mathrm{n}_\varphi : 0 \leq \varphi \leq 1, \varphi\ \ \omega\text{-psh}\Big\}.$$

The above notion is due to Kołodziej as an analogue to the capacity given by Bedford and Taylor in the local setting. Let us recall the following important result of Kołodziej.

Lemma 2.3 (see [12, Le. 2.2])**.** *Let φ_1, φ_2 be bounded ω-psh functions on X. Let s be a real number. Assume that the set $\{\varphi_1 - s < \varphi_2\}$ is nonempty and there are a positive constant A and an increasing function $h : (0, \infty) \to (1, \infty)$ such that*

$$\int_1^\infty \frac{dt}{th^{1/n}(t)} < \infty, \quad \int_K \omega^n_{\varphi_1} \leq \frac{A\mathrm{cap}_\omega(\mathrm{K})}{h\big([\mathrm{cap}_\omega(\mathrm{K})]^{-1/\mathrm{n}}\big)}$$

for every compact subset K of X. For $\epsilon \in (0, 1)$, put $c_\epsilon := \mathrm{cap}_\omega\big(\{\varphi_1 - \mathrm{s} - \epsilon < \varphi_2\}\big)$. Then we have

$$\int_{c_\epsilon^{-1/n}}^\infty \frac{dt}{th^{1/n}(t)} + h^{-1/n}(c_\epsilon^{-1/n}) \geq A'(1 + \|\varphi_2\|_{L^\infty})^{-1}\epsilon,$$

where A' is a positive constant depending only on A, n.

As introduced in [6], a positive measure μ is said to be *K-moderate* if there are positive constants A and δ_0 for which

$$\mu(K) \leq Ae^{-[\mathrm{cap}_\omega(\mathrm{K})]^{-\delta_0}} \tag{2.2}$$

for every Borel subset K of X. Recall that if μ has a Hölder continuous superpotential with Hölder exponent α and Hölder constant C, then μ is K-moderate by [6, Pro. 2.4]. Moreover, the constants A, δ_0 in (2.2) depend only on α, C, and X. The following result is crucial for our later proof.

Lemma 2.4. *Let φ_1, φ_2 be bounded ω-psh functions on X. Let s be a real number. Assume that the set $\{\varphi_1 - s < \varphi_2\}$ is nonempty and $\omega^n_{\varphi_1}$ is K-moderate. Let δ be a positive number in $(0,1)$. Then there exists a constant A' depending only on A, δ_0, δ, and n such that for any $\epsilon \in (0,1)$ we have*

$$\mathrm{cap}_\omega\big(\{\varphi_1 - \mathrm{s} - \epsilon < \varphi_2\}\big) \geq \mathrm{A}'(1 + \|\varphi_2\|_{\mathrm{L}^\infty})^{-\delta}\epsilon^\delta.$$

Proof. Using the inequality

$$e^{-t} \leq m!\, t^{-m}$$

for $t > 0$ and $m \in \mathbb{N}$, we get

$$e^{-[\mathrm{cap}_\omega(\mathrm{K})]^{-\delta_0}} \leq m! [\mathrm{cap}_\omega(\mathrm{K})]^{\mathrm{m}\delta_0}$$

for every $m \in \mathbb{N}$. Choose m such that $m\delta_0 \geq n\delta^{-1} + 1$. This combined with (2.2) for $\mu = \omega^n_{\varphi_1}$ gives

$$\int_K \omega^n_{\varphi_1} \leq A_1 [\mathrm{cap}_\omega(\mathrm{K})]^{\mathrm{n}\delta^{-1}+1} \tag{2.3}$$

for some constant A_1 depending only on A, n, δ_0, δ. Define $h(t) := \max\{t^{n^2\delta^{-1}}, 1\}$ for positive real numbers t. Put $c_\epsilon := \mathrm{cap}_\omega\big(\{\varphi_1 - \mathrm{s} - \epsilon < \varphi_2\}\big)$. Applying Lemma 2.3 to $h(t)$ and φ_1, φ_2 gives

$$\int_{c_\epsilon^{-1/n}}^\infty t^{-1} h^{-1/n}(t)dt + h^{-1/n}(c_\epsilon^{-1/n}) \gtrsim (1 + \|\varphi_2\|_{L^\infty})^{-1}\epsilon.$$

Then the desired inequality follows easily. The proof is finished. □

Lemma 2.5. *Let $\mu_1, \mu_2 \in \mathcal{P}_\alpha$ and φ_1, φ_2 be Hölder continuous solutions of* (1.1) *for μ_1, μ_2, respectively. Let $\beta \in (0, \frac{2\alpha}{n+1})$. Then we have*

$$\|\varphi_1 - \varphi_2\|_{L^1(X)} \leq C\, \mathrm{dist}(\mu_1, \mu_2)^{\beta 2^{-n}} \tag{2.4}$$

for some constant C independent of μ_1, μ_2.

Proof. By [2, Th. 1.2], we have

$$\int_X d(\varphi_1 - \varphi_2) \wedge d^c(\varphi_1 - \varphi_2) \wedge \omega^{n-1} \leq C\Big(\int_X (\varphi_1 - \varphi_2)(\omega^n_{\varphi_2} - \omega^n_{\varphi_1})\Big)^{2^{1-n}}, \tag{2.5}$$

for some constant C independent of μ_1, μ_2. Now using Poincaré's inequality (see [8, Th. 1, page 275]) for L^2-norm and the fact that φ_1, φ_2 are Hölder continuous with Hölder exponent β and a fixed Hölder constant, we get

$$\|\varphi_1 - \varphi_2\|_{L^1(X)} \lesssim \Big(\int_X (\varphi_1 - \varphi_2)(\omega^n_{\varphi_2} - \omega^n_{\varphi_1})\Big)^{2^{-n}} \lesssim \mathrm{dist}_\beta(\mu_1, \mu_2)^{2^{-n}}, \tag{2.6}$$

where

$$\mathrm{dist}_\beta(\mu_1, \mu_2) := \sup_{\|v\|_{\mathcal{C}^\beta} \leq 1} \big|\langle \mu_1 - \mu_2, v\rangle\big|.$$

Recall from [7, 16] that $\mathrm{dist}_\beta(\mu_1,\mu_2) \lesssim \mathrm{dist}^\beta(\mu_1,\mu_2)$ for $\beta \in [0,1]$. This together with (2.6) gives (2.4). The proof is finished. □

The following result is the interpolation inequality for Hölder norms of which we include a proof for the reader's convenience.

Lemma 2.6. *Let f be a Hölder continuous function in $\mathcal{C}^\beta(X)$ for some positive constant $\beta \in (0,1)$. Let ϵ be a positive number in $[0, 1-\beta]$. Then we have*

$$\|f\|_{\mathcal{C}^\beta} \le C \|f\|_{\mathcal{C}^0}^{\frac{\epsilon}{\beta+\epsilon}} \|f\|_{\mathcal{C}^{\beta+\epsilon}}^{\frac{\beta}{\beta+\epsilon}}$$

for some constant C depending only on X.

Proof. Recall that the $\mathcal{C}^\beta$-norm is defined in X by using a fixed cover of X by local charts. Thus without loss of generality, we can assume that $X = \mathbb{C}^n$. For $x, y \in \mathbb{C}^n$, we have

$$\frac{|f(x)-f(y)|}{|x-y|^\beta} = |f(x)-f(y)|^{\frac{\epsilon}{\beta+\epsilon}} \Big(\frac{|f(x)-f(y)|}{|x-y|^{\beta+\epsilon}}\Big)^{\frac{\beta}{\beta+\epsilon}} \le 2\|f\|_{\mathcal{C}^0}^{\frac{\epsilon}{\beta+\epsilon}} \|f\|_{\mathcal{C}^{\beta+\epsilon}}^{\frac{\beta}{\beta+\epsilon}}.$$

The proof is finished. □

End of the proof of Theorem 1.1. Let $\mu_1, \mu_2 \in \mathcal{P}_\alpha$ ($\mu_1 \neq \mu_2$) and φ_1, φ_2 be Hölder continuous solutions of (1.1) for μ_1, μ_2, respectively. Fix a constant $\beta \in (0, \frac{2\alpha}{n+1})$ and $\delta \in [0, \frac{2\alpha}{n+1} - \beta)$. By Lemma 2.2 and the definition of $\mathcal{P}_\alpha$, there is a positive constant $\tilde{C}$ independent of φ_1, φ_2 such that φ_1, φ_2 are Hölder continuous with Hölder exponent $(\beta+\delta)$ and Hölder constant $\tilde{C}$. Set

$$N(\varphi_1,\varphi_2) := \max\{\|\varphi_1-\varphi_2\|^\alpha_{L^1(X)}, \|\varphi_1-\varphi_2\|_{L^1(X)}\} \neq 0.$$

Fix a real number $\tilde{\delta}$ in $(0,1)$. In order to prove (1.2), it suffices to suppose from now on that $\mathrm{dist}(\mu_1,\mu_2)$ is small. As it will be clear later, we will need that $\mathrm{dist}(\mu_1,\mu_2)$ is less than a positive constant depending on $\tilde{\delta}$ but independent of μ_1, μ_2. By Lemma 2.5, the quantity $N(\varphi_1,\varphi_2)$ is also small. In what follows, we use the notation $\lesssim$ and $\gtrsim$ to indicate $\le$ and $\ge$, respectively, up to a multiplicative constant independent of μ_1, μ_2.

Let ϵ be a positive real number in $(0,1)$ to be chosen later. Put $E_\epsilon := \{\varphi_1 + \epsilon < \varphi_2\}$. On E_ϵ we have $\varphi_1 - \varphi_2 \le -\epsilon < 0$, hence $|\varphi_1-\varphi_2| \ge \epsilon$. It follows that

$$\int_{E_\epsilon} d\mu_1 \le \epsilon^{-1}\int_X |\varphi_1-\varphi_2| d\mu_1 \lesssim \epsilon^{-1} N(\varphi_1,\varphi_2) \tag{2.7}$$

by (2.1). Since $|\varphi_2| \le \tilde{C}$, for any ω-psh function φ on X such that $0 \le \varphi \le 1$, we have

$$E := \{\varphi_1 + (\tilde{C}+2)\epsilon < \epsilon\varphi + (1-\epsilon)\varphi_2\} \subset \{\varphi_1 + (\tilde{C}+2)\epsilon < \epsilon + \varphi_2 + \epsilon\tilde{C}\} = E_\epsilon.$$

This combined with the comparison principle gives

$$\int_E \omega^n_{\epsilon\varphi+(1-\epsilon)\varphi_2} \le \int_E \omega^n_{\varphi_1} \le \int_{E_\epsilon} \omega^n_{\varphi_1} = \int_{E_\epsilon} d\mu_1.$$

On the other hand, we also have $E_{2\epsilon(\tilde{C}+1)} \subset E$ and $\omega^n_{\epsilon\varphi+(1-\epsilon)\varphi_2} \ge \epsilon^n \omega^n_\varphi$. This yields

$$\epsilon^n \int_{E_{2\epsilon(\tilde{C}+1)}} \omega^n_\varphi \le \int_E \omega^n_{\epsilon\varphi+(1-\epsilon)\varphi_2} \le \int_{E_\epsilon} d\mu_1.$$

Combining the last inequality with (2.7), we obtain

$$\epsilon^n \int_{E_{2\epsilon(\tilde{C}+1)}} \omega_\varphi^n \lesssim \epsilon^{-1} N(\varphi_1, \varphi_2).$$

Taking the supremum over every φ in the last inequality implies

$$\mathrm{cap}_\omega\big(\mathrm{E}_{2\epsilon(\tilde{\mathrm{C}}+1)}\big) \lesssim \epsilon^{-\mathrm{n}-1}\mathrm{N}(\varphi_1, \varphi_2). \tag{2.8}$$

Choose $\epsilon := C_{\tilde{\delta}}\big(N(\varphi_1, \varphi_2)\big)^{1/(n+1+\tilde{\delta})} \in (0,1)$, where $C_{\tilde{\delta}} > 1$ is a constant large enough (depending on $\tilde{\delta}$) which is independent of φ_1, φ_2. Here recall that $N(\varphi_1, \varphi_2)$ was assumed to be small enough at the beginning of the proof.

We claim that $E_{2\epsilon(\tilde{C}+2)}$ is empty. Suppose the contrary. Thus applying Lemma 2.4 to $s := -2(\tilde{C}+2)\epsilon$ shows that $\mathrm{cap}_\omega\big(\mathrm{E}_{2\epsilon(\tilde{\mathrm{C}}+1)}\big) \geq \mathrm{A}_{\tilde{\delta}}\epsilon^{\tilde{\delta}}$ for some constant $A_{\tilde{\delta}}$ independent of φ_1, φ_2. This coupled with (2.8) gives

$$N(\varphi_1, \varphi_2) \gtrsim A_{\tilde{\delta}}\epsilon^{n+1+\tilde{\delta}} = A_{\tilde{\delta}} C_{\tilde{\delta}}^{n+1+\tilde{\delta}} N(\varphi_1, \varphi_2). \tag{2.9}$$

We get a contradiction because $C_{\tilde{\delta}}$ can be chosen such that $A_{\tilde{\delta}} C_{\tilde{\delta}} > 1$. Therefore, $E_{2\epsilon(\tilde{C}+2)}$ is empty. In other words, we have

$$\varphi_1 - \varphi_2 \gtrsim -\big(N(\varphi_1, \varphi_2)\big)^{1/(n+1+\tilde{\delta})}.$$

By swapping the roles of φ_1, φ_2 we also get

$$\varphi_2 - \varphi_1 \gtrsim -\big(N(\varphi_1, \varphi_2)\big)^{1/(n+1+\tilde{\delta})}.$$

This implies that

$$\|\varphi_1 - \varphi_2\|_{L^\infty(X)} \lesssim \big(N(\varphi_1, \varphi_2)\big)^{1/(n+1+\tilde{\delta})} \lesssim \|\varphi_1 - \varphi_2\|_{L^1(X)}^{\alpha/(n+1+\tilde{\delta})} \tag{2.10}$$

which is

$$\lesssim \mathrm{dist}(\mu_1, \mu_2)^{\alpha\beta 2^{-n}/(n+1+\tilde{\delta})}.$$

Now applying Lemma 2.6 to $f = \varphi_1 - \varphi_2$ and using (2.10) we obtain that for any $\delta \in [0, \frac{2\alpha}{n+1} - \beta)$,

$$\|\varphi_1 - \varphi_1\|_{\mathcal{C}^\beta} \lesssim \|\varphi_1 - \varphi_2\|_{L^\infty(X)}^{\frac{\delta}{\beta+\delta}} \lesssim \mathrm{dist}(\mu_1, \mu_2)^{\delta\alpha 2^{-n}(n+1+\tilde{\delta})^{-1}\frac{\beta}{\beta+\delta}}.$$

Letting $\delta \to (\frac{2\alpha}{n+1} - \beta)$ and $\tilde{\delta} \to 0$ gives the desired result. The proof is finished. ☐

Proposition 2.7. *Let $M, \{Y_\tau\}_{\tau\in M}$ and Ψ be as in the Introduction. Then the superpotential of μ_τ is Hölder continuous with uniformly bounded Hölder exponent and Hölder constant as τ varies in M.*

Proof. As already mentioned, the desired result can be deduced directly from [17]. We briefly explain it here for the reader's convenience. We need to prove (2.1) for μ_τ instead of μ and the constants C, α there must be independent of τ.

Since the problem is local, it is enough to work locally. Each Y_τ inherits the metric from Y. Fix $\tau \in M$ and a point $a \in Y_\tau$. The crucial point is that the data in [17, Le. 3.1] can be chosen uniformly in τ, a. To be precise, there exists a local chart (W, Φ) around a in X with $\Phi : W \to \mathbb{C}^n$ such that the following three properties hold:

(i) $W \cap K_\tau$ contains a ball $\mathbb{B}_{Y_\tau}(a, r)$ of radius r_a centered at a of Y_τ, where $r > 0$ is a constant independent of a, τ,

(ii) $\|\Phi\|_{\mathcal{C}^3}$ and $\|\Phi^{-1}\|_{\mathcal{C}^3}$ are bounded by a constant independent of a, τ,

(iii) $\Phi(W \cap K)$ is the graph over the unit ball $\mathbb{B}$ of $\mathbb{R}^n$ of a smooth map $h : \overline{\mathbb{B}} \to \mathbb{R}^n$ ($\mathbb{C}^n \approx \mathbb{R}^n + i\mathbb{R}^n$) such that $\|h\|_{\mathcal{C}^3}$ is bounded by a constant independent of a, τ and $D^j h(0) = 0$ for $j = 0, 1, 2$.

By Property (i), the number of local charts (W, Φ) needed to cover K_τ can be chosen to be a fixed number for every τ. On such a local chart, every constant in [17, Pro. 3.7] can be chosen to be the same for every τ, a. Now the rest of the proof is done as in [17]. This gives us constants C, α in (2.1) independent of τ. The proof is finished. □

Acknowledgment

The author would like to thank Ngoc Cuong Nguyen for fruitful discussions.

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