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EXISTENCE AND UNIQUENESS OF STEADY WEAK SOLUTIONS TO THE NAVIER–STOKES EQUATIONS IN $\mathbb{R}^2$

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ABSTRACT. The existence of weak solutions to the stationary Navier–Stokes equations in the whole plane $\mathbb{R}^2$ is proven. This particular geometry was the only case left open since the work of Leray in 1933. The reason is that due to the absence of boundaries the local behavior of the solutions cannot be controlled by the enstrophy in two dimensions. We overcome this difficulty by constructing approximate weak solutions having a prescribed mean velocity on some given bounded set. As a corollary, we obtain infinitely many weak solutions in $\mathbb{R}^2$ parameterized by this mean velocity, which is reminiscent of the expected convergence of the velocity field at large distances to any prescribed constant vector field. This explicit parameterization of the weak solutions allows us to prove a weak-strong uniqueness theorem for small data. The question of the asymptotic behavior of the weak solutions remains open however when the uniqueness theorem doesn't apply.

1. Introduction

We consider the stationary Navier–Stokes equations in an exterior domain $\Omega = \mathbb{R}^n \setminus \bar{B}$ where B is a bounded simply connected Lipschitz domain,

$$\Delta \boldsymbol{u} - \boldsymbol{\nabla} p = \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + \boldsymbol{f}\,, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0\,, \qquad \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{u}^*\,, \tag{1.1}$$

with a given forcing term $\boldsymbol{f}$ and a boundary condition $\boldsymbol{u}^*$ if B is not empty. Since the domain is unbounded, we add the boundary condition at infinity,

$$\lim_{|\boldsymbol{x}|\to\infty} \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_\infty\,, \tag{1.2}$$

where $\boldsymbol{u}_\infty \in \mathbb{R}^n$ is a constant vector. In his seminal work, Leray [13] proposed a three-step method to show the existence of weak solutions to this problem. First, the boundary conditions $\boldsymbol{u}^*$ and $\boldsymbol{u}_\infty$ are lifted by an extension $\boldsymbol{a}$ which satisfies the so-called extension condition. The second step is to show the existence of weak solutions in bounded domains. Finally, the third step is to define a sequence of invading bounded domains that coincide in the limit with the unbounded domain and show that the induced sequence of solutions converges in some suitable space. With this strategy, Leray [13] was able to construct weak solutions in domains with a compact boundary if the flux through each connected component of the

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boundary is zero. The extension of this result to the case where the fluxes are small was done by Galdi [5, Section X.4] in three dimensions and by Russo [17] in two dimensions. We note that by elliptic regularity, weak solutions are automatically two derivatives more regular than the data [5, Theorem X.1.1]. All these results about weak solutions have essentially only two drawbacks, both in two dimensions: the validity of (1.2) is not known and the method of Leray cannot be applied if $\Omega = \mathbb{R}^2$.

In three dimensions, the method of Leray can be used to prove the existence of a weak solution satisfying (1.2) for any $\boldsymbol{u}_\infty \in \mathbb{R}^3$. By assuming the existence of a strong solution satisfying various decay conditions at infinity, Kozono and Sohr [12] and Galdi [5, §X.3] proved the uniqueness of weak solutions satisfying the energy inequality. Moreover, the asymptotic behavior was determined by Galdi [5, Theorem X.8.1] if $\boldsymbol{u}_\infty \neq \boldsymbol{0}$ and by Korolev and Šverák [11, Theorem 1] if $\boldsymbol{u}_\infty = \boldsymbol{0}$ and the data are small enough. Therefore, in three dimensions the picture is pretty complete.

In two-dimensional exterior domains, the homogeneous Sobolev space $\dot{H}^1(\Omega)$ used in the construction of weak solutions is too weak to determine the validity of (1.2), because elements in this function space can even grow at infinity. Therefore, the results concerning the uniqueness and the asymptotic behavior of weak solutions in two dimensions are very limited. Concerning the asymptotic behavior, Gilbarg and Weinberger [7, 19] proved that either there exists $\boldsymbol{u}_0 \in \mathbb{R}^2$ such that

$$\lim_{r\to\infty} \fint_{\partial B_r} |\boldsymbol{u} - \boldsymbol{u}_0|^2 = 0 \qquad \text{or} \qquad \lim_{r\to\infty} \fint_{\partial B_r} |\boldsymbol{u}|^2 = \infty\,.$$

Later on Amick [1] showed that if $\boldsymbol{u}^* = \boldsymbol{f} = \boldsymbol{0}$, then $\boldsymbol{u} \in L^\infty(\Omega)$ so that the first alternative must apply for some $\boldsymbol{u}_0$. Nevertheless, the question if any prescribed value at infinity $\boldsymbol{u}_\infty$ can be obtained this way remains open in general. For small data and $\boldsymbol{u}_\infty \neq \boldsymbol{0}$, Finn and Smith [4] constructed strong solutions satisfying (1.2). By assuming that the domain is centrally symmetric, Guillod [8, Theorem 2.27] proved the existence of a weak solution with $\boldsymbol{u}_\infty = \boldsymbol{0}$. Under additional symmetry assumptions, the existence and asymptotic decay of solutions with $\boldsymbol{u}_\infty = \boldsymbol{0}$ was proven under suitable smallness assumptions [8, 16, 20, 21] or specific boundary conditions [10]. We refer the reader to Galdi [5, Chapter XII] and Guillod [8] for a more complete discussion on the asymptotic behavior of solutions in two-dimensional unbounded domains. The question of the uniqueness of weak solutions for small data is even more open in two-dimensional exterior domains. The reason is that the value at infinity $\boldsymbol{u}_\infty$ should be intuitively part of the data in order to expect uniqueness. The only known results in that direction are due to Yamazaki [21] and Nakatsuka [14], who proved the uniqueness of weak solutions satisfying the energy inequality under suitable symmetry and smallness assumptions.

The other main issue concerns the construction of weak solutions in $\Omega = \mathbb{R}^2$, which fails due to a fundamental issue with the function space [5, Remark X.4.4 and §XII.1]. More precisely the completion $\dot{H}^1_0(\Omega)$ of smooth compactly supported functions in the semi-norm of $\dot{H}^1(\Omega)$ can be viewed as a space of locally defined functions only if $\Omega \neq \mathbb{R}^2$. The example of Deny and Lions [3, Remarque 4.1] shows that the elements of $\dot{H}^1_0(\mathbb{R}^2)$ are equivalence classes and cannot be viewed as functions. The reason is that constant functions can be approximated by compactly supported functions in $\dot{H}^1(\mathbb{R}^2)$; hence the function cannot be locally bounded by its gradient. This can also be viewed as a consequence of the absence of Poincaré inequality in $\dot{H}^1(\mathbb{R}^2)$.

The main result of this paper (Theorem 2.6) is a modification of the method of Leray which allows us to construct weak solutions in $\Omega = \mathbb{R}^2$. The idea is to construct approximate solutions in invading balls having a prescribed mean on some fixed bounded set. This can be done by using the freedom in the choice of the boundary condition on the boundary of the balls. That way, the local properties of the approximate solutions are controlled and can be used to prove that the sequence of approximate solutions converges locally in L^p-spaces. The method we are using furnishes as a corollary infinitely many weak solutions parameterized by the mean $\boldsymbol{\mu} = ⨍_\omega \boldsymbol{u}$, where ω is a fixed bounded set of positive measure. Intuitively we have recovered the parameter $\boldsymbol{u}_\infty \in \mathbb{R}^2$, even if the validity of (1.2) remains open. However, the explicit parametrization by $\boldsymbol{\mu}$ can be used to prove a weak-strong uniqueness theorem for small solutions (Theorem 2.9). This is done in the spirit of what is known in three dimensions [5, Theorem X.3.2] and is the first general uniqueness result available in two dimensions. We remark that the existence of a parametrization of the two-dimensional weak solutions by two real parameters is open when $\partial\Omega \neq \emptyset$, and in this case it is not clear that the mean $\boldsymbol{\mu} = ⨍_\omega \boldsymbol{u}$ will be such a parametrization. A more detailed discussion of the results is added at the end of §2.

Notation. The open ball of radius n centered at the origin is denoted by B_n. For $\boldsymbol{x} \in \mathbb{R}^d$, we define $\langle \boldsymbol{x} \rangle = 1 + |\boldsymbol{x}|$ and the weight $\mathfrak{w}(\boldsymbol{x}) = \big[\langle \boldsymbol{x} \rangle \langle \log\langle \boldsymbol{x} \rangle\rangle\big]^{-1}$. The mean value of a vector field on a bounded set ω of positive measure is written as $⨍_\omega \boldsymbol{u} = \frac{1}{|\omega|}\int_\omega \boldsymbol{u}$. The space of smooth solenoidal functions having compact support in Ω is denoted by $C^\infty_{0,\sigma}(\Omega)$. We denote by $\dot{H}^1(\Omega)$ the linear space $\big\{\boldsymbol{u} \in L^1_{\mathrm{loc}}(\Omega) : \boldsymbol{\nabla}\boldsymbol{u} \in L^2(\Omega)\big\}$ with the semi-norm $\|\boldsymbol{u}\|_{\dot{H}^1(\Omega)} = \|\boldsymbol{\nabla}\boldsymbol{u}\|_{L^2(\Omega)}$. The subspace of weakly divergence-free vector fields in $\dot{H}^1(\Omega)$ is written as $\dot{H}^1_\sigma(\Omega)$. Let $\dot{H}^1_{0,\sigma}(\Omega)$ denote the completion of $C^\infty_{0,\sigma}(\Omega)$ in the semi-norm of $\dot{H}^1(\Omega)$.

2. Main results

We first recall the standard notion of weak solutions to the stationary Navier–Stokes equations:

Definition 2.1. Let $\Omega \subset \mathbb{R}^2$ be any Lipshitz domain (in particular $\Omega = \mathbb{R}^2$ is allowed). Given $\boldsymbol{u}^* \in W^{1/2,2}(\partial\Omega)$ and a rank-two tensor $\mathbf{F} \in L^2(\Omega)$, a vector field $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ is called a weak solution of the Navier–Stokes equations (1.1) in Ω with $\boldsymbol{f} = \boldsymbol{\nabla}\cdot\mathbf{F}$ if

(1) $\boldsymbol{u} \in \dot{H}^1_\sigma(\Omega)$;
(2) $\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{u}^*$ in the trace sense;
(3) $\boldsymbol{u}$ satisfies

$$\big\langle \boldsymbol{\nabla}\boldsymbol{u}, \boldsymbol{\nabla}\boldsymbol{\varphi}\big\rangle_{L^2(\Omega)} + \big\langle \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}, \boldsymbol{\varphi}\big\rangle_{L^2(\Omega)} = \big\langle \mathbf{F}, \boldsymbol{\nabla}\boldsymbol{\varphi}\big\rangle_{L^2(\Omega)} \tag{2.1}$$

for all $\boldsymbol{\varphi} \in C^\infty_{0,\sigma}(\Omega)$.

The existence of weak solutions in two-dimensional unbounded domains was first proved by Leray [13] for vanishing flux through the boundaries and was extended to the case of small fluxes by Russo [17]:

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^2$ be an exterior domain having a compact connected Lipschitz boundary $\partial\Omega \neq \emptyset$. Let $\boldsymbol{u}^* \in W^{1/2,2}(\partial\Omega)$ and $\mathbf{F} \in L^2(\Omega)$. If the flux*

$$\Phi = \int_{\partial\Omega} \boldsymbol{u}^* \cdot \boldsymbol{n}$$

satisfies $|\Phi| < 2\pi$, then there exists a weak solution $\boldsymbol{u} \in \dot{H}^1_\sigma(\Omega)$ of the Navier–Stokes equations (1.1) *in Ω.*

Remark 2.3. For $\partial\Omega \neq \emptyset$, if $\boldsymbol{f} \in L^2(\Omega)$ is a source term of compact support, then there exists $\mathbf{F} \in L^2(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{\nabla} \cdot \boldsymbol{F}$. See Lemma 3.5 for a more general result in this direction.

Remark 2.4. This result can be easily extended to the case where the boundary $\partial\Omega$ has finitely many connected components, provided the flux through each connected component is small enough.

Remark 2.5. The three-dimensional analogue of this theorem is valid even if $\partial\Omega = \emptyset$, i.e., if $\Omega = \mathbb{R}^3$; see Galdi [5, Theorem X.4.1].

As explained in the introduction, the method used to prove Theorem 2.2 fails for $\Omega = \mathbb{R}^2$. Our main result is the existence of infinitely many weak solutions in $\mathbb{R}^2$ for every given $\boldsymbol{F}$:

Theorem 2.6. *Let $\Omega = \mathbb{R}^2$ and let $\omega \subset \Omega$ be a bounded subset of positive measure. Let $\mathbf{F} \in L^2(\Omega)$ be a rank-two tensor. Then for any $\boldsymbol{\mu} \in \mathbb{R}^2$, there exists a weak solution $\boldsymbol{u} \in \dot{H}^1_\sigma(\Omega)$ of the Navier–Stokes equations* (1.1) *in Ω such that $\fint_\omega \boldsymbol{u} = \boldsymbol{\mu}$. Moreover,*

$$\left\|\boldsymbol{\nabla u}\right\|^2_{L^2(\Omega)} \leq \left\langle \mathbf{F}, \boldsymbol{\nabla u} \right\rangle_{L^2(\Omega)}, \tag{2.2}$$

so $\left\|\boldsymbol{\nabla u}\right\|_{L^2(\Omega)} \leq \left\|\mathbf{F}\right\|_{L^2(\Omega)}$.

Remark 2.7. For $\Omega = \mathbb{R}^2$, if $\boldsymbol{f} \in L^2(\Omega)$ is a source term of compact support and $\int_\Omega \boldsymbol{f} = \boldsymbol{0}$, then there exists $\mathbf{F} \in L^2(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{\nabla} \cdot \boldsymbol{F}$. See Lemma 3.6 for a more general result in this direction.

Remark 2.8. In this result the set ω can be easily replaced by a bounded and uniformly Lipschitz arc $\omega \subset \mathbb{R}^2$ of positive one-dimensional measure.

Finally, with our parametrization of weak solutions by the average $\boldsymbol{\mu}$, we can prove a weak-strong uniqueness theorem for small data:

Theorem 2.9. *Let $\Omega = \mathbb{R}^2$ and let $\omega \subset \Omega$ be a bounded subset of positive measure. Let $\boldsymbol{u}$ and $\tilde{\boldsymbol{u}}$ be two weak solutions of the Navier–Stokes equations* (1.1) *in Ω for the same source term $\boldsymbol{F} \in L^2(\Omega)$, having the same mean value $\fint_\omega \boldsymbol{u} = \fint_\omega \tilde{\boldsymbol{u}}$ and satisfying the energy inequality* (2.2). *There exists $\delta > 0$ depending only on ω such that if*

$$|\tilde{\boldsymbol{u}}(\boldsymbol{x}) - \boldsymbol{u}_\infty| \leq \frac{\delta}{\langle \boldsymbol{x} \rangle \langle \log\langle \boldsymbol{x} \rangle \rangle}, \tag{2.3}$$

for some $\boldsymbol{u}_\infty \in \mathbb{R}^2$, then $\boldsymbol{u} = \tilde{\boldsymbol{u}}$.

We now discuss our results in more detail. The space $\dot{H}^1(\Omega)$ is not a Banach space since the constant vector fields are in the kernel of the semi-norm, but $\dot{H}^1(\Omega)$ can be viewed as a sort of graded space. In the presence of a nontrivial boundary, this problem can be fixed by using the completion $\dot{H}^1_0(\Omega)$ of smooth compactly supported functions in the semi-norm of $\dot{H}^1(\Omega)$. Intuitively, there is no more freedom in the choice of the constant, since the elements of $\dot{H}^1_0(\Omega)$ are vanishing on the boundary $\partial\Omega$.

When the boundary is trivial, i.e., $\Omega = \mathbb{R}^n$, the boundary cannot serve as an anchor anymore to fix the problem of the constants. The solution of this problem now depends on the dimension. For $\Omega = \mathbb{R}^3$, the constants do not belong to the completion $\dot{H}^1_0(\Omega)$, the reason being the Sobolev embedding into $L^6(\Omega)$. Therefore, the space $\dot{H}^1(\Omega)$ is in some sense naturally graded by the constant at infinity $\boldsymbol{u}_\infty \in \mathbb{R}^3$ in three dimensions.

For $\Omega = \mathbb{R}^2$, the constants belong to the completion $\dot{H}^1_0(\Omega)$ of smooth compactly supported functions in the semi-norm of $\dot{H}^1(\Omega)$, so $\dot{H}^1_0(\Omega)$ is a space of equivalence classes defined by the relation of being equal up to a constant vector field. Therefore, $\dot{H}^1_0(\Omega)$ cannot be viewed as a space of locally defined functions. To overcome this difficulty, we choose to graduate the space $\dot{H}^1(\Omega)$ by the mean $\boldsymbol{\mu} \in \mathbb{R}^2$ of the vector field on ω. Intuitively, this is a recovery of the parameter $\boldsymbol{u}_\infty \in \mathbb{R}^2$, which is lost in two dimensions during the completion. This new way of parameterizing the function space in two dimensions is crucial to prove the existence of weak solutions and also for the weak-strong uniqueness result.

Concerning our weak-strong uniqueness result, we note that we don't except the existence of a solution $\tilde{\boldsymbol{u}}$ satisfying (2.3) for all $\boldsymbol{F} \in L^2(\Omega)$. In fact, we can easily construct counterexamples. For $\boldsymbol{u}_\infty \neq \boldsymbol{0}$, the derivative of a suitable smoothing of the Oseen fundamental solution will typically decay at infinity like $|\boldsymbol{x}|^{-1}$ in the wake and will be a weak solution for a particular forcing. For $\boldsymbol{u}_\infty = \boldsymbol{0}$, the smoothing of the exact solution $\boldsymbol{x}^\perp |\boldsymbol{x}|^{-2}$ will also be an exact solution decaying like $|\boldsymbol{x}|^{-1}$ for a forcing term of compact support. However, by using the asymptotic behavior proven by Babenko [2, Theorem 6.1], we can deduce some compatibility conditions on $\boldsymbol{f}$ such that the existence of a solution $\tilde{\boldsymbol{u}}$ satisfying (2.3) with $\boldsymbol{u}_\infty \neq \boldsymbol{0}$ can be deduced. For $\boldsymbol{u}_\infty = \boldsymbol{0}$, it was conjectured that some solutions could even decay like $|\boldsymbol{x}|^{-1/3}$ [8, §5.4]; however some compatibility conditions on $\boldsymbol{f}$ ensuring the existence of a solution satisfying (2.3) with $\boldsymbol{u}_\infty = \boldsymbol{0}$ are known [8, §3.6].

For two-dimensional exterior domains with $\partial\Omega \neq \emptyset$, we would *a priori* also expect the existence of infinitely many weak solutions parameterized by some parameter in $\mathbb{R}^2$. However, this question is open, and therefore no general weak-strong uniqueness result comparable to Theorem 2.9 is known if $\partial\Omega \neq \emptyset$. We remark that the method of proof used here for $\Omega = \mathbb{R}^2$ does not work if $\partial\Omega \neq \emptyset$ and that it is even not clear if the mean $\boldsymbol{\mu} \in \mathbb{R}^2$ will furnish a parametrization in this case.

The asymptotic behavior of the weak solutions in $\Omega = \mathbb{R}^2$ can obviously be determined when our weak-strong theorem is applicable, but otherwise we are not able to prove more than the best currently known results of Gilbarg and Weinberger [7, 19]. The result of Amick [1] cannot be used to prove the boundedness of the weak solutions, due to the fact that the maximum principle used in the proof does not hold on the region where $\boldsymbol{f}$ has support.

For $\Omega = \mathbb{R}^3$ and at any fixed force term $\boldsymbol{f}$, we expect the map $\boldsymbol{u}_\infty \in \mathbb{R}^3 \mapsto \boldsymbol{\mu} \in \mathbb{R}^3$ to be multivalued since nonuniqueness is expected for large data. Moreover, it is not clear if this map is surjective. In two dimensions, we might speculate the existence of a multivalued map $\boldsymbol{\mu} \in \mathbb{R}^2 \mapsto \boldsymbol{u}_\infty \in \mathbb{R}^2$ at fixed forcing $\boldsymbol{f}$, even if the asymptotic behavior of the weak solutions is unknown. However, it is not clear if one can find a nontrivial forcing $\boldsymbol{f}$ such that for any $\boldsymbol{u}_\infty \in \mathbb{R}^2$ a weak solution $\tilde{\boldsymbol{u}}$ satisfying the hypotheses of Theorem 2.9 can be proven. Therefore, we cannot prove that the mapping $\boldsymbol{\mu} \in \mathbb{R}^2 \mapsto \boldsymbol{u}_\infty \in \mathbb{R}^2$ is well-defined even for one nontrivial $\boldsymbol{f}$ (when $\boldsymbol{f} = \boldsymbol{0}$, the mapping is trivially the identity). Even if this could be proven, this is not clear if this well-defined map will be injective or surjective.

3. Function spaces

We first start with the following standard generalization of the Poincaré inequality; see for example Nečas [15, Theorems 1.5 and 1.9]:

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and let λ be a subset of positive measure of either Ω or $\partial\Omega$. Then, there exists $C > 0$ depending on Ω and λ such that*

$$\|\boldsymbol{u}\|_{L^2(\Omega)} \leq C \left(\|\boldsymbol{\nabla u}\|_{L^2(\Omega)} + \left| \fint_\lambda \boldsymbol{u} \right| \right),$$

for all $\boldsymbol{u} \in \dot{H}^1(\Omega)$.

Proof. First we note that if $\boldsymbol{u} \in \dot{H}^1(\Omega)$, then by the standard Poincaré inequality, $\boldsymbol{u} \in H^1(\Omega)$, so $\boldsymbol{u} \in L^1(\lambda)$ and the mean over λ is well-defined. We use a proof by contradiction. If the inequality is false, we can find a sequence $(\boldsymbol{u}_n)_{n\in\mathbb{N}} \in H^1(\Omega)$ such that $\|\boldsymbol{u}_n\|_{L^2(\Omega)} = 1$ and

$$\|\boldsymbol{\nabla u}_n\|_{L^2(\Omega)} + \left| \fint_\lambda \boldsymbol{u}_n \right| < \frac{1}{n}.$$

Since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, we can find a subsequence also denoted by $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ and $\boldsymbol{u} \in H^1(\Omega)$ such that $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$ weakly in $H^1(\Omega)$ and $\boldsymbol{u}_n \to \boldsymbol{u}$ strongly in $L^2(\Omega)$. Therefore,

$$\|\boldsymbol{\nabla u}\|_{L^2(\Omega)} \leq \liminf_{n\to\infty} \|\boldsymbol{\nabla u}_n\|_{L^2(\Omega)} = 0,$$

so $\boldsymbol{u}_n \to \boldsymbol{u}$ strongly in $H^1(\Omega)$ and $\boldsymbol{u}$ is a constant. We can show that

$$\fint_\lambda \boldsymbol{u} = \lim_{n\to\infty} \fint_\lambda \boldsymbol{u}_n = \boldsymbol{0},$$

and since λ has positive measure and Ω is connected, we obtain $\boldsymbol{u} = \boldsymbol{0}$, in contradiction to $\|\boldsymbol{u}\|_{L^2(\Omega)} = 1$. $\square$

In a second step, we determine a generalized Hardy inequality:

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^2$ be an exterior domain having a compact connected Lipschitz boundary (in particular $\Omega = \mathbb{R}^2$ is allowed), and let λ denote a bounded subset of positive measure of either Ω or $\partial\Omega$. There exists a constant $C > 0$ depending only on Ω and λ such that*

$$\|\boldsymbol{u}\mathfrak{w}\|_{L^2(\Omega)} \leq C \left(\|\boldsymbol{\nabla u}\|_{L^2(\Omega)} + \left| \fint_\lambda \boldsymbol{u} \right| \right),$$

for all $\boldsymbol{u} \in \dot{H}^1(\Omega)$, where

$$\mathfrak{w}(\boldsymbol{x}) = \frac{1}{\langle \boldsymbol{x} \rangle \langle \log\langle \boldsymbol{x} \rangle \rangle}, \qquad \langle \boldsymbol{x} \rangle = 1 + |\boldsymbol{x}|.$$

Proof. Let $R > 0$ be such that $\mathbb{R}^2 \setminus \Omega \subset B_R$ and $\lambda \subset B_R$. In this proof C denotes a positive constant depending only on λ and R, but which might change from line to line. Let χ be a smooth radial cutoff function such that $\chi(\boldsymbol{x}) = 1$ if $\boldsymbol{x} \in B_R$ and $\chi(\boldsymbol{x}) = 0$ if $\boldsymbol{x} \notin B_{2R}$. We consider the splitting $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2$, where $\boldsymbol{u}_1 = \chi \boldsymbol{u}$ and $\boldsymbol{u}_2 = (1-\chi)\boldsymbol{u}$. By using the generalized Poincaré inequality of Lemma 3.1, we first remark that

$$\|\boldsymbol{u}\|_{L^2(\Omega\cap B_{2R})} \leq C\left(\|\boldsymbol{\nabla u}\|_{L^2(\Omega\cap B_{2R})} + \left|\fint_\lambda \boldsymbol{u}\right|\right).$$

For the first part, we have

$$\begin{aligned}\|\boldsymbol{u}_1\mathfrak{w}\|_{L^2(\Omega)} = \|\chi\boldsymbol{u}\mathfrak{w}\|_{L^2(\Omega\cap B_{2R})} &\leq \|\chi\mathfrak{w}\|_{L^\infty(\Omega\cap B_{2R})}\|\boldsymbol{u}\|_{L^2(\Omega\cap B_{2R})} \\ &\leq C\left(\|\boldsymbol{\nabla u}\|_{L^2(\Omega)} + \left|\fint_\lambda \boldsymbol{u}\right|\right).\end{aligned}$$

For the second part, we first recall the standard Hardy inequality,

$$\left\|\frac{\boldsymbol{u}}{|\boldsymbol{x}|\log(R^{-1}|\boldsymbol{x}|)}\right\|_{L^2(\Omega\setminus B_R)} \leq \frac{2}{R}\|\boldsymbol{\nabla u}\|_{L^2(\Omega\setminus B_R)},$$

valid for all $\boldsymbol{u} \in H^1(\Omega \setminus B_R)$ having vanishing trace of ∂B_R; see for example Galdi [5, Theorem II.6.1]. Since there exists $C > 0$ such that

$$\mathfrak{w}(\boldsymbol{x}) = \frac{1}{\langle\boldsymbol{x}\rangle\langle\log\langle\boldsymbol{x}\rangle\rangle} \leq \frac{C}{|\boldsymbol{x}|\log(R^{-1}|\boldsymbol{x}|)},$$

for $|\boldsymbol{x}| > R$, we obtain

$$\|\boldsymbol{u}_2\mathfrak{w}\|_{L^2(\Omega)} = \|\boldsymbol{u}_2\mathfrak{w}\|_{L^2(\Omega\setminus B_R)} \leq C\|\boldsymbol{\nabla u}_2\|_{L^2(\Omega)}.$$

Since $\boldsymbol{\nabla u}_2 = (1-\chi)\boldsymbol{\nabla u} - \boldsymbol{\nabla}\chi \otimes \boldsymbol{u}$, we have

$$\begin{aligned}\|\boldsymbol{\nabla u}_2\|_{L^2(\Omega)} &\leq \|(1-\chi)\boldsymbol{\nabla u}\|_{L^2(\Omega)} + \|\boldsymbol{\nabla}\chi\otimes\boldsymbol{u}\|_{L^2(\Omega\cap B_{2R})} \\ &\leq \|1-\chi\|_{L^\infty(\Omega)}\|\boldsymbol{\nabla u}\|_{L^2(\Omega)} + \|\boldsymbol{\nabla}\chi\|_{L^\infty(\Omega\cap B_{2R})}\|\boldsymbol{u}\|_{L^2(\Omega\cap B_{2R})} \\ &\leq C\|\boldsymbol{\nabla u}\|_{L^2(\Omega)} + C\left(\|\boldsymbol{\nabla u}\|_{L^2(\Omega)} + \left|\fint_\lambda \boldsymbol{u}\right|\right).\end{aligned}$$

Therefore, putting all the bounds together, we have

$$\|\boldsymbol{u}\mathfrak{w}\|_{L^2(\Omega)} \leq \|\boldsymbol{u}_1\mathfrak{w}\|_{L^2(\Omega)} + \|\boldsymbol{u}_2\mathfrak{w}\|_{L^2(\Omega)} \leq C\left(\|\boldsymbol{\nabla u}\|_{L^2(\Omega)} + \left|\fint_\lambda \boldsymbol{u}\right|\right),$$

and the lemma is proven. □

In view of the result of Lemmas 3.1 and 3.2 with $\lambda = \partial\Omega$, we see that the semi-norm of $\dot{H}^1(\Omega)$ defines a norm on $C_0^\infty(\Omega)$ if $\partial\Omega \neq \emptyset$. Therefore, we have the following standard result; see for example Galdi [5] or Sohr [18]:

Proposition 3.3. *Let $\Omega \subset \mathbb{R}^2$ be a domain having a compact connected Lipschitz boundary $\partial\Omega \neq \emptyset$. Then the completion of $C_{0,\sigma}^\infty(\Omega)$ in the norm of $\dot{H}^1(\Omega)$ is the Hilbert space*

$$\dot{H}^1_{0,\sigma}(\Omega) = \left\{\boldsymbol{u} \in \dot{H}^1_\sigma(\Omega) : \Gamma_{\partial\Omega}\boldsymbol{u} = \boldsymbol{0}\right\},$$

with the inner product

$$\langle\boldsymbol{u},\boldsymbol{v}\rangle_{\dot{H}^1_{0,\sigma}(\Omega)} = \langle\boldsymbol{\nabla u},\boldsymbol{\nabla v}\rangle_{L^2(\Omega)}.$$

Moreover, $\dot{H}^1_{0,\sigma}(\Omega)$ has the following equivalent norms:

$$\left\|\boldsymbol{u}\right\|_{L^2(\Omega\cap B_R)} + \left\|\boldsymbol{\nabla u}\right\|_{L^2(\Omega)}\,,$$

for any $R > 0$ such $\partial\Omega \cap B_R \neq \emptyset$, and provided that Ω is an exterior domain

$$\left\|\boldsymbol{u}\mathfrak{w}\right\|_{L^2(\Omega)} + \left\|\boldsymbol{\nabla u}\right\|_{L^2(\Omega)}\,.$$

Proof. The proof that the completion of $C^\infty_{0,\sigma}(\Omega)$ in the norm of $\dot{H}^1(\Omega)$ is equal to $\dot{H}^1_{0,\sigma}(\Omega)$ is given in Galdi [5, Theorems II.7.3 and III.5.1] or in Sohr [18, Lemma III.1.2.1]. The equivalence of the norms follows from Lemmas 3.1 and 3.2 with $\lambda = \partial\Omega \cap B_R$, since λ has positive measure as a nonempty Lipschitz arc. □

When the boundary is trivial, i.e., $\Omega = \mathbb{R}^2$, the boundary cannot be used as an anchor point for the Poincaré inequality, and in particular the semi-norm of $\dot{H}^1(\Omega)$ does not define a norm on $C^\infty_0(\Omega)$. The idea is to fix some bounded subset $\omega \subset \Omega$ of positive measure so that $\dot{H}^1(\Omega)$ is a Hilbert space with the inner product

$$\left\langle \boldsymbol{\nabla u}, \boldsymbol{\nabla v}\right\rangle_{L^2(\Omega)} + \fint_\omega \boldsymbol{u} \cdot \fint_\omega \boldsymbol{v}\,.$$

For the case $\Omega = \mathbb{R}^2$ not covered by Proposition 3.3, we have the following result, which will play a crucial role in the construction of weak solutions in $\Omega = \mathbb{R}^2$:

Proposition 3.4. *Let $\Omega = \mathbb{R}^2$. Given a bounded subset $\omega \subset \Omega$ of positive measure, the completion of*

$$C^\infty_{0,\sigma}(\Omega,\omega) = \left\{\boldsymbol{\varphi} \in C^\infty_{0,\sigma}(\Omega) : \fint_\omega \boldsymbol{\varphi} = \boldsymbol{0}\right\}$$

in the norm of $\dot{H}^1(\Omega)$ is the Hilbert space

$$\dot{H}^1_{0,\sigma}(\Omega,\omega) = \left\{\boldsymbol{u} \in \dot{H}^1_\sigma(\Omega) : \fint_\omega \boldsymbol{u} = \boldsymbol{0}\right\}\,,$$

with the inner product

$$\left\langle \boldsymbol{u}, \boldsymbol{v}\right\rangle_{\dot{H}^1_{0,\sigma}(\Omega,\omega)} = \left\langle \boldsymbol{\nabla u}, \boldsymbol{\nabla v}\right\rangle_{L^2(\Omega)}\,.$$

Moreover, $\dot{H}^1_{0,\sigma}(\Omega,\omega)$ has the following equivalent norms:

$$\left\|\boldsymbol{u}\right\|_{L^2(B_R)} + \left\|\boldsymbol{\nabla u}\right\|_{L^2(\Omega)}\,,$$

for any $R > 0$ such that $\omega \subset B_R$, and

$$\left\|\boldsymbol{u}\mathfrak{w}\right\|_{L^2(\Omega)} + \left\|\boldsymbol{\nabla u}\right\|_{L^2(\Omega)}\,.$$

Proof. Let $\dot{H}^1_{0,\sigma}(\Omega,\omega)$ denote the completion of $C^\infty_{0,\sigma}(\Omega,\omega)$ in the norm of $\dot{H}^1(\Omega)$. First of all we remark that $\dot{H}^1_{0,\sigma}(\Omega,\omega) \subset \left\{\boldsymbol{u} \in \dot{H}^1_{0,\sigma}(\Omega) : \fint_\omega \boldsymbol{u} = \boldsymbol{0}\right\}$. Using the generalized Poincaré and Hardy inequalities (Lemmas 3.1 and 3.2), we have

$$\left\|\boldsymbol{u}\right\|^2_{L^2(B_R)} \le C\left(\left\|\boldsymbol{\nabla u}\right\|^2_{L^2(B_R)} + \left|\fint_\omega \boldsymbol{u}\right|^2\right)$$

and

$$\left\|\boldsymbol{u}\mathfrak{w}\right\|_{L^2(\Omega)} \le C\left(\left\|\boldsymbol{\nabla u}\right\|_{L^2(\Omega)} + \left|\fint_\omega \boldsymbol{u}\right|\right)\,,$$

for any $\boldsymbol{u} \in \dot{H}^1(\Omega)$, which show the claimed equivalence of the norms. Therefore, it only remains to prove that any $\boldsymbol{u} \in \dot{H}^1_{0,\sigma}(\Omega, \omega)$ can be approximated by functions in $C^\infty_{0,\sigma}(\Omega, \omega)$. The proof of this fact follows almost directly by using the proofs presented in Chapters II and III of Galdi [5], so we only sketch the main steps.

Let $\psi : \mathbb{R}^+ \to [0, 1]$ be a smooth cutoff function such that $\psi(r) = 1$ if $r \leq 1/2$ and $\psi(r) = 0$ if $r \geq 1$. For $n > 0$ large enough,

$$\psi_n(\boldsymbol{x}) = \psi\left(\frac{\log\langle\log\langle\boldsymbol{x}\rangle\rangle}{\log\langle\log\langle n\rangle\rangle}\right)$$

is a cutoff function such that $\psi_n(\boldsymbol{x}) = 0$ if $|\boldsymbol{x}| \geq n$ and $\psi_n(\boldsymbol{x}) = 1$ if $|\boldsymbol{x}| \leq \gamma_n$ where

$$\gamma_n = \exp\left(\sqrt{\langle\log\langle n\rangle\rangle} - 1\right) - 1\,.$$

Explicitly, we have

$$|\boldsymbol{\nabla}\psi_n(\boldsymbol{x})| \leq \frac{\|\psi'\|_\infty}{\log\langle\log\langle n\rangle\rangle}\mathfrak{w}(\boldsymbol{x}). \tag{3.1}$$

Therefore $\psi_n\boldsymbol{u}$ has compact support, vanishing mean on ω, belongs to $H^1(\Omega)$, and converges to $\boldsymbol{u}$ in $\dot{H}^1(\Omega)$ as $n \to \infty$ by using (3.1) and applying Lemma 3.2 (see [5, Theorems II.7.1 and II.7.2]). Moreover, $\psi_n\boldsymbol{u}$ is divergence-free except on the annulus $\gamma_n \leq |\boldsymbol{x}| \leq n$. There exists a corrector $\boldsymbol{w}_n \in \dot{H}^1(\Omega)$ having support in the annulus $\gamma_n \leq |\boldsymbol{x}| \leq n$ such that $\psi_n\boldsymbol{u} + \boldsymbol{w}_n$ is divergence-free and $\|\boldsymbol{w}_n\|_{\dot{H}^1(\Omega)} \leq C\,\|\boldsymbol{u}\cdot\boldsymbol{\nabla}\psi_n\|_{L^2(\Omega)}$ with $C > 0$ independent of n (see [5, Theorem III.3.1]). Therefore, $\psi_n\boldsymbol{u} + \boldsymbol{w}_n$ has support in B_n, is zero mean on ω, belongs to $\dot{H}^1_\sigma(\Omega)$, and converges to $\boldsymbol{u}$ in $\dot{H}^1_\sigma(\Omega)$ by (3.1) and Lemma 3.2. Now for any $n > 0$, there exists a smoothing $\boldsymbol{u}_n \in C^\infty_{0,\sigma}(\Omega)$ of $\psi_n\boldsymbol{u} + \boldsymbol{w}_n$ such that

$$\left\|\psi_n\boldsymbol{u} + \boldsymbol{w}_n - \boldsymbol{u}_n\right\|_{\dot{H}^1(\Omega)} + \left\|\psi_n\boldsymbol{u} + \boldsymbol{w}_n - \boldsymbol{u}_n\right\|_{L^2(B_n)} \leq \frac{1}{n}$$

(see [5, Theorems III.4.1 and III.4.2]). Hence we have

$$\left|\fint_\omega \boldsymbol{u}_n\right| = \left|\fint_\omega (\boldsymbol{u}_n - \psi_n\boldsymbol{u})\right| \leq \fint_\omega |\boldsymbol{u}_n - \psi_n\boldsymbol{u}| \leq |\omega|^{-1/2}\,\|\psi_n\boldsymbol{u} - \boldsymbol{u}_n\|_{L^2(\omega)} \leq \frac{1}{|\omega|^{1/2}\,n}\,.$$

Finally, it is not hard to find two explicit functions $\boldsymbol{v}_i \in C^\infty_{0,\sigma}(\Omega)$ such that $\fint_\omega \boldsymbol{v}_i = \boldsymbol{e}_i$ for $i = 1, 2$. Therefore $\boldsymbol{u}_n - (\boldsymbol{v}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{v}_2 \otimes \boldsymbol{e}_2)\cdot\fint_\omega \boldsymbol{u}_n \in C^\infty_{0,\sigma}(\Omega, \omega)$ converges to $\boldsymbol{u}$ in $\dot{H}^1_{0,\sigma}(\Omega, \omega)$ as $n \to \infty$. $\square$

Finally, we discuss conditions under which $\boldsymbol{f}$ can be represented as $\boldsymbol{f} = \boldsymbol{\nabla}\cdot\mathbf{F}$ with $\mathbf{F} \in L^2(\Omega)$ and in particular we prove the claims made in Remarks 2.3 and 2.7.

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^2$ be an exterior domain having a compact connected Lipschitz boundary $\partial\Omega \neq \emptyset$. Let $\boldsymbol{f} \in L^1_{\mathrm{loc}}(\Omega)$. If the linear form $\boldsymbol{\varphi} \mapsto \langle\boldsymbol{f}, \boldsymbol{\varphi}\rangle_{L^2(\Omega)}$ is continuous on $\dot{H}^1_{0,\sigma}(\Omega)$, then there exists $\boldsymbol{F} \in L^2(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{\nabla}\cdot\boldsymbol{F}$ in the following sense:*

$$\langle\boldsymbol{f}, \boldsymbol{\varphi}\rangle_{L^2(\Omega)} = -\langle\mathbf{F}, \boldsymbol{\nabla}\boldsymbol{\varphi}\rangle_{L^2(\Omega)}\,,$$

for all $\boldsymbol{\varphi} \in C^\infty_{0,\sigma}(\Omega)$. In particular this holds when $\boldsymbol{f}/\mathfrak{w} \in L^2(\Omega)$.

Proof. By using the Riesz representation theorem, there exists $\boldsymbol{u} \in \dot{H}^1_{0,\sigma}(\Omega)$ such that

$$\big\langle \boldsymbol{\nabla u}, \boldsymbol{\nabla \varphi} \big\rangle_{L^2(\Omega)} = \big\langle \boldsymbol{f}, \boldsymbol{\varphi} \big\rangle_{L^2(\Omega)} ,$$

for all $\boldsymbol{\varphi} \in \dot{H}^1_0(\Omega)$, and we can take $\mathbf{F} = -\boldsymbol{\nabla u}$. If $\boldsymbol{f}/\mathfrak{w} \in L^2(\Omega)$, then by Lemma 3.2 with $\lambda = \partial\Omega$, we have

$$\Big| \big\langle \boldsymbol{f}, \boldsymbol{\varphi} \big\rangle_{L^2(\Omega)} \Big| \leq \big\| \boldsymbol{f}/\mathfrak{w} \big\|_{L^2(\Omega)} \big\| \boldsymbol{\varphi}\mathfrak{w} \big\|_{L^2(\Omega)} \leq C \big\| \boldsymbol{\nabla \varphi} \big\|_{L^2(\Omega)} ,$$

so the linear form is continuous on $\dot{H}^1_0(\Omega)$. □

Lemma 3.6. *Let $\Omega = \mathbb{R}^2$ and let $\omega \subset \Omega$ be a bounded subset of positive measure. If the linear form $\boldsymbol{\varphi} \mapsto \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{L^2(\Omega)}$ is continuous on $\dot{H}^1_{0,\sigma}(\Omega, \omega)$ and $\int_\Omega \boldsymbol{f} = \boldsymbol{0}$, then there exists $\boldsymbol{F} \in L^2(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{\nabla} \cdot \boldsymbol{F}$ in the following sense:*

$$\big\langle \boldsymbol{f}, \boldsymbol{\varphi} \big\rangle_{L^2(\Omega)} = -\big\langle \mathbf{F}, \boldsymbol{\nabla \varphi} \big\rangle_{L^2(\Omega)} ,$$

for all $\boldsymbol{\varphi} \in C^\infty_{0,\sigma}(\Omega)$. In particular this holds when $\boldsymbol{f}/\mathfrak{w} \in L^2(\Omega)$ and $\int_\Omega \boldsymbol{f} = \boldsymbol{0}$.

Proof. By using the Riesz representation theorem, there exists $\boldsymbol{u} \in \dot{H}^1_{0,\sigma}(\Omega, \omega)$ such that

$$\big\langle \boldsymbol{\nabla u}, \boldsymbol{\nabla \psi} \big\rangle_{L^2(\Omega)} = \big\langle \boldsymbol{f}, \boldsymbol{\psi} \big\rangle_{L^2(\Omega)} ,$$

for all $\boldsymbol{\psi} \in \dot{H}^1_{0,\sigma}(\Omega, \omega)$. For any $\boldsymbol{\varphi} \in C^\infty_{0,\sigma}(\Omega)$, let $\boldsymbol{\psi} = \boldsymbol{\varphi} - \fint \boldsymbol{\varphi} \in \dot{H}^1_{0,\sigma}(\Omega, \omega)$, and therefore

$$\big\langle \boldsymbol{\nabla u}, \boldsymbol{\nabla \varphi} \big\rangle_{L^2(\Omega)} = \big\langle \boldsymbol{f}, \boldsymbol{\psi} \big\rangle_{L^2(\Omega)} = \big\langle \boldsymbol{f}, \boldsymbol{\varphi} \big\rangle_{L^2(\Omega)}$$

because $\int_\Omega \boldsymbol{f} = \boldsymbol{0}$. If in addition $\boldsymbol{f}/\mathfrak{w} \in L^2(\Omega)$, then by Lemma 3.2 with $\lambda = \omega$, we have

$$\Big| \big\langle \boldsymbol{f}, \boldsymbol{\psi} \big\rangle_{L^2(\Omega)} \Big| \leq \big\| \boldsymbol{f}/\mathfrak{w} \big\|_{L^2(\Omega)} \big\| \boldsymbol{\psi}\mathfrak{w} \big\|_{L^2(\Omega)} \leq C \big\| \boldsymbol{\nabla \psi} \big\|_{L^2(\Omega)} ,$$

for any $\boldsymbol{\psi} \in \dot{H}^1_{0,\sigma}(\Omega, \omega)$. □

Remark 3.7. The hypothesis $\int_\Omega \boldsymbol{f} = \boldsymbol{0}$ is needed only for $\Omega = \mathbb{R}^2$ and not if $\partial\Omega \neq \emptyset$. This fact is linked to the Stokes paradox, since the existence proof given below works equally well for the Stokes equation. For $\Omega = \mathbb{R}^2$, it is well known that the Stokes equations have a solution in $\dot{H}^1_\sigma(\Omega)$ if and only if $\int_\Omega \boldsymbol{f} = \boldsymbol{0}$. Otherwise, the solutions of the Stokes equations in $\Omega = \mathbb{R}^2$ grow like $\log |\boldsymbol{x}|$ at infinity; hence the Stokes equations have no solutions in $\dot{H}^1_\sigma(\Omega)$. If $\Omega \neq \mathbb{R}^2$, the Stokes equations always admit a solution in $\dot{H}^1_\sigma(\Omega)$ regardless of the mean of $\boldsymbol{f}$.

4. Proof of existence

The main idea to construct weak solutions in $\Omega = \mathbb{R}^2$ is to construct for each $n \in \mathbb{N}$ large enough a particular weak solution in the ball B_n having a prescribed mean on a bounded subset of positive measure $\omega \subset \Omega$. This can be done by choosing a suitable constant $\boldsymbol{c}_n$ on the artificial boundary ∂B_n.

Proposition 4.1. *Assume that the hypotheses of Theorem* 2.6 *hold. For any $\boldsymbol{\mu} \in \mathbb{R}^2$ and $n \in \mathbb{N}$ large enough such that $\omega \subset B_n$, there exist $\boldsymbol{c}_n \in \mathbb{R}^2$ and a weak solution $\boldsymbol{u}_n \in \dot{H}^1_\sigma(B_n)$ of the Navier–Stokes equations* (1.1) *in B_n such that:*

(1) $\boldsymbol{u}_n|_{\partial B_n} = \boldsymbol{\mu} + \boldsymbol{c}_n$ *in the trace sense;*

(2) $\big\| \boldsymbol{\nabla u}_n \big\|^2_{L^2(B_n)} = \big\langle \mathbf{F}, \boldsymbol{\nabla u}_n \big\rangle_{L^2(B_n)}$;

(3) $\fint_\omega \boldsymbol{u}_n = \boldsymbol{\mu}$.

Proof. For any vector field $\boldsymbol{v} \in L^1_{\mathrm{loc}}(\omega)$, we denote by $\bar{\boldsymbol{v}}$ the mean of $\boldsymbol{v}$ on ω, $\bar{\boldsymbol{v}} = \fint_\omega \boldsymbol{v} = \frac{1}{|\omega|}\int_\omega \boldsymbol{v}$. We look for a solution of the form $\boldsymbol{u}_n = \boldsymbol{\mu} + \boldsymbol{v}_n - \bar{\boldsymbol{v}}_n$ with $\boldsymbol{v}_n \in \dot{H}^1_{0,\sigma}(B_n)$ so that the third condition of the proposition automatically holds. We have $\boldsymbol{u}_n|_{\partial B_n} = \boldsymbol{\mu} - \bar{\boldsymbol{v}}_n$, so the first condition is satisfied by choosing $\boldsymbol{c}_n = -\bar{\boldsymbol{v}}_n$. Therefore, it remains to prove the existence of $\boldsymbol{v}_n \in \dot{H}^1_{0,\sigma}(B_n)$ such that

$$\big\langle \boldsymbol{\nabla} \boldsymbol{v}_n, \boldsymbol{\nabla}\boldsymbol{\varphi}\big\rangle_{L^2(B_n)} + \big\langle \big(\boldsymbol{\mu} + \boldsymbol{v}_n - \bar{\boldsymbol{v}}_n\big)\cdot \boldsymbol{\nabla}\boldsymbol{v}_n, \boldsymbol{\varphi}\big\rangle_{L^2(B_n)} = \big\langle \mathbf{F}, \boldsymbol{\nabla}\boldsymbol{\varphi}\big\rangle_{L^2(B_n)}, \tag{4.1}$$

for all $\boldsymbol{\varphi} \in C^\infty_{0,\sigma}(B_n)$.

Since

$$\Big|\big\langle \mathbf{F}, \boldsymbol{\nabla}\boldsymbol{\varphi}\big\rangle_{L^2(B_n)}\Big| \le \|\mathbf{F}\|_{L^2(B_n)} \|\boldsymbol{\nabla}\boldsymbol{\varphi}\|_{L^2(B_n)} \le \|\mathbf{F}\|_{L^2(\Omega)} \|\boldsymbol{\varphi}\|_{\dot{H}^1_{0,\sigma}(B_n)},$$

for all $\boldsymbol{\varphi} \in \dot{H}^1_{0,\sigma}(B_n)$, by using the Riesz representation theorem, there exists $\boldsymbol{R}_n \in \dot{H}^1_{0,\sigma}(B_n)$, such that

$$\big\langle \boldsymbol{R}_n, \boldsymbol{\varphi}\big\rangle_{\dot{H}^1_{0,\sigma}(B_n)} = \big\langle \mathbf{F}, \boldsymbol{\nabla}\boldsymbol{\varphi}\big\rangle_{L^2(B_n)},$$

for all $\boldsymbol{\varphi} \in C^\infty_{0,\sigma}(\Omega)$.

The bilinear map $\boldsymbol{\mathcal{B}}_n$ defined by

$$\big\langle \boldsymbol{\mathcal{B}}_n(\boldsymbol{v}, \boldsymbol{w}), \boldsymbol{\varphi}\big\rangle_{\dot{H}^1_{0,\sigma}(B_n)} = \big\langle (\boldsymbol{v} - \bar{\boldsymbol{v}})\cdot\boldsymbol{\nabla}\boldsymbol{w}, \boldsymbol{\varphi}\big\rangle_{L^2(B_n)}$$

is continuous on $L^4(B_n)$,

$$\begin{aligned}
\Big|\big\langle \boldsymbol{\mathcal{B}}_n(\boldsymbol{v}, \boldsymbol{w}), \boldsymbol{\varphi}\big\rangle_{\dot{H}^1_{0,\sigma}(B_n)}\Big| &\le \Big|\big\langle (\boldsymbol{v} - \bar{\boldsymbol{v}})\cdot\boldsymbol{\nabla}\boldsymbol{\varphi}, \boldsymbol{w}\big\rangle_{L^2(B_n)}\Big| \\
&\le \Big(\|\boldsymbol{v}\|_{L^4(B_n)} + \|\bar{\boldsymbol{v}}\|_{L^4(B_n)}\Big)\|\boldsymbol{w}\|_{L^4(B_n)}\|\boldsymbol{\varphi}\|_{\dot{H}^1_{0,\sigma}(B_n)} \\
&\le \left(1 + \frac{\pi n^2}{|\omega|}\right)\|\boldsymbol{v}\|_{L^4(B_n)}\|\boldsymbol{w}\|_{L^4(B_n)}\|\boldsymbol{\varphi}\|_{\dot{H}^1_{0,\sigma}(B_n)},
\end{aligned}$$

because

$$\|\bar{\boldsymbol{v}}\|_{L^4(B_n)} \le \pi^{1/4} n^{1/2}\,|\bar{\boldsymbol{v}}| \le \frac{\pi^{1/4} n^{1/2}}{|\omega|}\int_{B_n} |\boldsymbol{v}| \le \frac{\pi n^2}{|\omega|}\|\boldsymbol{v}\|_{L^4(B_n)}.$$

The linear map $\boldsymbol{\mathcal{L}}_n$ defined by

$$\big\langle \boldsymbol{\mathcal{L}}_n(\boldsymbol{v}), \boldsymbol{\varphi}\big\rangle_{\dot{H}^1_{0,\sigma}(B_n)} = \big\langle \boldsymbol{\mu}\cdot\boldsymbol{\nabla}\boldsymbol{v}, \boldsymbol{\varphi}\big\rangle_{L^2(B_n)}$$

is also continuous on $L^4(B_n)$,

$$\Big|\big\langle \boldsymbol{\mathcal{L}}_n(\boldsymbol{v}), \boldsymbol{\varphi}\big\rangle_{\dot{H}^1_{0,\sigma}(B_n)}\Big| \le \Big|\big\langle \boldsymbol{\mu}\cdot\boldsymbol{\nabla}\boldsymbol{\varphi}, \boldsymbol{v}\big\rangle_{L^2(B_n)}\Big| \le \|\boldsymbol{\mu}\|_{L^4(B_n)}\|\boldsymbol{v}\|_{L^4(B_n)}\|\boldsymbol{\varphi}\|_{\dot{H}^1_{0,\sigma}(B_n)}.$$

Therefore, the map $\boldsymbol{\mathcal{A}}_n : \dot{H}^1_{0,\sigma}(B_n) \to \dot{H}^1_{0,\sigma}(B_n)$ defined by $\boldsymbol{\mathcal{A}}_n(\boldsymbol{v}) = \boldsymbol{\mathcal{B}}_n(\boldsymbol{v}, \boldsymbol{v}) + \boldsymbol{\mathcal{L}}_n(\boldsymbol{v})$ is continuous on $\dot{H}^1_{0,\sigma}(B_n)$ when equipped with the L^4-norm, hence completely continuous on $\dot{H}^1_{0,\sigma}(B_n)$, since $\dot{H}^1_{0,\sigma}(B_n)$ is compactly embedded in $L^4(B_n)$.

We have

$$\begin{aligned}
\big\langle \boldsymbol{v}_n + \boldsymbol{\mathcal{A}}_n(\boldsymbol{v}_n) - \boldsymbol{R}_n, \boldsymbol{\varphi}\big\rangle_{\dot{H}^1_{0,\sigma}(B_n)} =& \big\langle \boldsymbol{\nabla}\boldsymbol{v}_n, \boldsymbol{\nabla}\boldsymbol{\varphi}\big\rangle_{L^2(B_n)} + \big\langle \big(\boldsymbol{\mu} + \boldsymbol{v}_n - \bar{\boldsymbol{v}}_n\big) \\
&\cdot \boldsymbol{\nabla}\boldsymbol{v}_n, \boldsymbol{\varphi}\big\rangle_{L^2(B_n)} - \big\langle \mathbf{F}, \boldsymbol{\nabla}\boldsymbol{\varphi}\big\rangle_{L^2(B_n)},
\end{aligned}$$

so the weak formulation (4.1) is equivalent to the functional equation

$$\boldsymbol{v}_n + \boldsymbol{\mathcal{A}}_n(\boldsymbol{v}_n) - \boldsymbol{R}_n = \boldsymbol{0} \tag{4.2}$$

in $\dot{H}^1_{0,\sigma}(B_n)$. From the Leray–Schauder fixed point theorem (see for example Gilbarg and Trudinger [6, Theorem 11.6]), to prove the existence of a solution to (4.2) it is sufficient to prove that the set of solutions $\boldsymbol{v}$ of the equation

$$\boldsymbol{v}_n + \lambda\left(\boldsymbol{\mathcal{A}}_n(\boldsymbol{v}_n) - \boldsymbol{R}_n\right) = \boldsymbol{0} \tag{4.3}$$

is uniformly bounded in $\lambda \in [0,1]$. To this end, we take the scalar product of (4.3) with $\boldsymbol{v}_n$,

$$\left\langle \boldsymbol{\nabla}\boldsymbol{v}_n, \boldsymbol{\nabla}\boldsymbol{v}_n \right\rangle_{L^2(B_n)} + \lambda\left\langle \left(\boldsymbol{\mu} + \boldsymbol{v}_n - \bar{\boldsymbol{v}}_n\right) \boldsymbol{\cdot} \boldsymbol{\nabla}\boldsymbol{v}_n, \boldsymbol{v}_n \right\rangle_{L^2(B_n)} = \lambda\left\langle \mathbf{F}, \boldsymbol{\nabla}\boldsymbol{v}_n \right\rangle_{L^2(B_n)} .$$

By integrating by parts, we obtain

$$\left\langle \boldsymbol{\nabla}\boldsymbol{v}_n, \boldsymbol{\nabla}\boldsymbol{v}_n \right\rangle_{L^2(B_n)} = \lambda\left\langle \mathbf{F}, \boldsymbol{\nabla}\boldsymbol{v}_n \right\rangle_{L^2(B_n)} ,$$

so

$$\left\|\boldsymbol{\nabla}\boldsymbol{v}_n\right\|_{L^2(B_n)} \leq \left\|\boldsymbol{F}\right\|_{L^2(B_n)} \leq \left\|\boldsymbol{F}\right\|_{L^2(\Omega)} .$$

□

Now we can prove the existence of weak solutions in $\Omega = \mathbb{R}^2$ by using the method of invading domains:

Proof of Theorem 2.6. By Proposition 4.1, for any $n \in \mathbb{N}$, there exists $\boldsymbol{c}_n \in \mathbb{R}^2$ and a weak solution $\boldsymbol{u}_n \in \dot{H}^1_\sigma(B_n)$ satisfying the three conditions of this proposition. We write $\boldsymbol{u}_n = \boldsymbol{\mu} + \boldsymbol{v}_n$, so extending $\boldsymbol{v}_n$ to Ω by setting $\boldsymbol{v}_n = \boldsymbol{c}_n$ on $\Omega \setminus B_n$, we have

$$\left\|\boldsymbol{\nabla}\boldsymbol{v}_n\right\|^2_{L^2(\Omega)} = \left\langle \mathbf{F}, \boldsymbol{\nabla}\boldsymbol{v}_n \right\rangle_{L^2(\Omega)} , \qquad \fint_\omega \boldsymbol{v}_n = \boldsymbol{0} ,$$

and $(\boldsymbol{v}_n)_{n\in\mathbb{N}}$ is bounded by $\|\mathbf{F}\|_{L^2(\Omega)}$ in the function space $\dot{H}^1_{0,\sigma}(\Omega,\omega)$ defined by Proposition 3.4. Therefore, there exists a subsequence also denoted by $(\boldsymbol{v}_n)_{n\in\mathbb{N}}$ which converges weakly to $\boldsymbol{v} \in \dot{H}^1_{0,\sigma}(\Omega,\omega)$. Let $\boldsymbol{u} = \boldsymbol{\mu} + \boldsymbol{v}$. We directly obtain that

$$\left\|\boldsymbol{\nabla}\boldsymbol{u}\right\|^2_{L^2(\Omega)} = \left\|\boldsymbol{\nabla}\boldsymbol{v}\right\|^2_{L^2(\Omega)} \leq \liminf_{n\to\infty}\left\|\boldsymbol{\nabla}\boldsymbol{v}_n\right\|^2_{L^2(\Omega)}$$

and

$$\lim_{n\to\infty}\left\langle \mathbf{F}, \boldsymbol{\nabla}\boldsymbol{v}_n \right\rangle_{L^2(\Omega)} = \left\langle \mathbf{F}, \boldsymbol{\nabla}\boldsymbol{v} \right\rangle_{L^2(\Omega)} = \left\langle \mathbf{F}, \boldsymbol{\nabla}\boldsymbol{u} \right\rangle_{L^2(\Omega)} ,$$

so the energy inequality (2.2) is proven.

We now prove that the limit $\boldsymbol{u}$ is a weak solution to the Navier–Stokes equations in Ω. Let $\boldsymbol{\varphi} \in C^\infty_{0,\sigma}(\Omega)$. There exists $m \in \mathbb{N}$ such that the support of $\boldsymbol{\varphi}$ is contained in B_m. In view of Proposition 3.4, $(\boldsymbol{v}_n)_{n\in\mathbb{N}}$ is bounded in $H^1(B_m)$, so there exists a subsequence also denoted by $(\boldsymbol{v}_n)_{n\in\mathbb{N}}$ which converges strongly to $\boldsymbol{v}$ in $L^4(B_m)$, since $H^1(B_m)$ is compactly embedded in $L^4(B_m)$. Since $\boldsymbol{u}_n = \boldsymbol{\mu} + \boldsymbol{v}_n$ is a weak solution in B_n, we have

$$\left\langle \boldsymbol{\nabla}\boldsymbol{u}_n, \boldsymbol{\nabla}\boldsymbol{\varphi} \right\rangle_{L^2(B_m)} + \left\langle \boldsymbol{u}_n \boldsymbol{\cdot} \boldsymbol{\nabla}\boldsymbol{u}_n, \boldsymbol{\varphi} \right\rangle_{L^2(B_m)} = \left\langle \mathbf{F}, \boldsymbol{\nabla}\boldsymbol{\varphi} \right\rangle_{L^2(B_m)} ,$$

for any $n \geq m$, and it only remains to show that this equation remains valid in the limit $n \to \infty$. Let $\boldsymbol{\psi} = \boldsymbol{\varphi} - \fint_\omega \boldsymbol{\varphi}$, where by Proposition 3.4, $\boldsymbol{\psi} \in \dot{H}^1_{0,\sigma}(\Omega,\omega)$. By definition of the weak convergence,

$$\begin{aligned}\lim_{n\to\infty}\left\langle \boldsymbol{\nabla}\boldsymbol{u}_n, \boldsymbol{\nabla}\boldsymbol{\varphi} \right\rangle_{L^2(B_m)} &= \lim_{n\to\infty}\left\langle \boldsymbol{v}_n, \boldsymbol{\psi} \right\rangle_{\dot{H}^1_{0,\sigma}(\Omega,\omega)} \\ &= \left\langle \boldsymbol{v}, \boldsymbol{\psi} \right\rangle_{\dot{H}^1_{0,\sigma}(\Omega,\omega)} = \left\langle \boldsymbol{\nabla}\boldsymbol{u}, \boldsymbol{\nabla}\boldsymbol{\varphi} \right\rangle_{L^2(B_m)} .\end{aligned}$$

Since $\boldsymbol{\varphi}$ has compact support in B_m, we have

$$\begin{aligned}
&\left|\left\langle \boldsymbol{u}_n \cdot \boldsymbol{\nabla} \boldsymbol{u}_n - \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}, \boldsymbol{\varphi}\right\rangle_{L^2(B_m)}\right| \\
&\quad \leq \left|\left\langle (\boldsymbol{u}_n - \boldsymbol{u}) \cdot \boldsymbol{\nabla} \boldsymbol{u}_n, \boldsymbol{\varphi}\right\rangle_{L^2(B_m)}\right| + \left|\left\langle \boldsymbol{u} \cdot (\boldsymbol{\nabla} \boldsymbol{u}_n - \boldsymbol{\nabla} \boldsymbol{u}), \boldsymbol{\varphi}\right\rangle_{L^2(B_m)}\right| \\
&\quad \leq \left|\left\langle (\boldsymbol{v}_n - \boldsymbol{v}) \cdot \boldsymbol{\nabla} \boldsymbol{v}_n, \boldsymbol{\varphi}\right\rangle_{L^2(B_m)}\right| + \left|\left\langle \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{\varphi}, \boldsymbol{v}_n - \boldsymbol{v}\right\rangle_{L^2(B_m)}\right| \\
&\quad \leq \left(\left\|\boldsymbol{\nabla} \boldsymbol{v}_n\right\|_{L^2(B_m)} \left\|\boldsymbol{\varphi}\right\|_{L^4(B_m)} + \left\|\boldsymbol{u}\right\|_{L^4(B_m)} \left\|\boldsymbol{\nabla} \boldsymbol{\varphi}\right\|_{L^2(B_m)}\right) \left\|\boldsymbol{v}_n - \boldsymbol{v}\right\|_{L^4(B_m)} ,
\end{aligned}$$

so

$$\lim_{n\to\infty} \left\langle \boldsymbol{u}_n \cdot \boldsymbol{\nabla} \boldsymbol{u}_n, \boldsymbol{\varphi}\right\rangle_{L^2(B_m)} = \left\langle \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}, \boldsymbol{\varphi}\right\rangle_{L^2(B_m)} ,$$

and $\boldsymbol{u}$ satisfies (2.1). □

5. Proof of uniqueness

We first start with the following approximation lemma:

Lemma 5.1. *For $\Omega = \mathbb{R}^2$, if $\tilde{\boldsymbol{v}} \in \dot{H}^1_\sigma(\Omega)$ satisfies $\tilde{\boldsymbol{v}}/\mathfrak{w} \in L^\infty(\Omega)$, then there exists a sequence $(\tilde{\boldsymbol{v}}_n)_{n\in\mathbb{N}} \subset C^\infty_{0,\sigma}(\Omega)$ such that $\tilde{\boldsymbol{v}}_n \to \tilde{\boldsymbol{v}}$ strongly in $\dot{H}^1_\sigma(\Omega)$ and $\boldsymbol{u} \otimes \tilde{\boldsymbol{v}}_n \to \boldsymbol{u} \otimes \tilde{\boldsymbol{v}}$ strongly in $L^2(\Omega)$ for any $\boldsymbol{u} \in \dot{H}^1_\sigma(\Omega)$.*

Proof. First of all we need a better Sobolev cutoff than the one used in the proof of Proposition 3.4. Let $\eta : \mathbb{R}^+ \to [0,1]$ be a smooth cutoff function such that $\eta(r) = 1$ if $r \leq 1/2$ and $\eta(r) = 0$ if $r \geq 1$. For $n > 0$ large enough,

$$\eta_n(\boldsymbol{x}) = \eta\left(\frac{\log\langle\log\langle\log\langle\boldsymbol{x}\rangle\rangle\rangle}{\log\langle\log\langle\log\langle n\rangle\rangle\rangle}\right)$$

is a cutoff function such that $\eta_n(\boldsymbol{x}) = 0$ if $|\boldsymbol{x}| \geq n$ and $\eta_n(\boldsymbol{x}) = 1$ if $|\boldsymbol{x}| \leq \gamma_n$ where

$$\gamma_n = \exp\left(\exp\left(\sqrt{\langle\log\langle\log\langle n\rangle\rangle\rangle} - 1\right) - 1\right) - 1 .$$

Explicitly, we have

$$\left|\boldsymbol{\nabla}\eta_n(\boldsymbol{x})\right| \leq \frac{\|\eta'\|_\infty}{\log\langle\log\langle\log\langle n\rangle\rangle\rangle} \frac{1}{\langle\boldsymbol{x}\rangle\langle\log\langle\boldsymbol{x}\rangle\rangle\langle\log\langle\log\langle\boldsymbol{x}\rangle\rangle\rangle} \tag{5.1}$$

and

$$\left|\boldsymbol{\nabla}^2\eta_n(\boldsymbol{x})\right| \leq \frac{4\,\|\eta'\|_\infty + 2\,\|\eta''\|_\infty}{\log\langle\log\langle\log\langle n\rangle\rangle\rangle} \frac{1}{\langle\boldsymbol{x}\rangle^2\langle\log\langle\boldsymbol{x}\rangle\rangle\langle\log\langle\log\langle\boldsymbol{x}\rangle\rangle\rangle} . \tag{5.2}$$

By the trace formula, $\tilde{\boldsymbol{v}}$ is integrable on the segment $[\boldsymbol{0}, \boldsymbol{x}]$ for any $\boldsymbol{x} \in \Omega$ since $\tilde{\boldsymbol{v}} \in H^1_{\mathrm{loc}}(\Omega)$, and we can define the stream function associated to $\tilde{\boldsymbol{v}}$ by the following curvilinear integral (see for example Galdi [5, Lemma IX.4.1]):

$$\tilde{\psi}(\boldsymbol{x}) = \int_{\boldsymbol{0}}^{\boldsymbol{x}} \tilde{\boldsymbol{v}}^\perp \cdot \mathrm{d}\boldsymbol{x} .$$

Since $\tilde{\boldsymbol{v}}/\mathfrak{w} \in L^\infty(\Omega)$, we obtain the following bound on the stream function:

$$\left|\tilde{\psi}(\boldsymbol{x})\right| \leq C \int_0^{|\boldsymbol{x}|} \frac{1}{\langle r\rangle\langle\log\langle r\rangle\rangle} \mathrm{d}r \leq C \log\langle\log\langle\boldsymbol{x}\rangle\rangle . \tag{5.3}$$

Now let $\tilde{\boldsymbol{v}}_n = \boldsymbol{\nabla}^\perp\left(\eta_n\tilde{\psi}\right)$. We have $\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}_n = (1-\eta_n)\,\tilde{\boldsymbol{v}} - \tilde{\psi}\boldsymbol{\nabla}^\perp\eta_n$, so

$$\left\|\boldsymbol{u} \otimes (\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}_n)\right\|_{L^2(\Omega)} \leq \left\|(1-\eta_n)\boldsymbol{u} \otimes \tilde{\boldsymbol{v}}\right\|_{L^2(\Omega)} + \left\|\tilde{\psi}\boldsymbol{u} \otimes \boldsymbol{\nabla}\eta_n\right\|_{L^2(\Omega)} .$$

The first term goes to zero as $n \to \infty$ since $\boldsymbol{u} \otimes \tilde{\boldsymbol{v}} \in L^2(\Omega)$ because $\boldsymbol{u}\mathfrak{w} \in L^2(\Omega)$ and $\tilde{\boldsymbol{v}}/\mathfrak{w} \in L^\infty(\Omega)$. Using the bound (5.1) on $\boldsymbol{\nabla}\eta_n$ and the bound (5.3) on $\tilde{\psi}$, we obtain

$$\bigl\|\tilde{\psi}\boldsymbol{u} \otimes \boldsymbol{\nabla}\eta_n\bigr\|_{L^2(\Omega)} \leq \frac{C}{\log\langle\log\langle\log\langle n\rangle\rangle\rangle}\bigl\|\boldsymbol{u}\mathfrak{w}\bigr\|_{L^2(\Omega)} ,$$

so the second term also goes to zero as $n \to \infty$, since $\boldsymbol{u}\mathfrak{w} \in L^2(\Omega)$ in view of Lemma 3.2. Finally, we have

$$\bigl\|\boldsymbol{\nabla}\tilde{\boldsymbol{v}} - \boldsymbol{\nabla}\tilde{\boldsymbol{v}}_n\bigr\|_{L^2(\Omega)} \leq \bigl\|(1-\eta_n)\boldsymbol{\nabla}\tilde{\boldsymbol{v}}\bigr\|_{L^2(\Omega)} + 2\bigl\|\boldsymbol{\nabla}\eta_n \otimes \tilde{\boldsymbol{v}}\bigr\|_{L^2(\Omega)} + \bigl\|\tilde{\psi}\boldsymbol{\nabla}^2\eta_n\bigr\|_{L^2(\Omega)} .$$

The first term goes to zero since $\boldsymbol{\nabla}\tilde{\boldsymbol{v}} \in L^2(\Omega)$. For the second term, using (5.1) we have

$$\bigl\|\boldsymbol{\nabla}\eta_n \otimes \tilde{\boldsymbol{v}}\bigr\|_{L^2(\Omega)} \leq \frac{C}{\log\langle\log\langle\log\langle n\rangle\rangle\rangle}\bigl\|\tilde{\boldsymbol{v}}\mathfrak{w}\bigr\|_{L^2(\Omega)} ,$$

and using (5.2) for the third term,

$$\bigl\|\tilde{\psi}\boldsymbol{\nabla}^2\eta_n\bigr\|_{L^2(\Omega)} \leq \frac{C}{\log\langle\log\langle\log\langle n\rangle\rangle\rangle}\bigl\|\langle\boldsymbol{x}\rangle^{-2}\bigr\|_{L^2(\Omega)} ,$$

so both converge to zero and $\tilde{\boldsymbol{v}}_n \to \tilde{\boldsymbol{v}}$ in $\dot{H}^1_\sigma(\Omega)$. Finally, the sequence $(\tilde{\boldsymbol{v}}_n)_{n\in\mathbb{N}}$ can be smoothed by using the standard mollification technique. □

Using the previous lemma, we can replace $\boldsymbol{\varphi}$ by $\tilde{\boldsymbol{v}}$ in the definition of the weak solution $\boldsymbol{u}$:

Lemma 5.2. *If $\boldsymbol{u}$ is a weak solution in $\Omega = \mathbb{R}^2$, then*

$$\bigl\langle\boldsymbol{\nabla}\boldsymbol{u}, \boldsymbol{\nabla}\tilde{\boldsymbol{v}}\bigr\rangle_{L^2(\Omega)} + \bigl\langle\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}, \tilde{\boldsymbol{v}}\bigr\rangle_{L^2(\Omega)} = \bigl\langle\mathbf{F}, \boldsymbol{\nabla}\tilde{\boldsymbol{v}}\bigr\rangle_{L^2(\Omega)} ,$$

for any $\tilde{\boldsymbol{v}} \in \dot{H}^1_\sigma(\Omega)$ satisfying $\tilde{\boldsymbol{v}}/\mathfrak{w} \in L^\infty(\Omega)$.

Proof. Let $(\tilde{\boldsymbol{v}}_n)_{n\in\mathbb{N}} \subset C^\infty_{0,\sigma}(\Omega)$ be the approximation of $\tilde{\boldsymbol{v}}$ constructed in Lemma 5.1. Since $\boldsymbol{u}$ is a weak solution, we have

$$\bigl\langle\boldsymbol{\nabla}\boldsymbol{u}, \boldsymbol{\nabla}\tilde{\boldsymbol{v}}_n\bigr\rangle_{L^2(\Omega)} + \bigl\langle\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}, \tilde{\boldsymbol{v}}_n\bigr\rangle_{L^2(\Omega)} = \bigl\langle\mathbf{F}, \boldsymbol{\nabla}\tilde{\boldsymbol{v}}_n\bigr\rangle_{L^2(\Omega)} . \tag{5.4}$$

Since

$$\Bigl|\bigl\langle\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}, \tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}_n\bigr\rangle_{L^2(\Omega)}\Bigr| \leq \bigl\|\boldsymbol{\nabla}\boldsymbol{u}\bigr\|_{L^2(\Omega)}\bigl\|\boldsymbol{u}\otimes(\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}_n)\bigr\|_{L^2(\Omega)} ,$$

by Lemma 5.1 we obtain the claimed result by passing to the limit in (5.4). □

We can also replace $\boldsymbol{\varphi}$ by $\boldsymbol{u}$ in the definition of the weak solution $\tilde{\boldsymbol{u}}$:

Lemma 5.3. *If $\tilde{\boldsymbol{u}} = \boldsymbol{u}_\infty + \tilde{\boldsymbol{v}}$ is a weak solution in $\Omega = \mathbb{R}^2$ with $\boldsymbol{u}_\infty \in \mathbb{R}^2$ and $\tilde{\boldsymbol{v}}/\mathfrak{w} \in L^\infty(\Omega)$, then*

$$\bigl\langle\boldsymbol{\nabla}\tilde{\boldsymbol{v}}, \boldsymbol{\nabla}\boldsymbol{u}\bigr\rangle_{L^2(\Omega)} - \bigl\langle\tilde{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\boldsymbol{u}, \tilde{\boldsymbol{v}}\bigr\rangle_{L^2(\Omega)} = \bigl\langle\mathbf{F}, \boldsymbol{\nabla}\boldsymbol{u}\bigr\rangle_{L^2(\Omega)} ,$$

for any $\boldsymbol{u} \in \dot{H}^1_\sigma(\Omega)$.

Proof. By Proposition 3.4, let $(\boldsymbol{u}_n)_{n\in\mathbb{N}} \subset C^\infty_{0,\sigma}(\Omega)$ be a sequence converging to $\boldsymbol{u}$ in $\dot{H}^1_\sigma(\Omega)$. Since $\tilde{\boldsymbol{u}} = \boldsymbol{u}_\infty + \tilde{\boldsymbol{v}}$ is a weak solution, we have

$$\bigl\langle\boldsymbol{\nabla}\tilde{\boldsymbol{v}}, \boldsymbol{\nabla}\boldsymbol{u}_n\bigr\rangle_{L^2(\Omega)} + \bigl\langle\tilde{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\tilde{\boldsymbol{v}}, \boldsymbol{u}_n\bigr\rangle_{L^2(\Omega)} = \bigl\langle\mathbf{F}, \boldsymbol{\nabla}\boldsymbol{u}_n\bigr\rangle_{L^2(\Omega)}$$

or, after an integration by parts,

$$\bigl\langle\boldsymbol{\nabla}\tilde{\boldsymbol{v}}, \boldsymbol{\nabla}\boldsymbol{u}_n\bigr\rangle_{L^2(\Omega)} - \bigl\langle\tilde{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\boldsymbol{u}_n, \tilde{\boldsymbol{v}}\bigr\rangle_{L^2(\Omega)} = \bigl\langle\mathbf{F}, \boldsymbol{\nabla}\boldsymbol{u}_n\bigr\rangle_{L^2(\Omega)} .$$

We can easily pass to the limit in the first and last terms. For the second term, we have

$$\left|\left\langle \tilde{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\left(\boldsymbol{u}-\boldsymbol{u}_n\right),\tilde{\boldsymbol{v}}\right\rangle_{L^2(\Omega)}\right|\le\left\|\tilde{\boldsymbol{u}}\otimes\tilde{\boldsymbol{v}}\right\|_{L^2(\Omega)}\left\|\boldsymbol{\nabla}\boldsymbol{u}-\boldsymbol{\nabla}\boldsymbol{u}_n\right\|_{L^2(\Omega)}\le C\left\|\boldsymbol{\nabla}\boldsymbol{u}-\boldsymbol{\nabla}\boldsymbol{u}_n\right\|_{L^2(\Omega)},$$

and the lemma is proven. □

We now prove the following consequence of the integration by parts:

Lemma 5.4. *For $\Omega=\mathbb{R}^2$, if $\tilde{\boldsymbol{v}}\in\dot{H}^1_\sigma(\Omega)$ satisfies $\tilde{\boldsymbol{v}}/\mathfrak{w}\in L^\infty(\Omega)$, then*

$$\left\langle \boldsymbol{u}\cdot\boldsymbol{\nabla}\tilde{\boldsymbol{v}},\tilde{\boldsymbol{v}}\right\rangle_{L^2(\Omega)}=0\,,$$

for any $\boldsymbol{u}\in\dot{H}^1_\sigma(\Omega)$.

Proof. Let $\left(\tilde{\boldsymbol{v}}_n\right)_{n\in\mathbb{N}}\subset C^\infty_{0,\sigma}(\Omega)$ be the approximation of $\tilde{\boldsymbol{v}}$ constructed in Lemma 5.1. By integrating by parts, we have

$$\left\langle \boldsymbol{u}\cdot\boldsymbol{\nabla}\tilde{\boldsymbol{v}},\tilde{\boldsymbol{v}}_n\right\rangle_{L^2(\Omega)}+\left\langle \boldsymbol{u}\cdot\boldsymbol{\nabla}\tilde{\boldsymbol{v}}_n,\tilde{\boldsymbol{v}}\right\rangle_{L^2(\Omega)}=0\,. \tag{5.5}$$

We have

$$\left|\left\langle \boldsymbol{u}\cdot\boldsymbol{\nabla}\tilde{\boldsymbol{v}},\boldsymbol{v}-\tilde{\boldsymbol{v}}_n\right\rangle_{L^2(\Omega)}\right|\le\left\|\boldsymbol{\nabla}\tilde{\boldsymbol{v}}\right\|_{L^2(\Omega)}\left\|\boldsymbol{u}\otimes\left(\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}_n\right)\right\|_{L^2(\Omega)}$$

and

$$\left|\left\langle \boldsymbol{u}\cdot\boldsymbol{\nabla}\left(\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}_n\right),\tilde{\boldsymbol{v}}\right\rangle_{L^2(\Omega)}\right|\le\left\|\boldsymbol{u}\otimes\tilde{\boldsymbol{v}}\right\|_{L^2(\Omega)}\left\|\boldsymbol{\nabla}\tilde{\boldsymbol{v}}-\boldsymbol{\nabla}\tilde{\boldsymbol{v}}_n\right\|_{L^2(\Omega)},$$

so by using Lemma 5.1, we can pass to the limit in (5.5) and the lemma is proven. □

We now can prove our weak-strong uniqueness results by using some standard method ([5, Theorem X.3.2]; [9, Theorem 6]):

Proof of Theorem 2.9. Let $\tilde{\boldsymbol{v}}=\tilde{\boldsymbol{u}}-\boldsymbol{u}_\infty$, $\boldsymbol{v}=\boldsymbol{u}-\boldsymbol{u}_\infty$, and $\boldsymbol{d}=\boldsymbol{u}-\tilde{\boldsymbol{u}}=\boldsymbol{v}-\tilde{\boldsymbol{v}}$. By Lemma 5.2, we have

$$\left\langle \boldsymbol{\nabla}\boldsymbol{v},\boldsymbol{\nabla}\tilde{\boldsymbol{v}}\right\rangle_{L^2(\Omega)}+\left\langle \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{v},\tilde{\boldsymbol{v}}\right\rangle_{L^2(\Omega)}=\left\langle \mathbf{F},\boldsymbol{\nabla}\tilde{\boldsymbol{u}}\right\rangle_{L^2(\Omega)},$$

and by Lemma 5.3,

$$\left\langle \boldsymbol{\nabla}\tilde{\boldsymbol{v}},\boldsymbol{\nabla}\boldsymbol{v}\right\rangle_{L^2(\Omega)}-\left\langle \tilde{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\boldsymbol{v},\tilde{\boldsymbol{v}}\right\rangle_{L^2(\Omega)}=\left\langle \mathbf{F},\boldsymbol{\nabla}\boldsymbol{u}\right\rangle_{L^2(\Omega)},$$

so we obtain

$$\begin{aligned}\left\|\boldsymbol{\nabla}\boldsymbol{d}\right\|^2_{L^2(\Omega)}&=\left\|\boldsymbol{\nabla}\boldsymbol{u}\right\|^2_{L^2(\Omega)}+\left\|\boldsymbol{\nabla}\tilde{\boldsymbol{u}}\right\|^2_{L^2(\Omega)}-\left\langle \boldsymbol{\nabla}\boldsymbol{v},\boldsymbol{\nabla}\tilde{\boldsymbol{v}}\right\rangle_{L^2(\Omega)}-\left\langle \boldsymbol{\nabla}\boldsymbol{v},\boldsymbol{\nabla}\tilde{\boldsymbol{v}}\right\rangle_{L^2(\Omega)}\\&=\left\|\boldsymbol{\nabla}\boldsymbol{u}\right\|^2_{L^2(\Omega)}-\left\langle \mathbf{F},\boldsymbol{\nabla}\tilde{\boldsymbol{u}}\right\rangle_{L^2(\Omega)}+\left\|\boldsymbol{\nabla}\tilde{\boldsymbol{u}}\right\|^2_{L^2(\Omega)}-\left\langle \mathbf{F},\boldsymbol{\nabla}\boldsymbol{u}\right\rangle_{L^2(\Omega)}\\&\quad+\left\langle \boldsymbol{d}\cdot\boldsymbol{\nabla}\boldsymbol{v},\tilde{\boldsymbol{v}}\right\rangle_{L^2(\Omega)}.\end{aligned}$$

Using the energy inequality (2.2) for both weak solutions and Lemma 5.4,

$$\left\|\boldsymbol{\nabla}\boldsymbol{d}\right\|^2_{L^2(\Omega)}\le\left\langle \boldsymbol{d}\cdot\boldsymbol{\nabla}\boldsymbol{v},\tilde{\boldsymbol{v}}\right\rangle_{L^2(\Omega)}=\left\langle \boldsymbol{d}\cdot\boldsymbol{\nabla}\boldsymbol{d},\tilde{\boldsymbol{v}}\right\rangle_{L^2(\Omega)}\le\left\|\boldsymbol{\nabla}\boldsymbol{d}\right\|_{L^2(\Omega)}\left\|\boldsymbol{d}\tilde{\boldsymbol{v}}\right\|_{L^2(\Omega)},$$

so by Lemma 3.2 we obtain

$$\left\|\boldsymbol{\nabla}\boldsymbol{d}\right\|_{L^2(\Omega)}\le\left\|\boldsymbol{d}\tilde{\boldsymbol{v}}\right\|_{L^2(\Omega)}\le\left\|\boldsymbol{d}\mathfrak{w}\right\|_{L^2(\Omega)}\left\|\tilde{\boldsymbol{v}}/\mathfrak{w}\right\|_{L^\infty(\Omega)}\le C\delta\left\|\boldsymbol{\nabla}\boldsymbol{d}\right\|_{L^2(\Omega)},$$

since by hypothesis $⨍_\omega\boldsymbol{d}=\boldsymbol{0}$. Therefore, for $\delta<C^{-1}$, $\boldsymbol{\nabla}\boldsymbol{d}=\boldsymbol{0}$, i.e., $\boldsymbol{d}=\boldsymbol{0}$. □

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