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AVOIDING ALGEBRAIC INTEGERS OF BOUNDED HOUSE IN ORBITS OF RATIONAL FUNCTIONS OVER CYCLOTOMIC CLOSURES

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Abstract. Let k be a number field with cyclotomic closure k^{c}, and let $h \in k^{\mathrm{c}}(x)$. For $A \geq 1$ a real number, we show that

$$\{\alpha \in k^{\mathrm{c}} : h(\alpha) \in \overline{\mathbb{Z}} \text{ has house at most } A\}$$

is finite for many h. We also show that for many such h the same result holds if $h(\alpha)$ is replaced by orbits $h(h(\cdots h(\alpha)))$. This generalizes a result proved by Ostafe that concerns avoiding roots of unity, which is the case $A = 1$.

1. Introduction

1.1. Rational functions and set avoidance.

We begin with the following general definition.

Definition 1.1. Let F be a subfield of $\mathbb{C}$, and let P be a subset of $\mathbb{C}$. Let $h \in F(x)$ be a rational function, and let h^n denote the function composition of h applied n times ($n = 0, 1, 2, \dots$).

- We say that h is *P-avoiding* (over F) if
$$\#\{\alpha \in F \mid h(\alpha) \in P\} < \infty.$$
- We say that h is *strongly P-avoiding* (over F) if
$$\#\{\alpha \in F \mid h^n(\alpha) \in P \text{ for some } n \geq 1\} < \infty.$$

Let $\mathbb{U} \subseteq \mathbb{C}$ denote the set of roots of unity and let k be a number field. We will denote its cyclotomic closure $k(\mathbb{U})$ by k^{c}. This paper will concern avoidance over k^{c}.

We say a rational function $h(x) \in k^{\mathrm{c}}(x)$ is *special* if h is conjugate, with respect to a Möbius transformation (i.e., via $\mathrm{PGL}_2(k^{\mathrm{c}})$), to either $\pm x^d$ or the Chebyshev polynomial $T_d(x)$ which is uniquely determined by the equation $T_d(\frac{1}{2}(t + t^{-1})) = \frac{1}{2}(t^d + t^{-d})$.

The question of $\mathbb{U}$-avoidance and strong $\mathbb{U}$-avoidance has been examined by Dvornicich and Zannier. For example, as a consequence of [2, Corollary 1], we have the following result.

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Theorem (From [2, Corollary 1]). *Let $h = p/q \in k^{\mathrm{c}}(x)$, where $p, q \in k^{\mathrm{c}}[x]$. Assume that $p(x) - y^m q(x)$ is irreducible over k^{c} for all positive integers $m \le \max(\deg p, \deg q)$. Then h is $\mathbb{U}$-avoiding over k^{c}.*

Ostafe [7] proved the following result for strong $\mathbb{U}$-avoidance.

Theorem ([7, Theorem 1.2]). *Let $h = p/q \in k(x)$, where $p, q \in k[x]$. Assume h is $\mathbb{U}$-avoiding over k^{c}, and $\deg p > \deg q + 1$. Assume also that $\max(\deg p, \deg q) \ge 2$ and $p(x) - y^m q(x)$ as a polynomial in x does not have a root in $k^{\mathrm{c}}(y)$ for all positive integers $m \le \deg(p)$. Then h is strongly $\mathbb{U}$-avoiding unless h is special.*

In this paper we investigate a generalization of these results proposed by Ostafe (see [7, §4]). In order to state it, we need to define the following.

Definition 1.2. The *house* of an algebraic number α, denoted $\overline{|\alpha|}$, is the maximum value of $|\beta|$ across the $\mathbb{Q}$-Galois conjugates β of α.

For $A \ge 1$ a real number, let P_A denote the set of algebraic integers α which have house at most A.

For example every algebraic integer has house at least 1, and by Kronecker's theorem (the main result of [5], see also [4]) we have $P_1 = \mathbb{U}$.

We answer the following question.

Question. For $A \ge 1$ and $h \in k^{\mathrm{c}}(x)$, under what conditions can one show that h is (strongly) P_A-avoiding?

1.2. **Summary of results.** The *degree* of a nonconstant rational function h with coefficients in some field F is defined to be $[F(x) : F(h(x))]$. Consequently, note that $\deg(h_1 \circ h_2) = \deg h_1 \deg h_2$. If h is written as a quotient of relatively prime polynomials p/q, then $\deg h = \max(\deg p, \deg q)$.

Our results on P_A-avoidance can be summarized as follows.

Theorem 1.3. *Let k be a number field, $A \ge 1$ and $\varepsilon > 0$. Let $h \in k^{\mathrm{c}}(x)$ be a rational function.*

- *Then h is P_A-avoiding unless there exists $S \in k^{\mathrm{c}}(x)$ such that $h(S(x))$ equals a Laurent polynomial with d terms, where*

$$d \ll_{k,\varepsilon} A^{2+\varepsilon}.$$

- *If $\deg h \gg_{k,A} 1$, then we can also assume $\deg S \le 2$.*

This theorem has an effective and more explicit form given as Theorem 2.5 and Theorem 2.7.

A corollary of Theorem 1.3 is the following.

Corollary 1.4. *Let k be a number field and $A \ge 1$. If h has more than two poles, then h is P_A-avoiding.*

Using this result, we will deduce the following generalization of a result of Ostafe [7, Theorem 1.2], and give a simple proof using Theorem 2.5.

Theorem 1.5. *Let $h = p/q \in k(x)$, where $p, q \in k[x]$. Let $A \ge 1$. Assume h is P_A-avoiding over k^{c}, and $\deg p > \deg q + 1$. Then h is strongly P_A-avoiding unless h is special.*

1.3. **Outline.** The rest of the paper is structured as follows. In Section 2, we state the Loxton theorem, namely Theorem 2.1, and use this to give a more precise version of Theorem 1.3 as Theorem 2.5 and Theorem 2.7. In Section 3, we introduce several auxiliary results which will be used in our proofs.

In Section 4 we prove Theorem 2.5 and Theorem 2.7, as well as Corollary 1.4; these are our results on P_A-avoidance. Finally, Section 5 gives the proof of Theorem 1.5, which is our result on strong P_A-avoidance.

2. Full statement of results on P_A-avoidance

In order to recall the full version of Theorem 1.3, we first need to state the following extension of a theorem of Loxton [6, Theorem 1].

Theorem 2.1 (Loxton theorem, [2, Theorem L]). *There exists a function $\mathscr{L}\colon \mathbb{R}_+ \to \mathbb{R}_+$ with the following property. For every number field k, we can fix a real number $B > 0$ and a finite subset $E \subseteq k$ of cardinality at most $[k : \mathbb{Q}]$ so that every algebraic integer α in k^{c} can be written as*

$$\sum_{i=1}^{d} e_i \xi_i,$$

where $e_i \in E$, $\xi_i \in \mathbb{U}$, and $d \leq \mathscr{L}(B \cdot \lceil\overline{\alpha}\rceil)$.

In light of this, it will be convenient to make the following definition.

Definition 2.2. For every number field k we fix a pair (B, E) (depending only on k) as above. We will call this the *Loxton pair* for k. The *Loxton function* $\mathscr{L}$ will also remain fixed through the paper.

Remark 2.3. The exact nature of $\mathscr{L}$ is not important for our purposes. However, it is possible to choose $\mathscr{L}(x) = O_\varepsilon(x^{2+\varepsilon})$. Moreover, in the case $k = \mathbb{Q}$ one can select $E = \{1\}$. See [6] for more details.

Definition 2.4. Let $h \in k^{\mathrm{c}}(x)$ and fix (B, E) a Loxton pair for k. Suppose that there exist a nonconstant $S \in k^{\mathrm{c}}(x)$, integers n_i, roots of unity $\beta_i \in \mathbb{U}$, and $e_i \in E$ which satisfy

$$\sum_{i=1}^{d} \beta_i e_i x^{n_i} = h(S(x)).$$

In this case, we call the rational function $\sum \beta_i e_i x^{n_i}$ a *witness* for h.

If $A \geq 1$ is a real number, the witness is called *A-short* if $d \leq \mathscr{L}(AB)$.

Observe that, if there exists a witness for h, then h is seen to not be P_A-avoiding for sufficiently large A, by simply selecting $x \in \mathbb{U}$. We will prove the following result.

Theorem 2.5. *Let $h(x) \in k^{\mathrm{c}}(x)$ be nonconstant, and $A \geq 1$. Then h is P_A-avoiding unless there exists an A-short witness for h.*

According to Remark 2.3 above, the case $k = \mathbb{Q}$ has a particularly nice phrasing.

Corollary 2.6. *Let $h(x) \in \mathbb{Q}^{\mathrm{c}}(x)$ be nonconstant and $A \geq 1$. Then h is P_A-avoiding unless there exists $S \in \mathbb{Q}^{\mathrm{c}}(x)$ such that $h(S(x))$ is equal to a Laurent polynomial $p \in \mathbb{Z}[\mathbb{U}][x, x^{-1}]$ with $|p(1)| \ll_\varepsilon A^{2+\varepsilon}$.*

As stated, these results do not give any bound on the size of the degree of a witness. However, the following theorem shows that "most" of $h(x) \in k^{\mathrm{c}}(x)$ are in fact P_A-avoiding.

Theorem 2.7. *Let k be a number field with Loxton pair (B, E). Let $A \geq 1$ and let $h(x) \in k^{\mathrm{c}}(x)$ be nonconstant. Suppose that*

- $\deg h > 2016 \cdot 5^{\mathscr{L}(AB)+1}$, *or*
- *h is a polynomial and* $\deg h > (2\mathscr{L}(AB) + 1)^2$.

Then h is P_A-avoiding unless it has an A-short witness $h(S(x))$ for which $\deg S \leq 2$.

Remark 2.8. In fact, if $h \in k^{\mathrm{c}}[x]$ is a polynomial which is not P_A-avoiding, one can find an A-short witness of the form $h(ax + b + cx^{-1})$ for some $a, b, c \in k^{\mathrm{c}}$ (see Theorem 3.3).

Remark 2.9. The constants involved in Theorem 2.7 come from Fuchs-Zannier [3], reproduced in the next section as Theorem 3.3.

3. Background

To prove the main result, we will need other auxiliary results, which we collect in this section.

3.1. **Tools from arithmetic geometry.** In what follows, fix k a number field, and $\mathbb{G}_{\mathrm{m}} = \operatorname{Spec} k[x, x^{-1}]$ as usual. By a *torsion coset* of $\mathbb{G}_{\mathrm{m}}^d$, we mean a translate $\beta \cdot T$ of a subtorus T (i.e., a connected algebraic group) by a torsion point β of $\mathbb{G}_{\mathrm{m}}^d$.

Theorem 3.1 ([2, Torsion Points Theorem])**.** *Let V be an algebraic subvariety of $\mathbb{G}_{\mathrm{m}}^d$ defined over $\overline{\mathbb{Q}}$. Then the Zariski closure of the set of torsion points in V is a finite union of torsion cosets of $\mathbb{G}_{\mathrm{m}}^d$.*

We also use a special case of [2, Theorem 1].

Theorem 3.2. *Let k be a number field. Let V/k be an affine variety irreducible over k^{c} and let*

$$\pi : V \to \mathbb{G}_{\mathrm{m}}^r$$

be a morphism of finite degree, defined over k. Assume the set of torsion points of $\pi(V(k^{\mathrm{c}}))$ is Zariski-dense in $\mathbb{G}_{\mathrm{m}}^r$.

Then, there exists an isogeny $\mu : \mathbb{G}_{\mathrm{m}}^r \to \mathbb{G}_{\mathrm{m}}^r$ and a birational map $\rho : \mathbb{G}_{\mathrm{m}}^r \dashrightarrow V$, both defined over k^{c}, such that the diagram

$$\begin{array}{ccc} \mathbb{G}_{\mathrm{m}}^r & \overset{\rho}{\dashrightarrow} & V \\ & \searrow^{\mu} & \downarrow \pi \\ & & \mathbb{G}_{\mathrm{m}}^r \end{array}$$

commutes (over k^{c}).

Proof. We define the set

$$J = \{\eta \in V(k^{\mathrm{c}}) : \pi(\eta) \text{ is a torsion point of } \mathbb{G}_{\mathrm{m}}^r\}.$$

Thus $\pi(J)$ consists exactly of all torsion points of $\pi(V(k^{\mathrm{c}}))$, so it is Zariski-dense by hypothesis. Since π is of finite degree, it follows that J is Zariski-dense in V as well. Then we can apply [2, Theorem 1], where the torsion coset T in question is the entire $\mathbb{G}_{\mathrm{m}}^r$. $\square$

3.2. **Results on compositions of rational functions.** We recall the following results of Fuchs and Zannier [3]. These results hold in much more generality if k^c is replaced by any field of characteristic zero, but we will not need that generality for our purposes.

Theorem 3.3 ([3, Main Theorem and Theorem 2])**.** *Let $p, q, h \in k^c(x)$ be rational functions with $p = h \circ q$, Denote by ℓ the sum of the number of terms in the numerator and denominator of p.*

- *Assume q is not of the shape $\lambda(ax^n + bx^{-n})$ for $a, b \in k^c$, $\lambda \in \mathrm{PGL}_2(k^c)$, $n \in \mathbb{Z}_{>0}$. Then,*
$$\deg h \leq 2016 \cdot 5^{\ell}.$$
- *Suppose $p \in k^c[x, x^{-1}] \setminus k^c[x]$ is a Laurent polynomial with ℓ nonconstant terms for some $\ell \geq 0$. Suppose moreover that $h \in k^c[x]$ is a polynomial and $q \in k^c[x, x^{-1}]$, where $q(x)$ is not of the shape $ax^n + b + cx^{-n}$ for $a, b, c \in k^c$, $n \in \mathbb{Z}_{>0}$. Then,*
$$\deg h \leq 2(2\ell - 1)(\ell - 1).$$

Corollary 3.4 ([3, Corollary on pg. 177])**.** *Let $q \in k^c(x)$ be nonconstant, and let $h \in k^c(x)$ with $\deg h \geq 3$ be not special. Then for any integer $n \geq 3$, the sum of the number of terms in the numerator and denominator of the rational function $h^n \circ q$ is at least*
$$\log_5 \left(\frac{(\deg h)^{n-2}}{2016} \right).$$

3.3. **Estimates on sizes of orbits.** We will use the following result, which is based on [7, §1.3].

Lemma 3.5. *Let k be a number field and let $h = p/q \in k(x)$ be a rational function. Assume $\deg p > \deg q + 1$.*

Then, there exist a real number $T > 0$ and an integer D (depending only on h) with the following properties. For any algebraic number α,

- *If $\overline{|h^n(\alpha)|} \leq A$ for some $n \geq 1$, then*
$$\overline{|h^j(\alpha)|} \leq \max(T, A) \quad \text{for } j = 0, \ldots, n-1.$$
- *If $h^n(\alpha)$ is an algebraic integer for some $n \geq 1$, then $Dh^j(\alpha)$ is an algebraic integer for $j = 0, 1, \ldots, n-1$.*

Proof. Suppose that $h^n(\alpha) = \gamma$.

First, since $\deg p - \deg q \neq 1$ we can pick $0 \neq c \in \overline{\mathbb{Q}}$ (depending only on h) such that
$$h(x) = c^{-1} \cdot \widetilde{h}(cx)$$
and moreover $\widetilde{h}$ is "monic" in the sense that $\widetilde{h} = \widetilde{p}/\widetilde{q}$ and
$$\widetilde{p}(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0,$$
$$\widetilde{q}(x) = x^e + b_{e-1}x^{e-1} + \cdots + b_0.$$
(It is possible that $c \notin k$; in this case we enlarge k to contain c.) Now, for any $j = 0, \ldots, n$ we have
$$h^j(x) = c^{-1} \cdot \widetilde{h}^j(cx).$$
In particular, $\widetilde{h}^j(c\alpha) = c\gamma$.

The first part now follows from applying [7, Corollary 2.7], to cA, $c\alpha$, and $\widetilde{h}$, using the condition $\deg p - \deg q > 1$.

We proceed to the second part. Assume γ is an algebraic integer. Note that by replacing the value of n, it suffices just to show that $D\alpha$ is an algebraic integer for some integer D depending only on h.

Let ν be an arbitrary finite place of k. Then [7, Corollary 2.5] implies that if $\|c\alpha\|_\nu > \max\{1, \|a_i\|_\nu, \|b_i\|_\nu\}$, then the sequence

$$\left\|\widetilde{h}^j(c\alpha)\right\|_\nu \qquad \text{for } j = 0, 1, 2, \dots$$

is strictly increasing. Thus, in particular we must have

$$\|c\alpha\|_\nu \le \max\left(1, \|a_i\|_\nu, \|b_i\|_\nu, \|c\gamma\|_\nu\right)$$

or else we contradict the fact that $\widetilde{h}^j(c\alpha) = c\gamma$.

Now, let D be an integer for which Dc^{-1}, $Dc^{-1}a_i$, $Dc^{-1}b_i$ are all algebraic integers. Multiplying the previous inequality by Dc^{-1}, we obtain

$$\begin{aligned} \|D\alpha\|_\nu &\le \max\left(\left\|Dc^{-1}\right\|_\nu, \left\|Dc^{-1}a_i\right\|_\nu, \left\|Dc^{-1}b_i\right\|_\nu, \|D\gamma\|_\nu\right) \\ &\le 1. \end{aligned}$$

Since this is true for every finite place ν, it follows that $D\alpha$ is an integer. Moreover, since D depends only on c, a_i, b_i and not on γ, it follows that D depends only on h, which proves our assertion. □

4. Proof of results on P_A-avoidance

Proof of Theorem 2.5. Assume h is not P_A-avoiding, so $h(k^{\mathrm{c}})$ contains infinitely many elements of P_A. By Theorem 2.1 and the pigeonhole principle, we can fix $d \le \mathscr{L}(AB)$ and $e_i \in E$ such that there exist infinitely many elements $y \in k^{\mathrm{c}}$ and $\xi_1, \dots, \xi_d \in \mathbb{U}$ satisfying

$$h(y) = \sum_{i=1}^{d} e_i \xi_i.$$

Take $\mathbb{G}_{\mathrm{m}}^{d+1}$ equipped with coordinates $(x_1, \dots, x_d, y)$. Letting $h = p/q$ for $p, q \in k^{\mathrm{c}}[x]$, consider the subvariety

$$V \subseteq \mathbb{G}_{\mathrm{m}}^{d+1}$$

defined by the equation

$$p(y) = q(y) \sum_{i=1}^{d} e_i x_i.$$

Moreover, let $\mathbb{U}_d$ denote the set of torison points of $\mathbb{G}_{\mathrm{m}}^d$ and let $\Pi : V \to \mathbb{G}_{\mathrm{m}}^d$ be the projection onto the first d coordinates. We now consider the following iterative procedure. Initially, let

$$W_0 = V, \quad \boldsymbol{\beta}_0 = \mathbf{1} \in \mathbb{G}_{\mathrm{m}}^d, \quad \text{and } T_0 = \mathbb{G}_{\mathrm{m}}^d$$

so the torsion coset $\boldsymbol{\beta}_0 T_0$ is all of $\mathbb{G}_{\mathrm{m}}^d$. So we have $\Pi(W_0) \subseteq \boldsymbol{\beta}_0 T_0$ and $\#(\Pi(W_0) \cap \mathbb{U}_d) = \infty$. Then we recursively perform the following procedure for $i = 0, 1, 2, \dots$.

- Consider the infinite set $\boldsymbol{\beta}_i^{-1}\Pi(W_i) \cap \mathbb{U}_d \subseteq T_i$. By Theorem 3.1 applied to the subvariety T_i, its Zariski closure consists of finitely many torsion cosets. Hence by pigeonhole principle, we may pick a particular torsion coset, say $\boldsymbol{\beta}' T_{i+1}$, containing infinitely many elements of $\mathbb{U}_d$. Now set $\boldsymbol{\beta}_{i+1} = \boldsymbol{\beta}_i\boldsymbol{\beta}'$. Then we conclude that $\boldsymbol{\beta}_{i+1}T_{i+1}$ is the closure of some infinite subset of $\Pi(W_i) \cap \mathbb{U}_d$.
- Now consider the preimage $\Pi^{-1}(\boldsymbol{\beta}_{i+1}T_{i+1})$, which is a closed subvariety of W_i. Then by pigeonhole principle, we can set W_{i+1} to be any irreducible component of W_i such that $\#(\Pi(W_{i+1}) \cap \mathbb{U}_d) = \infty$. Of course by construction $\Pi(W_{i+1}) \subseteq \boldsymbol{\beta}_{i+1}T_{i+1}$.

From this we have constructed

$$V = W_0 \supseteq W_1 \supseteq \cdots$$

a decreasing sequence of subvarieties of V, with W_i irreducible for $i \geq 1$. For dimension reasons, this sequence must eventually stabilize. Thus the torsion coset $\boldsymbol{\beta}_i T_i$ stabilizes too. So we conclude there exists

- an *irreducible* affine subvariety $W \subseteq V$,
- a particular torsion coset $\boldsymbol{\beta}T \subseteq \mathbb{G}_{\mathrm{m}}^d$, where $\boldsymbol{\beta} = (\beta_1, \dots, \beta_d) \in \mathbb{U}^d$ and T is a torus, and
- $Z := \Pi(W) \cap \mathbb{U}_d$ a set of torsion points of $\mathbb{G}_{\mathrm{m}}^d$

such that

$$\Pi(W) \subseteq \boldsymbol{\beta}T, \qquad \overline{Z} = \boldsymbol{\beta}T, \qquad \text{and} \quad \#Z = \infty.$$

(In the case V is already an irreducible subvariety, then $W = V$, the torsion coset $\boldsymbol{\beta}T$ is exactly $\mathbb{G}_{\mathrm{m}}^d$, and $Z = \mathbb{U}_d$. On the other hand if V is not irreducible, then the W_i start to decrease after the first step.)

Let $r := \dim T$; note that $r \geq 1$ since T contains the infinite set Z.

We now wish to apply Theorem 3.2. Consider the composed map $\pi : W \to \mathbb{G}_{\mathrm{m}}^r$ defined by taking φ as below:

$$\begin{array}{ccccc} W & \xrightarrow{\quad\varphi\quad} & T & \xrightarrow[\simeq]{\psi} & \mathbb{G}_{\mathrm{m}}^r \\ (x_1, \dots, x_d, y) & \longmapsto & (\beta_1^{-1}x_1, \dots, \beta_d^{-1}x_d). & & \end{array}$$

From the fact that $\overline{Z} = \boldsymbol{\beta} \cdot T$, we conclude that the set of torsion points in $\pi(W)$ is Zariski-dense in $\mathbb{G}_{\mathrm{m}}^r$. Applying Theorem 3.2, there exist an isogeny $\mu : \mathbb{G}_{\mathrm{m}}^r \to \mathbb{G}_{\mathrm{m}}^r$ and a birational map $\rho : \mathbb{G}_{\mathrm{m}}^r \dashrightarrow W$ such that the diagram

$$\begin{array}{ccccc} \mathbb{G}_{\mathrm{m}}^r & \overset{\rho}{\dashrightarrow} & W & \xrightarrow{\varphi} & T \\ & \searrow^{\mu} & \downarrow{\pi} & \nearrow_{\psi^{-1}} & \\ & & \mathbb{G}_{\mathrm{m}}^r & & \end{array}$$

commutes.

Assume

$$\rho(\boldsymbol{x}) = (R_1(\boldsymbol{x}), \dots, R_d(\boldsymbol{x}), R(\boldsymbol{x}))$$

for rational functions $R_1, \dots, R_d, R$ (here $\boldsymbol{x} \in \mathbb{G}_{\mathrm{m}}^r$); then

$$\varphi(\rho(\boldsymbol{x})) = \left(\beta_1^{-1} R_1(\boldsymbol{x}), \dots, \beta_d^{-1} R_d(\boldsymbol{x}), R(\boldsymbol{x})\right).$$

Now, the right-hand side of $\varphi \circ \rho = \psi^{-1} \circ \mu$ is the composition of an isogeny and an isomorphism, thus (for instance by [1, Proposition 3.2.17]), we recover that $R_i(\boldsymbol{x}) = \beta_i \boldsymbol{x}^{\boldsymbol{v}_i}$ for some vectors $\boldsymbol{v}_i \in \mathbb{Z}^r$ which are linearly independent (and in particular nonzero).

Thus

$$\rho(\boldsymbol{x}) = (\beta_1 \boldsymbol{x}^{\boldsymbol{v}_1}, \dots, \beta_d \boldsymbol{x}^{\boldsymbol{v}_d}, R(\boldsymbol{x}))$$

and we obtain an identity

$$h(R(\boldsymbol{x})) = \sum_{i=1}^{d} e_i \cdot \beta_i \boldsymbol{x}^{\boldsymbol{v}_i}.$$

Since the $\boldsymbol{v_i}$ are independent, it follows that one can specialize $\boldsymbol{x}$ to a choice of the form $\boldsymbol{x} = (x^{c_1}, \dots, x^{c_r})$ for some integers $c_i \in \mathbb{Z}$ so that the terms $\boldsymbol{x}^{\boldsymbol{v_i}}$ are pairwise distinct. Thus we finally obtain

$$h(S(x)) = \sum_{i=1}^{d} \beta_i e_i x^{n_i},$$

where S is a rational function (defined by $S(x) := R(x^{n_r}, \dots, x^{c_r})$), and the right-hand side is nonconstant in x. This is the desired A-short witness. □

Proof of Theorem 2.7. First suppose $h(x) \in k^{\mathrm{c}}(x)$. Then by Theorem 2.5, h is P_A-avoiding unless we have an identity

$$h(S(x)) = \sum_{i=1}^{d} \beta_i e_i x^{n_i},$$

where the right-hand side has at most $d \le \mathscr{L}(A \cdot B)$ terms.

First assume that $S = \mu(ax^n + bx^{-n})$ for some $\mu \in \mathrm{PGL}_2(k)$. Set now $\widetilde{S} = \mu(ax + bx^{-1})$, $\deg \widetilde{S} = 2$. We now see that

$$h(\widetilde{S}(x))$$

is an A-short witness, establishing the theorem.

Otherwise Theorem 3.3 applies with $\ell = d + 1$, and we deduce that

$$\deg h \le 2016 \cdot 5^{d+1}$$

which contradicts the first hypothesis of Theorem 2.7. This implies one direction.

In the case $h \in k^{\mathrm{c}}[x]$, we repeat the same argument, applying the second part of Theorem 3.3. (That S is a Laurent polynomial follows from the fact that it cannot have any nonzero poles, in light of the right-hand side having the same property.) □

Proof of Corollary 1.4. Suppose by contradiction h is not P_A-avoiding; then by Theorem 2.5 there is an A-short witness and we may write

$$h(S(x)) = \sum_{i} \beta_i e_i x^{n_i}.$$

View this as an identity of rational functions in $\mathbb{C}(x)$.

On the one hand, since $S \in \mathbb{C}(x)$ is a nonconstant rational function, its range in $\mathbb{C}$ omits at most one point of $\mathbb{C}$. Since h has at least three poles, it follows that there is an $x_0 \neq 0$ such that $S(x_0)$ is a pole of h.

On the other hand, the only possible pole of the right-hand side is $x = 0$, which is the desired contradiction. ☐

5. Proof of results on strong P_A-avoidance

Proof of Theorem 1.5. Since h is given to be P_A-avoiding, it suffices to show that for a given $\gamma \in P_A$, there are only finitely many $\alpha \in k^{\mathrm{c}}$ such that $h^n(\alpha) = \gamma$ for some $n \geq 1$.

Assume by contradiction there are infinitely many pairs (α, n) such that $h^n(\alpha) = \gamma$. Select $T > 0$ and $D \in \mathbb{Z}$ by Lemma 3.5, and let

$$C := D \max(T, A).$$

We make the following claim.

Claim. For any integer N, $D \cdot h^N(x)$ is not weakly P_C-avoiding.

To see this, discard the finitely many pairs with $n \leq N$, and consider only those with $n > N$. Then by applying Lemma 3.5 to such pairs (α, n) with $n > N$, there are infinitely many α such that $D \cdot h^N(\alpha)$ is an algebraic integer; moreover, the house of $D \cdot h^N(\alpha)$ is at most $D \cdot \max(T, A) = C$, giving the claim.

Consequently, by Theorem 2.5 for every integer N there exists a C-short witness. In other words, for all $N \geq 1$ there exists $S \in k^{\mathrm{c}}(x)$ such that

$$D \cdot h^N(S(x)) = \sum_{i=1}^{d} \beta_i e_i x^{n_i},$$

where $d \leq \mathscr{L}(BC) = \mathscr{L}(BD \max(T, A))$.

By hypothesis, $\deg h \geq 2$. Assume that $\deg h \geq 3$. Since we are given that h is special, by Corollary 3.4, h^N has at least $\log\left(\frac{(\deg h)^{N-2}}{2016}\right)$ terms, which gives a contradiction if we take

$$N > 2 + \log_{\deg h}\left(2016 \cdot 5^{\mathscr{L}(BD \max(T,A))}\right).$$

For $\deg h = 2$ one can apply the same argument replacing h with $h \circ h$. ☐

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