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STRONG COMPARISON PRINCIPLE FOR p-HARMONIC FUNCTIONS IN CARNOT-CARATHEODORY SPACES

LUCA CAPOGNA AND XIAODAN ZHOU

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ABSTRACT. We extend Bony's propagation of support argument to C^1 solutions of the nonhomogeneous subelliptic p-Laplacian associated to a system of smooth vector fields satisfying Hörmander's finite rank condition. As a consequence we prove a strong maximum principle and strong comparison principle that generalize results of Tolksdorf.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be an open and connected set, and consider a family of smooth vector fields $X_1, \cdots, X_m$ in $\mathbb{R}^n$ satisfying Hörmander's finite rank condition [6],

$$\text{rank Lie}[X_1, \cdots, X_m](x) = n, \tag{1.1}$$

for all $x \in \Omega$. We set $Xu = (X_1u, \cdots, X_mu)$ for any function $u : \Omega \to \mathbb{R}$ for which the expression is meaningful.

In this paper we will prove a strong comparison principle for solutions of the class of quasilinear, degenerate elliptic equations

$$L_pu = \sum_{j=1}^{m} X_j^*(A_j(Xu)) = f(x, u) \tag{1.2}$$

satisfying the structure conditions (3.1) and which includes the p-Laplacian, in the range $p > 1$, associated to $X_1, \cdots, X_m$ and to the Lebesgue measure dx in $\mathbb{R}^n$. Note that in (1.2) we have let1 X_j^* denote the L^2 adjoint of the operator X_j with respect to the Lebesgue measure; namely, if $X_j = \sum_{i=1}^m a_i^j(x)\partial_{x_i}$ is a smooth vector field and u is a smooth function, then the adjoint operator X_j^* is given by

$$X_j^*u = \sum_{i=1}^{m} \partial_{x_i}(a_i^j u) = -X_ju - \sum_{i=1}^{m} \partial_{x_i}(a_i^j)u.$$

Set $d_j(x) = -\sum_{i=1}^m \partial_{x_i}(a_i^j(x))$ and $d(x) = (d_1(x), \cdots, d_m(x))$. Note that d_j is a smooth function.

We explicitly note that all the results in this paper continue to hold if one substitutes the Lebesgue measure dx with any other measure $d\mu = \lambda(x)dx$ with

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$\lambda \in C^1$ density function. In particular the results apply in any sub-Riemannian manifold for solutions of the subelliptic p-Laplacian associated to a smooth volume form.

In addition to the structure conditions (3.1), our strong comparison principle holds under the following hypothesis:

$$\begin{aligned} &\text{(i)} \;\; \partial_u f \leq 0 \text{ in } \Omega, \\ &\text{(ii)} \;\; |f(x, u_2 + \epsilon) - f(x, u_2)| \leq L\epsilon, \text{ for any } \epsilon \in [0, \epsilon_0], x \in \Omega \end{aligned} \tag{1.3}$$

for some positive constants L, ϵ_0. Our main result is the following.

Theorem 1 (Strong comparison principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be a connected open set and consider two weak solutions $u_1 \in C^1(\bar{\Omega})$ and $u_2 \in C^2(\bar{\Omega})$ of* (1.2) *in Ω, with $|Xu_2| \geq \delta$ in Ω for some $\delta > 0$. We assume that the structure conditions* (3.1) *and the hypothesis* (1.3) *are satisfied. If*

$$u_1 \geq u_2 \text{ in } \Omega,$$

then either $u_1 = u_2$ or

$$u_1 > u_2 \text{ in } \Omega.$$

As will be evident from the proof, the regularity assumptions and the lower bound on $|Xu_2|$ are required only in a neighborhood of the contact set. The lower bound is not required in the nondegenerate case $\kappa > 0$. Note that, as in the Euclidean setting, one cannot relax the conditions on u_1, u_2, and f unless more hypotheses are added.

We also prove a nonhomogenous strong maximum/minimum principle. We suppose that f satisfies the following conditions: for all $x \in \Omega$ and $u \in \mathbb{R}$,

$$\begin{aligned} &\text{(i)} \;\; \partial_u f \leq 0, \\ &\text{(ii)} \;\; |f(x, u)| \leq \bar{C}(\kappa + |u|)^{p-2}|u| \end{aligned} \tag{1.4}$$

for some positive constant $\bar{C}$ and κ as in the structure conditions (3.1).

Theorem 2 (Strong minimum principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be a connected open set and consider a weak solution $u \in C^1(\bar{\Omega})$ of* (1.2) *in Ω. We assume that the structure conditions* (3.1) *and the hypothesis* (1.4) *hold. If*

$$u \geq 0 \text{ in } \Omega,$$

then either $u = 0$ or

$$u > 0 \text{ in } \Omega.$$

The proof of these results is at the end of Section 3. Theorems 1 and 2 extend to the subelliptic setting the strong maximum and comparison principles proved by Tolksdorff in [7, Propositions 3.2.2 and 3.3.2].

In the subelliptic setting Theorem 1 seems to be new even in the homogeneous case $f = 0$. In terms of previous literature on this subject: we recall that the case $p = 2$ was established through geometric methods by Bony in his landmark paper [2]. A proof of the strong maximum principle for the subelliptic p-Laplacian in H-type groups can be found in [8]. We note however that at the conclusion of that proof the authors claim that one can always fit a gauge ball tangentially at every point of the set where the solution attains the maximum. This statement is not proved in [8], and since gauge balls have zero curvature at the poles, we do not see how the statement can be proved.

A strong comparison and maximum principle for smooth solutions of the subelliptic p-Laplacian and of the horizontal mean curvature operator has been recently proved by Cheng, Chiu, Hwang, and Yang in their preprint [4]. Their proof is based on a linearization approach which is different from our arguments; however it also ultimately relies on Bony's argument and holds in every sub-Riemannian manifold. In comparison to the present paper, on the one hand our results hold for solutions which do not have to be smooth necessarily,[1] but for the comparison principle we require one of the two solutions to have nonvanishing horizontal gradient. On the other hand, while we only deal with the p-Laplacian, in [4] the authors also establish far-reaching results for the mean curvature operator, including some special cases where $|Xv_2|$ is allowed to vanish in a controlled fashion and still conclude a comparison principle.

The technical core of the proofs in the present paper is in Lemma 7 and consists of an adaptation of Bony's argument to our nonlinear setting. In his proof of the strong maximum principle [2], Bony introduced generalizations of certain standard results in differential calculus to a nonsmooth setting, namely, a notion of tangent vector that is appropriate for any closed set, not just for C^1 smooth sets. He then established that the integral lines of smooth tangent vector fields remain within the set, and so do all the integral lines of their brackets. The key step in his proof is the observation that all horizontal vector fields (out of which the operator is built) are tangent to the set where the maximum of a solution is achieved. This immediately implies that this set is either empty or the whole domain.

In closing we note that both in the elliptic and in the subelliptic case, a corresponding strong maximum principle for the homogenous problem $f = 0$ can be established immediately from the Harnack inequality (see for instance [1], [5], [3]), as well as with small modifications of the argument presented here. However, while in the linear setting one can deduce the strong comparison principle from the strong maximum principle, this is no longer the case in the nonlinear setting, where a new approach is needed.

2. Bony's propagation of support technique

Tolksdorf's argument in [7, 3.3.2] breaks down in the subelliptic setting, due to the fact that the horizontal gradient of the barrier functions typically used in this proof may vanish. The same problem occurs also in the linear setting, for $p = 2$. To deal with this issue we follow the outline of the proof of the strong maximum principle for sub-Laplacians, from Bony's paper [2], and adapt it to our nonlinear and nonhomogeneous setting.

We begin by recalling from [2, Definition 2.1] the definition of a nonzero vector $\mathbf{v}$ orthogonal to a set $F \subset \mathbb{R}^n$ at a point $y \in \partial F$.

Definition 1. Let F be a relatively closed subset of Ω. We say that a vector $\mathbf{v} \in \mathbb{R}^n \setminus \{0\}$ is (exterior) normal to F at a point $y \in \Omega \cap \partial F$ if

$$\overline{B(y + \mathbf{v}, |\mathbf{v}|)} \subset (\Omega \setminus F) \cup \{y\},$$

[1] We recall that in general p-harmonic functions do not enjoy more regularity than the Hölder continuity of their gradient.

where $B(y+\mathbf{v},|\mathbf{v}|)$ denotes the Euclidean ball centered in $u+\mathbf{v}$, with radius equal to the Euclidean norm of $\mathbf{v}$. If this inclusion holds, we write $\mathbf{v} \perp F$ at y. Set

$$F^* = \{y \in \Omega \cap \partial F \colon \text{there exists } \mathbf{v} \text{ such that } \mathbf{v} \perp F \textit{ at } y\}.$$

Note that when Ω is connected and $\emptyset \neq F \neq \Omega$, we have $F^* \neq \emptyset$.

We list in the following some of the results and definitions from [2] that play a role in our proof.

Definition 2. Let X be a vector field in Ω and let $F \subset \Omega$ be a closed set. We say that X is tangent to F if, for all $x_0 \in F^*$ and all vectors v normal to F at x_0, one has that their Euclidean product vanishes; i.e., $\langle X(x_0), v\rangle = 0$.

The following results are from [2, Theoremes 2.1 and 2.2]:

Theorem 3. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $F \subset \Omega$ be a closed subset. Let X be a Lipschitz vector field in Ω. If X is tangent to F, then all its integral curves that intersect F are entirely contained in F.*

Note that the converse of this result is also true and follows from a direct computation.

Theorem 4. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $F \subset \Omega$ be a closed subset. Let $X_1, \cdots, X_m$ be smooth vector fields in Ω. If $X_1, \cdots, X_m$ are tangent to F, then so is the Lie algebra they generate.*

As a corollary, if $X_1, \cdots, X_m$ satisfy Hörmander finite rank condition (1.1) and are all tangent to F, then every curve that touches F is entirely contained in F, so that either F is the empty set or $F = \Omega$.

3. A Hopf-type comparison principle and proof of Theorem 1

First we state precisely the structure conditions imposed on the left hand side of (1.2). The functions A_j satisfy the following ellipticity and growth condition: For $p > 1$, for a.e. $\xi \in \mathbb{R}^m$ and for every $\eta \in \mathbb{R}^m$,

$$\begin{aligned} &\sum_{i,j=1}^m \frac{\partial A_j}{\partial \xi_i}(\xi)\eta_i\eta_j \geq \beta(\kappa+|\xi|)^{p-2}|\eta|^2, \\ &\sum_{i,j=1}^m |\frac{\partial A_j}{\partial \xi_i}(\xi)| \leq \gamma(\kappa+|\xi|)^{p-2} \end{aligned} \tag{3.1}$$

for some positive constants β, γ, and for $\kappa \geq 0$.

One can easily deduce that there exist positive constants λ, C such that for all $\xi \in \mathbb{R}^m$,

$$\langle A_j(\xi) - A_j(\xi'), \xi - \xi'\rangle \geq \lambda \begin{cases} (1+|\xi|+|\xi'|)^{p-2}|\xi-\xi'|^2 & \text{if} \quad p \leq 2, \\ |\xi - \xi'|^p & \text{if} \quad p \geq 2, \end{cases} \tag{3.2}$$

and

$$|A_j(\xi)| \leq C(\kappa+|\xi|)^{p-2}|\xi|. \tag{3.3}$$

The subelliptic p-Laplacian

$$L_p u = \sum_{j=1}^m X_j^*(|Xu|^{p-2}X_j u)$$

corresponds to the choice $A_j(\xi) = |\xi|^{p-2}\xi_j$ for $j = 1, \cdots, m$.

We will need the following immediate consequence of the monotonicity inequality (3.2).

Lemma 5 (Weak comparison principle). *Let $\Omega \subset \mathbb{R}^n$ be an open and connected set and let $v_1, v_2 \in C^1(\Omega)$ satisfy in a weak sense*

$$\begin{cases} L_p v_2 \le f(x, v_2) & \text{in } \Omega, \\ L_p v_1 \ge f(x, v_1) & \text{in } \Omega, \end{cases} \tag{3.4}$$

with A_j satisfying the structure conditions (3.1) *and $\partial_u f(x, u) \le 0$. If $v_2 \le v_1$ in $\partial\Omega$, then $v_2 \le v_1$ in Ω.*

Proof. Given an arbitrary $\epsilon > 0$, we define $E_\epsilon = \{x \in \Omega | v_2(x) > v_1(x) + \epsilon\}$. Assume that $E_\epsilon \ne \emptyset$; then $\overline{E_\epsilon} \subset \Omega$. For all $\varphi \in C_c^1(\Omega)$, we have

$$\int_\Omega \langle A_j(Xv_2), X\varphi\rangle \le \int_\Omega f(x, v_2)\varphi,$$
$$\int_\Omega \langle A_j(Xv_1), X\varphi\rangle \ge \int_\Omega f(x, v_1)\varphi.$$

Subtracting the above two inequalities and setting $\varphi(x) = \max\{v_2(x) - v_1(x) - \epsilon, 0\}$, as a consequence of (i) in (1.3) one has

$$\int_{E_\epsilon} \langle A_j(Xv_2) - A_j(Xv_1), X(v_2 - v_1)\rangle \le \int_{\{v_2 > v_1 + \epsilon\}} (f(x, v_2) - f(x, v_1))(v_2 - v_1 - \epsilon) \le 0.$$

By (3.2), this inequality holds if and only if $X(v_2 - v_1) = 0$. Thus, $v_2 = v_1 + C$ in E_ϵ. The fact that $v_2 = v_1 + \epsilon$ on ∂E_ϵ implies that $C = \epsilon$. It follows that $v_2 \le v_1 + \epsilon$ in Ω. Letting $\epsilon \to 0$, we get $v_2 \le v_1$ in Ω. □

Next, we prove an analogue of the classical Hopf comparison principle: Given a subsolution v_2 and a supersolution v_1 such that $v_2 \le v_1$, every vector field $X_1, \cdots, X_m$ must be tangent to the contact set $F = \{v_2 = v_1\}$.

Lemma 6 (A Hopf-type comparison principle). *Let $\Omega \subset \mathbb{R}^n$ be an open and connected set and let $v_1 \in C^1(\Omega)$, $v_2 \in C^2(\Omega)$ with $|Xv_2| \ge \delta$ in Ω satisfy*

$$\begin{cases} v_2 \le v_1 & \text{in } \Omega, \\ L_p v_2 \le f(x, v_2) & \text{in } \Omega. \\ L_p v_1 \ge f(x, v_1) & \text{in } \Omega. \end{cases} \tag{3.5}$$

Set $F = \{x \in \Omega : v_2(x) = v_1(x)\}$. If the structure conditions (3.1) *and hypothesis* (1.3) *are satisfied and $\emptyset \ne F \ne \Omega$, then for every $y \in F^*$ and $\mathbf{v} \perp F$ at y, it follows that*

$$\langle X_i(y), \mathbf{v}\rangle = 0$$

for all $i = 1, \cdots, m$.

Proof. We argue by contradiction and suppose that there exist $y \in F^*$, a vector $\mathbf{v} \perp F$ at y, and $i \in \{1, \cdots, m\}$ such that $\langle X_i(y), \mathbf{v}\rangle \ne 0$. Let $z = y + \mathbf{v}$ and $r = |\mathbf{v}|$. We define

$$\sigma_i(x) := \langle X_i(x), x - z\rangle$$

and a vector field $\sigma(x) = (\sigma_1(x), \cdots, \sigma_m(x))$. Note that this is a smooth vector field on Ω and $\sigma_i(y) \ne 0$.

We denote by $|x-z|$ the Euclidean distance between the points x, z and proceed to define $\tilde{b}(x) = e^{-\alpha|x-z|^2}$ and

$$b(x) = \alpha^{-2}(\tilde{b}(x) - e^{-\alpha r^2})$$

in Ω where the value of the positive constant α is to be determined later. Choose a neighborhood V of y such that $0 < |\sigma(x)|$ for $x \in \overline{V} \subset \Omega$ and denote by M_1, M_2, M_3, M_4 positive constants depending on v_2 and F, such that for every $x \in \overline{V}$ one has $|X_j\sigma_i(x)| \le M_1$, $|X_jX_i(b+v_2)(x)| \le M_2$, and $M_4 \le |\sigma(x)| \le M_3$ for $i, j = 1, \cdots, m$.

By a direct calculation, one can deduce that

$$X_i b(x) = -2\alpha^{-1}\tilde{b}(x)\sigma_i(x),$$

$$|Xb(x)| = 2\alpha^{-1}\tilde{b}(x)|\sigma(x)| = 2\alpha^{-1}\tilde{b}(x)\Big(\sum_{i=1}^m \sigma_i(x)^2\Big)^{1/2},$$

$$X_jX_ib(x) = \tilde{b}(x)(4\sigma_j\sigma_i - 2\alpha^{-1}X_j\sigma_i(x)).$$

Substituting the identities above in the expression for $L_p b$ yields

$$\begin{aligned} L_p b(x) &= -\sum_{j=1}^m\sum_{i=1}^m \frac{\partial A_j}{\partial \xi_i}(Xb)X_jX_ib + \sum_{j=1}^m d_jA_j(Xb) \\ &= -\tilde{b}(x)\sum_{i,j=1}^m \Big(4\frac{\partial A_j}{\partial \xi_i}(Xb)\sigma_j\sigma_i - 2\alpha^{-1}\frac{\partial A_j}{\partial \xi_i}(Xb)X_j\sigma_i\Big) + \sum_{j=1}^m d_jA_j(Xb). \end{aligned}$$

Applying the structure conditions (3.1) and (3.3) of A_j, it follows that for every $x \in \overline{V}$,

$$\begin{aligned} L_p b(x) &= -\tilde{b}(x)\sum_{i,j=1}^m \Big(4\frac{\partial A_j}{\partial \xi_i}(Xb)\sigma_j\sigma_i - 2\alpha^{-1}\frac{\partial A_j}{\partial \xi_i}(Xb)X_j\sigma_i\Big) + \sum_{j=1}^m d_jA_j(Xb) \\ &\le -\tilde{b}(x)\Big(4\beta(\kappa+|Xb|)^{p-2}|\sigma|^2 - 2\alpha^{-1}M_1\gamma(\kappa+|Xb|)^{p-2}\Big) \\ &\quad + \sum_{j=1}^m |d_j|C(\kappa+|Xb|)^{p-2}|Xb| \\ &= -\tilde{b}(x)(\kappa+|Xb|)^{p-2}\Big(4\beta|\sigma|^2 - 2\alpha^{-1}M_1\gamma - C\alpha^{-1}|\sigma(x)|\Big). \end{aligned}$$

Similarly,

$$\begin{aligned} \sum_{i,j=1}^m \frac{\partial A_j}{\partial \xi_i}(Xv_2)X_jX_ib &\ge \tilde{b}(x)(\kappa+|Xv_2|)^{p-2}\Big(4\beta|\sigma|^2 - 2\alpha^{-1}M_1\gamma\Big) \\ &\ge \tilde{b}(x)(\kappa+|Xv_2|)^{p-2}\Big(4\beta M_4^2 - 2\alpha^{-1}M_1\gamma\Big). \end{aligned}$$

In view of the nonvanishing hypothesis on $|Xv_2|$, there exist α_1 and a positive constant ϵ_1 such that for $\alpha \ge \alpha_1$ and $x \in \overline{V}$,

$$|Xb(x)| \le \frac{1}{2}|Xv_2(x)|,$$

$$L_p b(x) \le 0,$$

and

$$\sum_{i,j=1}^{m} \frac{\partial A_j}{\partial \xi_i}(Xv_2)X_jX_ib(x) \geq \epsilon_1 \tilde{b}(x). \tag{3.6}$$

Since $A_j(\xi)$ is smooth in $\mathbb{R}^n \setminus \{0\}$, there exist positive constants C, ϵ_2 such that

$$\sum_{i,j=1}^{m} |\frac{\partial A_j}{\partial \xi_i}(X(b+v_2)) - \frac{\partial A_j}{\partial \xi_i}(Xv_2)| \leq C|Xb| \leq \epsilon_2 \alpha^{-1}\tilde{b}(x) \tag{3.7}$$

for $x \in \overline{V}$. Thus,

$$\begin{aligned}
L_p(b+v_2) &= -\sum_{i,j=1}^{m} \frac{\partial A_j}{\partial \xi_i}(X(b+v_2))X_jX_i(b+v_2) + \sum_{j=1}^{m} d_jA_j(Xb+Xv_2) \\
&= -\sum_{i,j=1}^{m} \Big(\frac{\partial A_j}{\partial \xi_i}(X(b+v_2)) - \frac{\partial A_j}{\partial \xi_i}(Xv_2) + \frac{\partial A_j}{\partial \xi_i}(Xv_2)\Big)X_jX_i(b+v_2) \\
&+ \sum_{j=1}^{m} d_jA_j(Xb+Xv_2) \\
&= -\sum_{i,j=1}^{m} \Big(\frac{\partial A_j}{\partial \xi_i}(X(b+v_2)) - \frac{\partial A_j}{\partial \xi_i}(Xv_2)\Big)X_jX_i(b+v_2) \\
&- \sum_{i,j=1}^{m} \frac{\partial A_j}{\partial \xi_i}(Xv_2)X_jX_ib - \sum_{i,j=1}^{m} \frac{\partial A_j}{\partial \xi_i}(Xv_2)X_jX_iv_2 + \sum_{j=1}^{m} d_jA_j(Xb+Xv_2) \\
&\leq M_2\epsilon_2\alpha^{-1}\tilde{b}(x) - \epsilon_1\tilde{b}(x) + L_pv_2 - \sum_{j=1}^{m} d_jA_j(Xv_2) + \sum_{j=1}^{m} d_jA_j(Xb+Xv_2) \\
&\leq (-\epsilon_1 + M_2\epsilon_2\alpha^{-1})\tilde{b}(x) + f(x,v_2) + \sum_{j=1}^{m} |d_j||A_j(Xb+Xv_2) - A_j(Xv_2)| \\
&\leq (-\epsilon_1 + M_2\epsilon_2\alpha^{-1} + C\alpha^{-1}|\sigma(x)|)\tilde{b}(x) + f(x,v_2) \\
&\leq (-\epsilon_1 + M_2\epsilon_2\alpha^{-1} + C\alpha^{-1}|\sigma(x)|)\tilde{b}(x) + |f(x,b+v_2) - f(x,v_2)| \\
&\quad + f(x,b+v_2) \\
&\leq (-\epsilon_1 + M_2\epsilon_2\alpha^{-1} + C\alpha^{-1}|\sigma(x)|)\tilde{b}(x) + L|b| + f(x,b+v_2),
\end{aligned}$$

where the last inequality follows from (ii) in (1.3). We can now choose $\alpha \geq \alpha_1$ such that $L_p(b+v_2) \leq f(x,b+v_2)$ on $\overline{V}$.

Next, we let $U = V \cap B(z,r)$ and express its boundary as the union of two components

$$\partial U = \Gamma_1 \cup \Gamma_2,$$

where $\Gamma_1 = \overline{B(z,r)} \cap \partial V$ and $\Gamma_2 = \overline{V} \cap \partial B(z,r)$.

For $x \in \Gamma_1 \subset \Omega \setminus F$, we have $v_2(x) < v_1(x)$. Choose α sufficiently large so that $v_2(x) + b(x) \leq v_1(x)$ on Γ_1 and $L_p(v_2+b) \leq f(x,b+v_2)$ on U. On the other hand, since $b(x) = 0$ when $x \in \Gamma_2$, the estimate $v_2(x) + b(x) \leq v_1(x)$ also holds on Γ_2.

Thus one eventually obtains

$$\begin{cases} v_2 + b \le v_1 & \text{in } \partial U, \\ L_p(v_2+b) \le f(x, b+v_2) & \text{in } U, \\ L_p v_1 \ge f(x, v_1) & \text{in } U. \end{cases} \tag{3.8}$$

The weak comparison principle in Lemma 5 implies that $v_2 + b \le v_1$ in U. Since y is a maximum point of $v_2 - v_1$ in Ω, necessarily its gradient at y must vanish; i.e., $\nabla(v_2 - v_1)(y) = 0$. Finally we invoke the C^1 regularity of v_1 near the contact set and we observe that

$$\begin{aligned} 0 = \langle \mathbf{v}, \nabla(v_2 - v_1)(y)\rangle &= \lim_{t\to 0^+} \frac{v_2(y+t\mathbf{v}) - v_1(y+t\mathbf{v}) - (v_2(y) - v_1(y))}{t} \\ &\le -\langle \mathbf{v}, \nabla b(y)\rangle \\ &= -2\alpha^{-1} r^2 e^{-\alpha r^2} < 0. \end{aligned}$$

Since we have arrived at a contradiction the proof is complete. □

By a similar argument, a Hopf-type maximum/minimum principle can be established.

Lemma 7 (A Hopf-type minimum principle). *Let $\Omega \subset \mathbb{R}^n$ be an open and connected set and let $v \in C^2(\Omega)$ satisfy*

$$\begin{cases} v \ge 0 & \text{in } \Omega, \\ L_p v \ge f(x, v) & \text{in } \Omega. \end{cases} \tag{3.9}$$

Set $F = \{x \in \Omega : v(x) = 0\}$. If the structure conditions (3.1) *and hypothesis* (1.4) *are satisfied and $\emptyset \ne F \ne \Omega$, then for every $y \in F^*$ and $\mathbf{v} \perp F$ at y, it follows that*

$$\langle X_i(y), \mathbf{v}\rangle = 0$$

for all $i = 1, \cdots, m$.

Proof. We argue by contradiction and suppose that there exist $y \in F^*$, a vector $\mathbf{v} \perp F$ at y, and $i \in \{1, \cdots, m\}$ such that $\langle X_i(y), \mathbf{v}\rangle \ne 0$. Let $z = y + \mathbf{v}$ and $r = |\mathbf{v}|$. We define

$$\sigma_i(x) := \langle X_i(x), x - z\rangle$$

and a vector field $\sigma(x) = (\sigma_1(x), \cdots, \sigma_m(x))$. Note that this is a smooth vector field on Ω and $\sigma_i(y) \ne 0$.

We denote by $|x - z|$ the Euclidean distance between the points x, z and proceed to define $\tilde{b}(x) = e^{-\alpha|x-z|^2}$ and

$$b(x) = k(\tilde{b}(x) - e^{-\alpha r^2})$$

in Ω where the value of the positive constants k and α are to be determined later. Choose a neighborhood V of y such that $0 < |\sigma(x)|$ for $x \in \overline{V} \subset \Omega$ and denote by M_1, M_2, M_3 positive constants depending on v_2 and F, such that for every $x \in \overline{V}$ one has $|X_j \sigma_i(x)| \le M_1$ and $M_2 \le |\sigma(x)| \le M_3$ for $i, j = 1, \cdots, m$.

By elementary calculations and (3.3), we get

$$\begin{aligned} L_p b(x) &= -\sum_{j=1}^{m}\sum_{i=1}^{m}\frac{\partial A_j}{\partial \xi_i}(Xb)X_jX_ib + \sum_{j=1}^{m} d_jA_j(Xb) \\ &\le -k\tilde{b}(x)\alpha^2(\kappa+|Xb|)^{p-2}\Big(4\beta|\sigma|^2 - 2\alpha^{-1}|X\sigma|\gamma - 2\alpha^{-1}C\sup_{\overline{V}}|d||\sigma(x)|\Big) \\ &= -k\tilde{b}(x)\alpha^2(\kappa+2\alpha|\sigma(x)|k\tilde{b}(x))^{p-2}\Big[4\beta M_2^2 - 2\alpha^{-1}M_1\gamma - CM_3\alpha^{-1}\Big]. \end{aligned}$$

Choosing α sufficiently large, we get that

$$\begin{aligned} L_p b(x) &\le -\alpha\beta|b(x)|(\kappa+|b(x)|)^{p-2} \\ &\le -\bar{C}|b(x)|(\kappa+|b(x)|)^{p-2} \le f(x,b(x)) \end{aligned}$$

for every $x \in \overline{V}$. Next, we let $U = V \cap B(z,r)$ and express its boundary as the union of two components

$$\partial U = \Gamma_1 \cup \Gamma_2,$$

where $\Gamma_1 = \overline{B(z,r)} \cap \partial V$ and $\Gamma_2 = \overline{V} \cap \partial B(z,r)$.

For $x \in \Gamma_1 \subset \Omega \setminus F$, we have $v(x) > 0$. Choose k sufficiently small so that $b(x) \le v(x)$ on Γ_1. On the other hand, since $b(x) = 0$ when $x \in \Gamma_2$, the estimate $b(x) \le v(x)$ also holds on Γ_2. Thus one eventually obtains

$$\begin{cases} b(x) \le v(x) & \text{in } \partial U, \\ L_p(b) \le f(x,b) & \text{in } U, \\ L_p v \ge f(x,v) & \text{in } U. \end{cases} \tag{3.10}$$

The weak comparison principle in Lemma 5 implies that $b(x) \le v(x)$ in U. Since y is a minimum point of $v(x)$ in Ω, necessarily its gradient at y must vanish; i.e., $\nabla v(y) = 0$. Finally we observe that in view of the C^1 regularity of v, one has

$$\begin{aligned} 0 = \langle \mathbf{v}, \nabla v(y)\rangle &= \lim_{t\to 0^+}\frac{v(y+t\mathbf{v}) - v(y)}{t} \\ &\ge \lim_{t\to 0^+}\frac{b(y+t\mathbf{v}) - b(y)}{t} \\ &= 2k\alpha r^2 e^{-\alpha r^2} > 0, \end{aligned}$$

arriving at a contradiction. □

In view of the Hopf-type comparison principle and Theorem 4, we deduce that the contact set $F = \{v_2 = v_1\}$ must be either all of Ω or the empty set, thus completing the proof of the strong comparison principle in Theorem 1.

Likewise, the strong maximum principle Theorem 2 follows from the Hopf-type maximum principle.

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Department of Mathematical Sciences, Worcester Polytechnic Institute, Worcester, Massachusetts 01609

Email address: lcapogna@wpi.edu

Department of Mathematical Sciences, Worcester Polytechnic Institute, Worcester, Massachusetts 01609

Email address: xzhou3@wpi.edu