

ON GEODESIC RAY BUNDLES IN HYPERBOLIC GROUPS

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(Communicated by David Futer)

ABSTRACT. We construct a Cayley graph $\mathbf{Cay}_S(\Gamma)$ of a hyperbolic group Γ such that there are elements $g, h \in \Gamma$ and a point $\gamma \in \partial_\infty \Gamma = \partial_\infty \mathbf{Cay}_S(\Gamma)$ such that the sets $\mathcal{RB}(g, \gamma)$ and $\mathcal{RB}(h, \gamma)$ in $\mathbf{Cay}_S(\Gamma)$ of vertices along geodesic rays from g, h to γ have infinite symmetric difference, thus answering a question of Huang, Sabok, and Shinko.

1. INTRODUCTION

To every infinite finite valence tree T we can associate a boundary at infinity $\partial_\infty T$ corresponding to ends of infinite rays. $\partial_\infty T$ is homeomorphic to a Cantor set. A metric space is called δ -hyperbolic if, roughly speaking, up to an error term δ it has a tree-like structure. Analogously to a tree, to a δ -hyperbolic space X , one can assign a Gromov boundary $\partial_\infty X$ which is a compact, metrizable, yet oftentimes exotic set corresponding to equivalence classes of ends of infinite *geodesic* rays. A group Γ is called *hyperbolic* if one of its Cayley graphs is a δ -hyperbolic metric space for some $\delta \geq 0$. In this case to Γ we can assign a canonical Gromov boundary $\partial_\infty \Gamma$ on which Γ acts nontrivially. The deep connections between the properties of $\partial_\infty \Gamma$ and the group Γ makes it highly a structured, and therefore fascinating, object to study.

In [HSS17] Huang, Sabok, and Shinko investigate Borel equivalence relations on $\partial_\infty \Gamma$. They show that if Γ is a hyperbolic group with the additional property that Γ acts properly discontinuously and cocompactly on a CAT(0) cube complex, i.e., Γ is *cubulated*, then the action of Γ on its boundary $\partial_\infty \Gamma$ is hyperfinite. This generalizes a result of Dougherty, Jackson, and KeCHRIS [DJK94, Corollary 8.2] from the class of free groups to the larger class of cubulated hyperbolic groups.

Although the result of [HSS17] feels like it should be true for all hyperbolic groups, an additional cubulation requirement is needed to prove a key lemma, [HSS17, Lemma 1.3], which states that for any two vertices x, y of a δ -hyperbolic CAT(0) cube complex C and for any point $\gamma \in \partial_\infty C$ the sets, called *ray bundles*, $\mathcal{RB}(x, \gamma)$ and $\mathcal{RB}(y, \gamma)$ of vertices of C that occur along geodesic rays from x and y (respectively) to $\gamma \in \partial_\infty C$ have finite symmetric difference. We call this the *boundary coherence property*. Precise definitions are given in Section 2.

It is known that the boundary coherence property does not hold for δ -hyperbolic graphs in general (see Section 3). The authors pose [HSS17, Question 1.4], which asks whether requiring a graph to be the Cayley graph of a hyperbolic group implies the boundary coherence property. Not only would a positive answer immediately

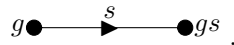
imply that the action of any hyperbolic group Γ on $\partial_\infty\Gamma$ is hyperfinite, but this is also a very natural question to ask from the point of view of geometric group theory. In this paper we give the following negative answer.

Theorem A (see Corollary 4.3). *The Cayley graph $X = \mathbf{Cay}_{\{p,q,s\}}(\mathbb{F}_2)$ corresponding to presentation $\langle p, q, s \mid s^{-2}ps^2q \rangle$ of \mathbb{F}_2 , the free group of rank 2, does not satisfy the boundary coherence property.*

This example, if anything, reinforces the relevance of [HSS17, Lemma 1.3]. The methods of this paper will be familiar to geometric group theorists, but, since this paper is aimed at a broader audience, necessary background is included to make it self-contained. That being said, the reader is expected to know the following notions from topology: group presentations, fundamental groups, the Seifert–van Kampen theorem, and universal covering spaces.

2. HYPERBOLIC GROUPS AND THEIR BOUNDARY

The author recommends [Aea91] for an accessible yet thorough treatment of the topics in this section. Given a group Γ and a generating set S of Γ we can construct a Cayley graph $\mathbf{Cay}_S(\Gamma)$ which is a directed graph whose vertices are the elements of Γ and for each $g \in \Gamma$ and $s \in S$ we draw the edge



By declaring each edge to be an isometric copy of the closed unit interval, we make graphs into connected metric spaces via the path metric. If X is a graph we say that a path starting at a vertex v and ending at a vertex u is *geodesic* if it is the shortest possible path between u, v . Typically there will be multiple geodesics between a pair of vertices. A metric space is δ -*hyperbolic* if it has the following property: for any three vertices u, v, w , if α, β , and γ are geodesics from u to v , v to w , and w to u , respectively, then α is contained in a δ -neighbourhood of $\beta \cup \gamma$. If a group Γ has a δ -hyperbolic Cayley graph with respect to one finite generating set, then for any other finite generating set the corresponding Cayley graph will also be δ' -hyperbolic, though possibly with $\delta' \neq \delta$. Such a group will therefore be called a *hyperbolic group*.

For example, if A is a finite set of symbols and $\mathbb{F}(A)$ is the free group on A , then, taking A as a generating set of $\mathbb{F}(A)$, the Cayley graph $\mathbf{Cay}_A(\mathbb{F}(A))$ is a regular tree with valence $|A|$. In particular, for any geodesics α, β , and γ as above, $\alpha \subset \beta \cup \gamma$ so that $\mathbf{Cay}_A(\mathbb{F}(A))$ is in fact 0-hyperbolic.

Let us now give a precise definition of the Gromov boundary $\partial_\infty X$ of a δ -hyperbolic graph X . A *geodesic ray* is a continuous map

$$\rho : [0, \infty) \rightarrow X$$

such that for every pair of positive integers $m < n$, $\rho(m)$ is a vertex and the segment $\rho([n, m])$ is a geodesic. $\partial_\infty X$ is the set of geodesic rays of X modulo the relation: $\rho \sim \rho' \Leftrightarrow$ there is some $R \geq 0$ such that $\rho([0, \infty))$ is contained in an R -neighbourhood of $\rho'([0, \infty))$ and $\rho'([0, \infty))$ is contained in an R -neighbourhood of $\rho([0, \infty))$.

Definition 2.1 (Ray bundle). If x is a vertex of a δ -hyperbolic graph X and $\gamma \in \partial_\infty X$, then the *ray bundle* $\mathcal{RB}(x, \gamma)$ from x to γ is the set of vertices of X that

lie in a geodesic ray ρ with $\rho(0) = x$ that is in the \sim -equivalence class corresponding to $\gamma \in \partial_\infty X$.

If $\rho \sim \rho'$, then, as a consequence of δ -hyperbolicity, the rays should eventually travel uniformly close together and one would expect ray bundles with the same endpoint in $\partial_\infty X$ to overlap.

Definition 2.2 (Boundary coherence property). A δ -hyperbolic graph X has the *boundary coherence property* if for any vertices x, y of X and any $\gamma \in \partial X$, the ray bundles $\mathcal{RB}(x, \gamma)$ and $\mathcal{RB}(y, \gamma)$ have finite symmetric difference.

If Γ is a hyperbolic group, then we define $\partial_\infty \Gamma = \partial_\infty \mathbf{Cay}_S(\Gamma)$ for some finite generating set S of Γ . We recommend the following exercises:

- If $\Gamma = \mathbb{F}(A)$ is a free group as above, then $\partial \Gamma$ is naturally identified with a Cantor set.
- If $\Gamma = \langle a \rangle \oplus \langle b \rangle$, the free abelian group of rank two (which is not a hyperbolic group), then $\partial_\infty \mathbf{Cay}_A(\Gamma)$ can be identified with the circle at infinity for \mathbb{R}^2 , but the action of Γ (induced by translating rays) yields a trivial action on $\partial_\infty \Gamma$.

That $\partial_\infty \Gamma$, thus given, is well defined, nontrivial, canonical for Γ , and admits a nontrivial Γ action is a consequence of δ -hyperbolicity. The reader may consult [GdlH90, §§6–8] or [KB02] for a complete treatment of the topic.

3. THE BAD LADDER

Consider the infinite graph \mathfrak{L} consisting of two *sides*, copies of \mathbb{R} , with a vertex at each integer point, and countably many *rungs*, edges connecting vertices at corresponding integral vertices on each side. Add a vertex to the middle of each rung. The resulting graph \mathfrak{L} is shown in Figure 1.

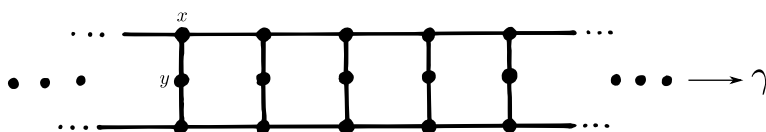


FIGURE 1. A ladder with a vertex x on a side and a vertex y in the middle of a rung.

We note that any two geodesic rays either go to the left or to the right, and if they go in the same direction, they remain at a bounded distance. It follows that $\partial_\infty \mathfrak{L}$ consists of two points.

Proposition 3.1. *Let x be a vertex on a side of \mathfrak{L} , let y be a vertex in the middle of a rung, and let $\gamma \in \partial_\infty \mathfrak{L} = \{\pm\infty\}$ correspond to one of the ends of the ladder. Then the sets $\mathcal{RB}(x, \gamma)$ and $\mathcal{RB}(y, \gamma)$ have infinite symmetric difference. In particular, \mathfrak{L} doesn't have the boundary coherence property.*

Proof. Without loss of generality we may assume that γ corresponds to $+\infty$. As any geodesic ρ travels towards γ , it must eventually stay within one of the sides of \mathfrak{L} . If ρ originates at x , then it is allowed to travel once through a rung to reach the other side. It follows that every vertex on a rung that is “greater” than x is in $\mathcal{RB}(x, \gamma)$. If ρ originates at y in the middle of a rung, then once it enters a

side s_1 it is no longer able to switch because if that happens, then there is some initial segment ρ' of ρ whose length does not realize the distance between y and the first point it encounters in $s_2 \neq s_1$. See Figure 1. It follows that $\mathcal{RB}(y, \gamma)$ doesn't contain any vertices contained in rungs; thus the two sets have infinite symmetric difference. \square

Although \mathfrak{L} is a hyperbolic graph, due to its nonhomogeneity, it cannot be the Cayley graph of a group. We will now construct the Cayley graph of a group, in fact a free group, which contains a ladder \mathfrak{L} as a convex subgraph. That is to say any geodesic connecting two points on the ladder inside this larger graph must stay within the ladder. To show this we must reach a sufficiently complete understanding of the geometry of a Cayley graph. Although it is not invoked explicitly, the proof is informed by the Bass–Serre theory of groups acting on trees and corresponding decompositions into graphs of spaces [SW79, Ser03].

4. EMBEDDING BAD LADDERS INTO CAYLEY GRAPHS

We will take some liberties with notation and identify group presentations with the groups they present. First consider the presentation

$$\Gamma_0 = \langle p, q, t \mid t^{-1}ptq^{-1} \rangle \approx \mathbb{F}_2.$$

For any group presentation, there is a standard construction known as a *presentation complex*, which is a CW-complex $\mathcal{P}(\Gamma_0)$ obtained by gluing polygons (corresponding to relations) to graphs (whose edges correspond to generators) in such a way (as a consequence of the Seifert–van Kampen Theorem) that $\pi_1(\mathcal{P}(\Gamma_0)) \approx \Gamma_0$.

In this case the presentation complex $\mathcal{P}(\Gamma_0)$ consists of a graph with one vertex, three directed edges labelled p, q, t , and a square along whose boundary the word $t^{-1}ptq^{-1}$ can be read. This word specifies the identifying map between the boundary of the square and a closed loop in the graph, making the latter nullhomotopic. As a topological space $\mathcal{P}(\Gamma_0)$ can also be obtained by taking a cylinder $A = [-1, 1] \times S^1$, picking a point on each boundary component, and identifying them. This is shown in Figure 2.

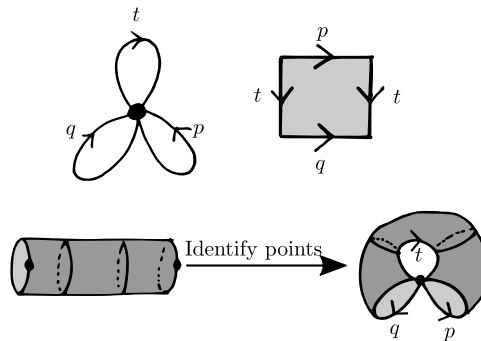


FIGURE 2. The presentation complex $\mathcal{P}(\Gamma_0)$

Remark 4.1. The 1-skeleton of the universal cover $\widetilde{\mathcal{P}(\Gamma_0)}$ corresponds to the Cayley graph $\mathbf{Cay}_{\{p,q,t\}}(\mathbb{F}_2)$, i.e., the Cayley graph relative to the generating set explicitly given by the group presentation. This is true for any presentation complex.

The universal cover $\widetilde{\mathcal{P}(\Gamma_0)}$ is a tree of spaces obtained by taking an infinite collection of copies of strips corresponding to connected components of the lift $A \subset \mathcal{P}(\Gamma_0)$ in $\widetilde{\mathcal{P}(\Gamma_0)}$ attached by points. We call these *pq-strips*. This is shown in Figure 3. There is also a collection of bi-infinite lines in $\widetilde{\mathcal{P}(\Gamma_0)}$ along which we read $\dots ttt\dots$; we call these *t-lines*. Now consider the amalgamated free product:

$$\Gamma_1 = \Gamma_0 *_{t=s^2} \langle s \rangle = \langle p, q, t, s \mid t^{-1}ptq^{-1}, s^2t^{-1} \rangle \approx \mathbb{F}_2$$

corresponding to adjoining a square root s to the basis element $t \in \mathbb{F}_2$. By the Seifert–van Kampen Theorem, it can be realized as the fundamental group of a space \mathcal{P}_1 , which is not a presentation complex, obtained by taking a copy of $\mathcal{P}(\Gamma_0)$, a circle $C = S^1$, and attaching another cylinder $D = [-1, 1] \times S^1$ so that the attaching map $\{-1\} \times S^1 \rightarrow \mathcal{P}(\Gamma_0)$ wraps with degree 1 around the loop corresponding to the edge with label t and the other attaching map $\{1\} \times S^1 \rightarrow C$ wraps with degree 2. See Figure 4.

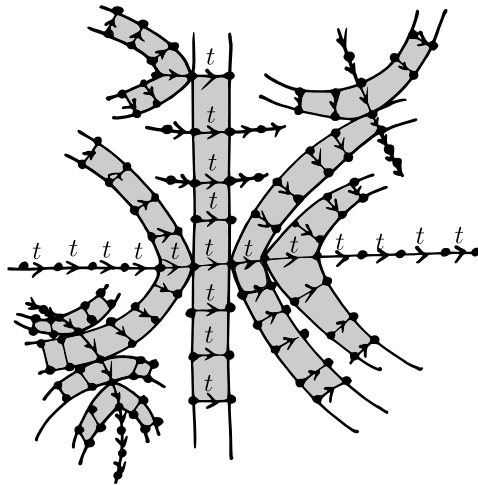


FIGURE 3. The universal cover of $\mathcal{P}(\Gamma_0)$. Decorated edges are labelled t . *pq*-strips are shaded grey.

In the universal covering space $\widetilde{\mathcal{P}_1}$, $\mathcal{P}(\Gamma_0) \subset \mathcal{P}_1$ lifts to a countable collection of disjoint copies of $\mathcal{P}(\Gamma_0)$ called Γ_0 -pieces and the circle $C \subset \mathcal{P}_1$ lifts to a countable collection of disjoint lines called *C-lines*. The connected components of lifts of the cylinder D are called *D-strips*, copies of $[-1, 1] \times \mathbb{R}$ connecting *t*-lines in Γ_0 -pieces to *C*-lines. In particular, each *C*-line is attached to two *D*-strips. Globally, the universal cover has the structure of a tree of spaces. See Figure 5.

Our final presentation Γ is obtained via the following Tietze transformation:

$$\Gamma_1 = \langle p, q, t, s \mid t^{-1}ptq^{-1}, s^2t^{-1} \rangle \approx \langle p, q, s \mid s^{-2}ps^2q \rangle = \Gamma.$$

This Tietze transformation corresponds to the fact that, since $s^2 = t$, we can remove t from the generating set. Geometrically the resulting presentation complex is obtained by collapsing the $[-1, 1]$ factor in the cylinder $D = S^1 \times [-1, 1] \subset \mathcal{P}_1$ to a point. See Figure 6. The universal cover of the presentation complex $\widetilde{\mathcal{P}(\Gamma)}$ can be obtained by taking the Γ_0 -pieces in $\widetilde{\mathcal{P}_1}$, subdividing each *t*-labelled edge into a

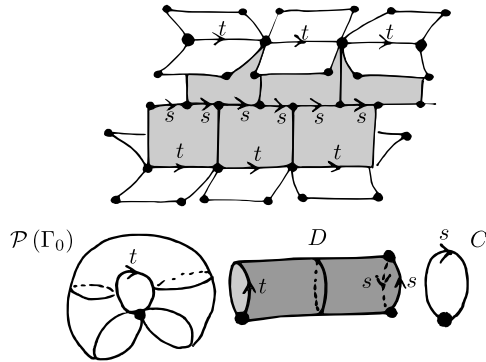


FIGURE 4. On top, a portion of the universal cover $\widetilde{\mathcal{P}}_1$. Below, how the CW-complex \mathcal{P}_1 is obtained from \mathcal{P}_0 .

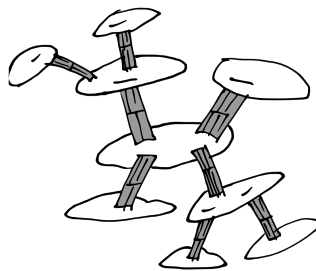


FIGURE 5. A portion of $\widetilde{\mathcal{P}}_1$ depicted as a tree of spaces obtained by attaching Γ_0 -pieces to D -strips (shown in grey) along t -lines. Note that although drawn as “pancakes”, the Γ_0 -pieces are actually copies of the space shown in Figure 3.

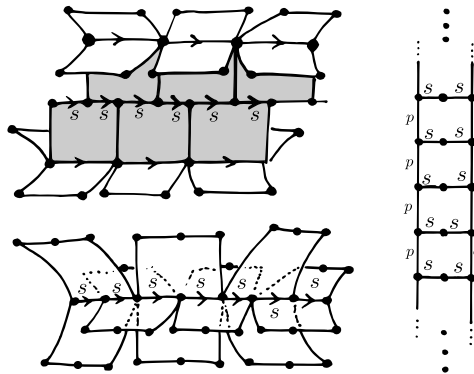


FIGURE 6. Collapsing D -strips (shaded grey) onto lines as seen from the universal cover, and the resulting pq -ladders, contained in a Γ_0 -piece.

length 2 edge path labelled ss , replacing t -lines with s -lines, and then identifying two s -lines in different Γ_0 -pieces if they are both connected by D -strips to the same

C -line. In this way $\widetilde{\mathcal{P}}_1$ has a large scale tree of spaces structure obtained by taking the resulting Γ_0 -pieces and attaching them along s -lines. Furthermore, we observe that each Γ_0 -piece contains a ladder \mathfrak{L} obtained by gluing together squares labelled $s^{-2}ps^2q$ along segments labelled s^2 . We call such a ladder a pq -ladder. See Figure 6.

In this way $\widetilde{\mathcal{P}}(\Gamma)$ admits a depth 2 hierarchical decomposition as a tree of spaces. At the top level we have Γ_0 -pieces connected along s -lines as a tree of spaces, and then the Γ_0 -pieces themselves are trees of pq -ladders, connected by vertices.

Proposition 4.2. *A pq -ladder \mathfrak{L} is convex in the 1-skeleton of $\widetilde{\mathcal{P}}(\Gamma)$. Furthermore, any geodesic ray starting in \mathfrak{L} and going to one of the ends of \mathfrak{L} must stay in \mathfrak{L} .*

Proof. Let \mathfrak{p} be the Γ_0 -piece containing a pq -ladder \mathfrak{L} . Let $u, v \in \mathfrak{L}$ be vertices and let ρ be a geodesic connecting u and v .

Claim 1: ρ cannot exit \mathfrak{p} 1. Suppose towards the contrary that this was the case. Then, by the tree of spaces structure, ρ must exit \mathfrak{p} at some point a contained in some s -line \mathfrak{s} , and then re-enter \mathfrak{p} at some other point $b \in \mathfrak{s}$ in the same s -line. It follows that if ρ is geodesic it cannot exit \mathfrak{p} because the subsegment $\rho([n_a, n_b])$, where $\rho(n_a) = a, \rho(n_b) = b$, can be replaced the strictly shorter segment from a to b contained within \mathfrak{s} .

Claim 2: If ρ stays in the Γ_0 -piece \mathfrak{p} , it cannot exit \mathfrak{L} 1. Indeed each piece consists of a tree of pq -ladders connected by points; it is the same space as the one shown in Figure 3 except that each edge labelled t has been replaced by a path of length 2 labelled ss . If ρ leaves \mathfrak{L} at some vertex p , then to re-enter \mathfrak{L} it must pass through p again, contradicting that it is geodesic.

The convexity of \mathfrak{L} now follows. This implies that any infinite path that stays in the p or q side of \mathfrak{L} is a geodesic ray. It remains to show that any geodesic ray starting at $x \in \mathfrak{L}$ and going to $\gamma \in \partial_\infty \mathfrak{L}$ stays in \mathfrak{L} . Let ρ be one such geodesic ray and let $\beta : [0, \infty) \rightarrow \widetilde{\mathcal{P}}(\Gamma)$ be another arc-length parameterized geodesic ray from x to γ . Suppose that β exits \mathfrak{L} at the point $\beta(N)$.

By the convexity of \mathfrak{L} , β cannot re-enter \mathfrak{L} , but it could still travel close to it. By definition of the Gromov boundary there must be some bound R such that for all $z, d(\beta(z), \rho) \leq R$. However, since β is geodesic and arc-length parameterized, $d(\beta(N + M), \beta(N)) = M$, and since the shortest path from $\beta(N + M)$ to \mathfrak{L} must pass through $\beta(N)$, we conclude that

$$d(\beta(N + M), \rho) \geq d(\beta(N + M), \mathfrak{L}) = M.$$

Since β is an infinite ray we may take $M > R$ which yields a contradiction. □

Propositions 3.1 and 4.2 immediately imply the main result.

Corollary 4.3. *Let $\Gamma = \langle p, q, s \mid s^{-2}ps^2q \rangle \approx \mathbb{F}_2$ and let $X = \mathbf{Cay}_{\{p,q,s\}}(\Gamma)$. If $\gamma \in \partial_\infty X = \partial_\infty \Gamma$ corresponds to an end of a pq -ladder \mathfrak{L} , x is a vertex contained in a side of \mathfrak{L} , and y is a vertex contained in a rung of \mathfrak{L} , then $\mathcal{RB}(x, \gamma)$ and $\mathcal{RB}(y, \gamma)$ have infinite symmetric difference.*

4.1. A one-ended example. The example we just gave is somewhat unsatisfying since it is a free group. We will outline another construction, which was the original example found by the author. This group is not free since it is one-ended, which

in the torsion-free case means it does not decompose as a nontrivial free product. Consider first the presentation

$$\Sigma_0 = \langle a, b, p, q, t \mid abpa^{-1}b^{-1}q, ptqt^{-1} \rangle.$$

This is an explicit decomposition of Σ_0 as an HNN extension of a free group of rank 3 and the presentation complex is homeomorphic to an orientable closed surface of genus 2. We then repeat the construction in the previous section,

$$\begin{aligned} \Sigma_1 &= \langle a, b, p, q, t \mid abpa^{-1}b^{-1}q, ptqt^{-1} \rangle *_{t= s^2} \langle s \rangle \\ &\approx \langle a, b, p, q, s \mid abpa^{-1}b^{-1}q, ps^2qs^{-2} \rangle = \Sigma, \end{aligned}$$

to embed a bad ladder into the Cayley graph corresponding to the presentation Σ . Since Σ_0 is a closed surface group, therefore one-ended, and $\langle s \rangle$ cannot act with an infinite orbit on a tree if s^2 fixes a point, [Tou15, Theorem 3.1] implies that Σ is one-ended. In particular, Σ is not free, but it is torsion-free since it is constructed by a sequence of amalgamations of torsion-free groups. Hyperbolicity follows from the combination theorems [BF92, BF96, KM98].

Again the universal cover is a tree of spaces obtained by gluing hyperbolic planes \mathbb{H}^2 along s -lines, and the proof is similar to Proposition 4.2. The first claim goes through as is; we leave the proof of Claim 2 (convexity of pq -ladders) as an exercise in small cancellation theory (one can use [MW02, Theorem 9.4]).

4.2. Cubulating bad ladders. A bad ladder consists of a chain of hexagons glued along edges. As an illustration of [HSS17, Lemma 1.3], observe that if we cubulate a bad ladder, i.e., make it into a cube complex (see Figure 7), the conclusion of Proposition 3.1 no longer holds. In fact both groups shown in this paper can be cubulated and therefore do not give counterexamples to the conjecture that the action of *every* hyperbolic group Γ on $\partial_\infty\Gamma$ is hyperfinite, a conjecture that this author believes to be true.

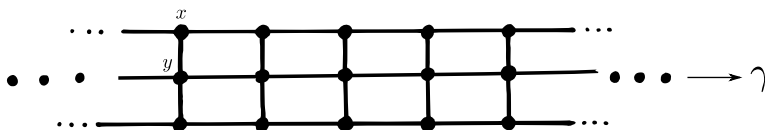


FIGURE 7. A cubulated bad ladder

ACKNOWLEDGMENTS

I first wish to thank Michael Hull and Jindrich Zapletal for the invitation to the South Eastern Logic Symposium 2017, which greatly increased my appreciation of the contemporary work of descriptive set theorists. I also wish to thank Marcin Sabok for posing this question about symmetric differences of ray bundles, specifically about the embedability of bad ladders into Cayley graphs, and for an interesting discussion, encouragement, and feedback. I am also grateful to Bob Gilman and Paul Schupp for conversations that confirmed that the main result of this paper is probably not a trivial consequence of what is known about the automaticity of the language of geodesics in hyperbolic groups. Finally, I wish to thank the anonymous referee who gave some good suggestions on the exposition.

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