PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 10, October 2018, Pages 4535–4548
http://dx.doi.org/10.1090/proc/14122
Article electronically published on June 28, 2018

DISCRETE TOPOLOGICAL COMPLEXITY

D. FERNÁNDEZ-TERNERO, E. MACÍAS-VIRGÓS, E. MINUZ, AND J. A. VILCHES

(Communicated by Michael A. Mandell)

Abstract. We introduce a notion of discrete topological complexity in the setting of simplicial complexes, using only the combinatorial structure of the complex and replacing the concept of homotopy by that of contiguous simplicial maps. We study the links of this new invariant with those of simplicial category and topological complexity.

1. Introduction

Topological complexity was introduced by Farber [6] as a topological invariant intended to solve problems such as motion planning in robotics. For this purpose one needs an algorithm that is capable to compute, for each pair of points of the so-called configuration space of a mechanical or physical device, a path connecting them in a continuous way. Farber's key idea was to interpret that algorithm in terms of a section of the so-called path-fibration, which is a well-known map in algebraic topology.

The aim of the present paper is to establish a discrete version of this approach. Discretization is interesting because many motion planning methods transform a continuous problem into a discrete one. While it is agreed that finite simplicial complexes are the proper setting to develop a discrete version of topology, the main technical difficulty is to avoid the construction of a path-space $\mathrm{P}K$ associated to the simplicial complex K. We were able to do so, by using a different but equivalent characterization of topological complexity, as we explain in Section 2.

The contents of the paper are as follows. In Section 2 we recall the basic notions of topological complexity, Švarc genus, and simplicial contiguity. In Section 3 we introduce the definition of discrete topological complexity $\mathrm{TC}(K)$ of a simplicial complex K and we prove that this new invariant only depends on the strong homotopy type of K, as defined by Barmak and Minian [3, 4]. In Section 4 we compare the new invariant with $\mathrm{scat}(K)$, the simplicial Lusternik-Schnirelmann category of K, which was defined by us in two previous papers [8, 9] and has been studied later by other authors [2, 13]. This comparison gives a simplicial version of two of Farber's well-known results [6]. Finally, in Section 5, $\mathrm{TC}(K)$ is compared with

Received by the editors November 24, 2017, and, in revised form, January 15, 2018, February 1, 2018, and February 6, 2018.

2010 *Mathematics Subject Classification.* Primary 55U10; Secondary 55M30, 55S40.

The first and the fourth authors were partially supported by MINECO Spain Research Project MTM2015-65397-P and Junta de Andalucía Research Groups FQM-326 and FQM-189.

The second author was partially supported by MINECO Spain Research Project MTM2016-78647-P and FEDER and by Xunta de Galicia GPC2015/006.

The third author was partially supported by DFF-Research Project Grants from the Danish Council for Independent Research.

the topological complexity $\mathrm{TC}(|K|)$ of the geometric realization $|K|$ of the complex K. The paper ends with the computation of the discrete topological complexity of certain families of graphs, namely trees and wedges of circles. Several explicit examples are given.

2. Preliminaries

2.1. **Topological complexity.** We include here some motivational remarks. Farber's topological complexity [6, 7] is a particular case of the Švarc genus or sectional category of a map [5, 16].

Definition 2.1. The *Švarc genus* $\mathrm{secat}(f)$ of a map $f\colon X \to Y$ is the minimum integer number $n \geq 0$ such that the codomain Y can be covered by open sets $V_0, \dots, V_n$ with the property that over each V_j there exists a local section s_j of f (that is, a continuous map $s_j\colon V_j \to X$ such that $f \circ s_j = \iota_j$, where $\iota_j\colon V_j \subset Y$ is the inclusion).

Definition 2.2. The *topological complexity* of a topological space X is $\mathrm{TC}(X) = \mathrm{secat}(\pi)$, where $\pi\colon \mathrm{P}X \to X \times X$ is the path fibration, that is, the map sending an arbitrary path $\gamma\colon [0,1] \to X$ into the pair $(\gamma(0), \gamma(1))$ formed by the initial and the final points of the path.

Remark 2.3. We adopt the normalized version of concepts such as Švarc genus, topological complexity, and LS-category, in such a way that contractible spaces have category zero. This convention is often used, as in [5], as well as in our papers [8, 9] and we will maintain it here. However, a non-normalized definition may appear sometimes in other papers, as Farber did in [6].

An important result is that for some topological spaces (including the geometric realization of any finite simplicial complex) the topological complexity can be computed by taking *closed* subspaces instead of open subspaces. This is discussed in [7, Chap. 4].

Now we proceed to modify the definition of sectional category in order to get a weaker notion, more suited for working modulo homotopy.

Definition 2.4. The *homotopic Švarc genus* of the map $f\colon X \to Y$, denoted by $\mathrm{hsecat}(f)$, is the minimum integer number $n \geq 0$ such that there exists an open covering $Y = V_0 \cup \dots \cup V_n$ of the codomain, with the property that for each V_j there exists a local *homotopic* section s_j, that is, a continuous map $s_j\colon V_j \to X$ such that there is a homotopy $f \circ s_j \simeq \iota_j$, where $\iota_j\colon V_j \subset Y$ is the inclusion.

Clearly, $\mathrm{hsecat}(f) \leq \mathrm{secat}(f)$. For a particular class of maps both invariants coincide.

Proposition 2.5. *If* $\pi\colon X \to Y$ *is a fibration* (*that is, a map with the homotopy lifting property*), *then* $\mathrm{hsecat}(\pi) = \mathrm{secat}(\pi)$. *In particular, this is true for the path fibration* $\pi\colon \mathrm{P}X \to X \times X$.

The proof is easy.

Now, it is well known that any map factors, up to homotopy equivalence, through a fibration. We will apply this to the particular case of the diagonal map $\Delta_X\colon X \to X \times X$.

Proposition 2.6. *There is a homotopy equivalence $X \simeq \mathrm{P}X$ such that the diagram in Figure 1 commutes up to homotopy (the maps are $c(x) = x$, the constant path, and $\alpha(\gamma) = \gamma(0)$, the initial point).*

$$\begin{tikzcd}
X \arrow[r, shift left, "c"] \arrow[dr, "\Delta_X"'] & \mathrm{P}X \arrow[l, shift left, "\alpha"] \arrow[d, "\pi"] \\
& X \times X
\end{tikzcd}$$

FIGURE 1

Corollary 2.7. *The maps π and Δ_X have the same homotopic Švarc genus, and both coincide with the topological complexity of X,*

$$\mathrm{hsecat}(\Delta_X) = \mathrm{hsecat}(\pi) = \mathrm{secat}(\pi) = \mathrm{TC}(X).$$

The following proposition gives two conditions which are equivalent to the existence of sections. The proof is an exercise.

Proposition 2.8. *Let $U \subset X \times X$ be a subspace. The following conditions are equivalent:*

(1) *there is a section $s_U : U \to \mathrm{P}X$ of the path fibration π,*
(2) *the restrictions to U of the projections $p_1, p_2 : X \times X \to X$ are homotopic maps,*
(3) *either $p_{1|U}$ or $p_{2|U}$ is a section (up to homotopy) of the diagonal map $\Delta_X : X \to X \times X$.*

2.2. **Simplicial complexes.** We refer the reader to Kozlov's book [12] for a modern survey of simplicial complexes and to Spanier's book [14] for the classical notions of simplicial map, simplicial approximation, and contiguity.

Recall that two maps $\varphi, \psi : K \to L$ are contiguous (denoted $\varphi \sim_c \psi$) if $\varphi(\sigma) \cup \psi(\sigma)$ is a simplex of L, for any simplex σ of K. Being in the same contiguity class, denoted by $\varphi \sim \psi$, means that there is a sequence of simplicial maps $h_i : K \to L$, with $i = 0, \ldots, m$, such that $h_0 = \varphi$, $h_m = \psi$, and the maps h_i and h_{i+1} are contiguous.

Let K be a finite abstract simplicial complex. Let $K^2 = K \Pi K$ be the *categorical product* as defined in [12, Definition 4.25]. The set of vertices is

$$V(K^2) = V(K) \times V(K),$$

and the simplices of K^2 are defined by the rule: $\sigma \in K^2$ if and only if $\pi_1(\sigma)$ and $\pi_2(\sigma)$ belong to K, where $\pi_1, \pi_2 : K^2 \to K$ are the projections.

Let $\varphi : K \to L$ be a simplicial map. We define the simplicial map

$$\varphi^2 = \varphi \,\Pi\, \varphi : K^2 \to L^2$$

by giving its value on each vertex, namely

$$\varphi^2(v, w) = (\varphi(v), \varphi(w)).$$

A very important property for our purposes is:

Proposition 2.9. *If $\varphi, \psi : K \to L$ are simplicial maps in the same contiguity class, then the maps $\varphi^2 \sim \psi^2$ are in the same contiguity class.*

Proof. We can assume without loss of generality that $\varphi \sim_c \psi$. Let

$$\sigma = \{(v_1, w_1), \dots, (v_n, w_n)\}$$

be a simplex in K^2. By definition, that means that $\pi_1(\sigma) = \{v_1, \dots, v_n\}$ and $\pi_2(\sigma) = \{w_1, \dots, w_n\}$ are simplices of K. Then

$$\varphi(\pi_1(\sigma)) \cup \psi(\pi_1(\sigma)) = \{\varphi(v_1), \dots, \varphi(v_n), \psi(v_1), \dots, \psi(v_n)\}$$

belongs to L. Analogously $\varphi(\pi_2(\sigma)) \cup \psi(\pi_2(\sigma))$ belongs to L. This is enough to prove that $\varphi^2(\sigma) \cup \psi^2(\sigma)$ belongs to L^2. □

Remark 2.10. There is another notion of simplicial product, the so-called *direct product* $K \times K$, where it is necessary to fix an order on $V(K)$. The difference with $K \Pi K$ is that the geometric realization $|K \times K|$ is homeomorphic to $|K| \times |K|$, while $|K \Pi K|$ has only the homotopy type of the latter, but is not homeomorphic to it. For instance, $|\Delta^1| \times |\Delta^1|$ is homeomorphic to $|\Delta^2|$ but $|\Delta^1 \Pi \Delta^1|$ is homeomorphic to $|\Delta^3|$. We use the categorical product because Proposition 2.9 would be true for the direct product only if the maps φ, ψ preserve the order.

Remark 2.11. Recently, González [10] introduced a combinatorial version $\mathrm{SC}(K)$ of the topological complexity which is based on a simplicial analog of part (2) of Proposition 2.8 and serves to compute $\mathrm{TC}(|K|)$. However, his notion is based on the direct product $K \times K$ and it seems not easy to compare it with our notion of simplicial complexity.

3. Discrete topological complexity

In Section 2.1 we have explained the reason of the following definitions, which avoid the need of a simplicial version $\mathrm{P}K$ of the path space.

3.1. **Farber subcomplexes.** Let $\Omega \subset K^2$ be a simplicial subcomplex of the categorical product $K^2 = K \Pi K$ and let $\iota_\Omega \colon \Omega \subset K^2$ be the inclusion map. Let $\Delta \colon K \to K^2$ be the diagonal map $\Delta(v) = (v, v)$.

Definition 3.1. We say that $\Omega \subset K^2$ is a *Farber subcomplex* if there exists a simplicial map $\sigma \colon \Omega \subset K^2 \to K$ such that $\Delta \circ \sigma \sim \iota_\Omega$.

The map σ will be called a *local homotopic section* of the diagonal, where "homotopic" must be understood in the sense of belonging to the same contiguity class.

Definition 3.2. The *discrete topological complexity* $\mathrm{TC}(K)$ of the simplicial complex K is the least integer $n \geq 0$ such that K^2 can be covered by $n+1$ Farber subcomplexes.

In other words, $\mathrm{TC}(K) \leq n$ if and only if $K^2 = \Omega_0 \cup \cdots \cup \Omega_n$, and there exist simplicial maps $\sigma_j \colon \Omega_j \to K$ such that $\Delta \circ \sigma_j \sim \iota_j$, where $\iota_j \colon \Omega_j \subset K^2$, for $j = 0, \dots, n$, are inclusions.

Sometimes we shall call $\mathrm{TC}(K)$ the *simplicial complexity* of K (not to be confused with the notion $\mathrm{SC}(K)$ defined by González in [10]). Notice that $\mathrm{TC}(K)$ is defined in purely combinatorial terms, involving neither the geometric realization $|K|$ of the complex, nor the notion of topological homotopy, nor that of simplicial approximation.

3.2. **Motion planning.** Farber's complexity is a topological invariant introduced to solve problems in robotics such as motion planning [7]. In this section we explain how our notion of discrete topological complexity is related to the motion planning problem on a simplicial complex.

Let $\Omega \subset K^2$ be a Farber simplicial subcomplex and let $\sigma\colon \Omega \to K$ be the associated section (up to contiguity class) of the diagonal, that is, such that $\Delta\circ\sigma \sim \iota_\Omega$. Then for each pair of vertices $x, y \in K$ such that $(x,y) \in \Omega$, the vertex $\sigma(x,y)$ is an *intermediate point* between x and y in the following sense: consider the sequence of contiguous maps

$$\Delta\circ\sigma = h_0 \sim_c \cdots \sim_c h_j \sim_c \cdots \sim_c h_m = \iota_\Omega.$$

Denote $h_j(x,y) = (x_j, y_j)$. Then $x_m = x$, $y_m = y$, and $x_0 = \sigma(x,y) = y_0$. That means that we have a sequence of vertices

$$x = x_m, \dots, x_0 = \sigma(x,y) = y_0, \dots, y_m = y. \tag{3.1}$$

Moreover, contiguity implies that two consecutive points in the above sequence belong to the same simplex: in fact, since $h_j \sim_c h_{j+1}$, the vertices $h_j(x,y) = (x_j, y_j)$ and $h_{j+1}(x,y) = (x_{j+1}, y_{j+1})$ generate a simplex of K^2 (that is, they are either equal or the vertices of an edge). By definition of the product K^2, this means that the vertices x_j and x_{j+1} (resp., y_j and y_{j+1}) generate a simplex of K. Hence the sequence (3.1) gives an edge-path on K connecting the vertices x and y.

3.3. **Invariance.** Let K be a finite simplicial complex. Recall from [3, 4] that a vertex $v \in K$ is *dominated* by another vertex $v' \neq v$ if every maximal simplex that contains v also contains v'. In this case we say that there is an *elementary strong collapse* from K to $K \setminus v$, where we denote by $K \setminus v$ the *deletion* of the vertex v, that is, the full subcomplex of K spanned by the vertices different from v. More generally, there is a *strong collapse* from a complex K to a subcomplex L if there exists a sequence of elementary strong collapses that starts in K and ends in L. The inverse of a strong collapse is a *strong expansion* and two finite complexes K and L are said to have the same *strong homotopy type*, denoted by $K \sim L$, if there is a sequence of strong collapses and strong expansions between them.

A beautiful result from Barmak and Minian [3, 4] states that two simplicial complexes K, L have the same strong homotopy type if and only if there exist simplicial maps $\varphi\colon K \to L$ and $\psi\colon L \to K$ such that $\varphi\circ\psi \sim 1_L$ and $\psi\circ\varphi \sim 1_K$ (recall that $\sim$ means "being in the same contiguity class").

Theorem 3.3. *The discrete topological complexity is an invariant of the strong homotopy type. That is, $K \sim L$ implies $\mathrm{TC}(K) = \mathrm{TC}(L)$.*

Proof. From Proposition 2.9 we have

$$\varphi^2\circ\psi^2 = (\varphi\circ\psi)^2 \sim (1_L)^2 = 1_{L^2}$$

and analogously $\psi^2\circ\varphi^2 \sim 1_{K^2}$, so we have $K^2 \sim L^2$. Moreover, the diagram in Figure 2 verifies $\Delta_L\circ\varphi = \varphi^2\circ\Delta_K$ and $\Delta_K\circ\psi = \psi^2\circ\Delta_L$.

Now let $\Omega \subset K^2$ be a Farber subcomplex of K^2, that is, there exists a simplicial map $\sigma\colon \Omega \to K$ such that $\Delta_K\circ\sigma \sim \iota_\Omega$. Then the inverse image

$$\Lambda = (\psi^2)^{-1}(\Omega) \subset L^2$$

is a Farber subcomplex of L^2, because the map (see Figure 2)

$$\lambda = \varphi\circ\sigma\circ{\psi^2}_{|\Lambda}\colon \Lambda \subset L^2 \to L$$

$$\begin{tikzcd}
 & K \arrow[r, shift left, "\varphi"] \arrow[d, "\Delta_K"'] & L \arrow[l, shift left, "\psi"] \arrow[d, "\Delta_L"] & \\
\Omega \arrow[r, hook] \arrow[ur, bend left, "\sigma"] & K^2 \arrow[r, shift left, "\varphi^2"] & L^2 \arrow[l, shift left, "\psi^2"] & \Lambda \arrow[l, hook'] \arrow[ul, dotted, bend right, "\lambda"']
\end{tikzcd}$$

FIGURE 2

verifies

$$\begin{aligned}
\Delta_L \circ \lambda =& \Delta_L \circ \varphi \circ \sigma \circ \psi^2 \circ \iota_\Lambda \\
=& \varphi^2 \circ \Delta_K \circ \sigma \circ \psi^2 \circ \iota_\Lambda \sim \varphi^2 \circ \iota_\Omega \circ \psi^2 \circ \iota_\Lambda \\
=& (\varphi^2 \circ \psi^2)_{|\Lambda} \sim 1_{L^2} \circ \iota_\Lambda \\
=& \iota_\Lambda.
\end{aligned}$$

Let $\mathrm{TC}(K) \leq n$, that is, there exists a covering $K = \Omega_0 \cup \cdots \cup \Omega_n$ where each Ω_j is a Farber subcomplex. Then the subcomplexes $\Lambda_j = (\psi^2)^{-1}(\Omega_j)$, for $j = 0, \ldots, n$, form a Faber covering of L^2, hence $\mathrm{TC}(L) \leq n$. The other inequality is proved in the same way. $\square$

We have the following characterization of Farber subcomplexes, which is the simplicial version of Proposition 2.8.

Theorem 3.4. *Let $\Omega \subset K^2$ be a subcomplex of the categorical product. The following conditions are equivalent:*

1. *Ω is a Farber subcomplex,*
2. *the restrictions to Ω of the projections are in the same contiguity class, that is, $(\pi_1)_{|\Omega} \sim (\pi_2)_{|\Omega}$,*
3. *either $(\pi_1)_{|\Omega}$ or $(\pi_2)_{|\Omega}$ is a section (up to contiguity) of the diagonal map $\Delta\colon K \to K^2$.*

Proof.
(1) $\Rightarrow$ (2) If $\Omega \subset K^2$ is a Farber subcomplex, then there exists $\sigma\colon \Omega \to K$ such that $\Delta \circ \sigma \sim \iota_\Omega$. But $\Delta \circ \sigma$ is the map (σ, σ) defined by $\omega \in \Omega \mapsto (\sigma(\omega), \sigma(\omega))$. On the other hand $\iota_\Omega = (\pi_1 \circ \iota_\Omega, \pi_2 \circ \iota_\Omega)$. Then

$$(\sigma, \sigma) \sim (\pi_1 \circ \iota_\Omega, \pi_2 \circ \iota_\Omega)$$

which implies, by composing with the projections, that

$$(\pi_1)_{|\Omega} = \pi_1 \circ \iota_\Omega \sim \sigma \sim \pi_2 \circ \iota_\Omega = (\pi_2)_{|\Omega}.$$

(2) $\Rightarrow$ (3) If $(\pi_1)_{|\Omega} \sim (\pi_2)_{|\Omega}$, define $\sigma\colon \Omega \to K$ by $\sigma = (\pi_1)_{|\Omega}$. Then $\iota_\Omega(x, y) = (x, y)$, for $(x, y) \in \Omega$, while $(\Delta \circ \sigma)(x, y) = (x, x)$. We have by hypothesis

$$\iota_\Omega = ((\pi_1)_{|\Omega}, (\pi_2)_{|\Omega}) \sim ((\pi_1)_{|\Omega}, (\pi_1)_{|\Omega}) = \Delta \circ \sigma.$$

(3) $\Rightarrow$ (1) If $\sigma = (\pi_i)_{|\Omega}$ verifies $\Delta \circ \sigma \sim \iota_\Omega$, then Ω is a Farber subcomplex, by definition. $\square$

4. Relationship with simplicial LS-category

One of Farber's most known results relates topological complexity to a well-known classical invariant, the Lusternik-Schnirelmann category of a space [5]. In this section we get analogous results for the discrete setting, by using the simplicial LS-category of a simplicial complex introduced by the authors in [8, 9].

Other related concepts like Tanaka's combinatorial strong category [15] or Aaronson-Scoville's category [1] will not be considered here.

4.1. Comparison with the category of K.

Definition 4.1. Let K be an abstract simplicial complex. A subcomplex $L \subset K$ is *categorical* if the inclusion $\iota_L \colon L \subset K$ belongs to the contiguity class of some constant map $L \to K$, that is, $\iota_L \sim *$. The (normalized) simplicial *LS-category* $\operatorname{scat} K$ of the simplicial complex K is the minimum number $m \geq 0$ such that there are categorical subcomplexes $L_0, \dots, L_m$ which cover K, that is, $K = L_0 \cup \cdots \cup L_m$.

Remark 4.2. As explained in [9], a categorical subcomplex may not be strongly collapsible in itself, but it must be in the ambient complex. Equivalently, it is the inclusion ι_L, and not the identity 1_L, which belongs to the contiguity class of a constant map.

The first inequality proved by Farber directly compares the topological complexity $\mathrm{TC}(X)$ of a space with the LS-category $\operatorname{cat} X$. We shall prove that this result also holds in the discrete setting.

Theorem 4.3. *For any abstract simplicial complex we have*

$$\operatorname{scat} K \leq \mathrm{TC}(K).$$

Proof. If $\mathrm{TC}(K) \leq n$, let $K^2 = \Omega_0 \cup \cdots \cup \Omega_n$ be a covering by Farber subcomplexes. Fix a base point $v_0 \in K$ and let $i_0 \colon K \to K^2$ be the simplicial map $i_0(w) = (v_0, w)$. Then, let us take the inverse images

$$\Sigma_j = (i_0)^{-1}(\Omega_j) \subset K, \quad j = 0, \dots, n.$$

Since $K = \Sigma_0 \cup \cdots \cup \Sigma_n$, if we prove that each Σ_j is a categorical subcomplex, then we can conclude that $\operatorname{scat} K \leq n$, and the result follows.

Let $\Omega \subset K^2$ be a Farber subcomplex, with a local section $\sigma \colon \Omega \to K$, and let

$$\Sigma = (i_0)^{-1}(\Omega) \subset K.$$

We shall prove that the inclusion $\iota_\Sigma \colon \Sigma \subset K$ belongs to the contiguity class of the constant map $v_0 \colon \Sigma \to K$, so we shall obtain that Σ is a categorical subcomplex of K.

Since $\Delta_K \circ \sigma \sim \iota_\Omega$, there is a sequence of simplicial maps $\psi_i \colon \Omega \to K^2$, $i = 1, \dots, m$, such that

$$\Delta_K \circ \sigma = \psi_1 \sim_c \cdots \sim_c \psi_m = \iota_\Omega. \tag{4.1}$$

Then, by composition,

$$\pi_1 \circ \psi_1 \circ i_0 \circ \iota_\Sigma \sim_c \cdots \sim_c \pi_1 \circ \psi_m \circ i_0 \circ \iota_\Sigma,$$

where, for every $w \in \Sigma$,

$$\pi_1 \circ \psi_1 \circ i_0 \circ \iota_\Sigma(w) = \pi_1 \circ \Delta_K \circ \sigma \circ i_0(w) = \sigma(v_0, w),$$

and

$$\pi_1 \circ \psi_m \circ i_0 \circ \iota_\Sigma(w) = \pi_1 \circ \iota_\Omega(v_0, w) = v_0.$$

On the other hand,

$$\pi_2 \circ \psi_1 \circ i_0 \circ \iota_\Sigma \sim_c \cdots \sim_c \pi_2 \circ \psi_m \circ i_0 \circ \iota_\Sigma, \tag{4.2}$$

where, for every $w \in \Sigma$,

$$\pi_2 \circ \psi_m \circ i_0 \circ \iota_\Sigma(w) = \pi_2 \circ \iota_\Omega(v_0, w) = w,$$

and

$$\pi_2 \circ \psi_1 \circ i_0 \circ \iota_\Sigma(w) = \pi_2 \circ \Delta_K \circ \sigma \circ i_0(w) = \sigma(v_0, w).$$

From (4.1) and (4.2) it follows

$$v_0 \sim \sigma(v_0, w) \sim w \quad \forall w \in \Sigma,$$

or equivalently, $v_0 \sim \iota_\Sigma$, hence Σ is a categorical subcomplex. □

4.2. **Comparison with the category of K^2.** The second comparison result by Farber in [6] is between $\mathrm{TC}(X)$ and $\mathrm{cat}(X \times X)$. We shall prove that it is also true in the discrete setting.

We start by a technical lemma whose proof is left to the reader.

Lemma 4.4. *The abstract simplicial complex K is edge-path connected if and only if two arbitrary constant maps $L \to K$ are in the same contiguity class.*

The following theorem uses the normalized versions of LS-category and topological complexity.

Theorem 4.5. *If K is an edge-path connected complex, then* $\mathrm{TC}(K) \leq \mathrm{scat}(K^2)$.

Proof. Let $\mathrm{scat}(K^2) = n$ and let $K^2 = \Omega_0 \cup \cdots \Omega_n$ be a categorical covering of K^2. If we are able to prove that each $\Omega = \Omega_j$, for $j = 0, \ldots, n$, is a Farber subcomplex, then we will have $\mathrm{TC}(K) \leq n$, thus proving the theorem.

By definition the inclusion $\iota_\Omega \colon \Omega \subset K^2$ verifies $\iota_\Omega \sim *$, where $* \colon \Omega \to K^2$ is some constant map (v_0, w_0). Since the complex is path-connected we can choose the point $*$ verifying $w_0 = v_0$.

By definition of contiguity class, since $\iota_\Omega \sim *$, there is a sequence of simplicial maps, each one contiguous to the next one,

$$\iota_\Omega = \varphi_1 \sim_c \cdots \sim_c \varphi_m = (v_0, v_0),$$

with $\varphi_j \colon \Omega \to K^2$. Let $\pi_1 \colon K^2 \to K$ be the projection onto the first factor; then each $\pi_1 \circ \varphi_j \colon \Omega \to K$ is contiguous to $\pi_1 \circ \varphi_{j+1}$. Hence

$$\pi_1 \circ \iota_\Omega \sim \pi_1 \circ \varphi_m = v_0. \tag{4.3}$$

Analogously, let $\pi_2 \colon K^2 \to K$ be the projection onto the second factor; then

$$\pi_2 \circ \iota_\Omega \sim \pi_2 \circ \varphi_m = v_0 \tag{4.4}$$

by means of the sequence $\pi_2 \circ \varphi_j$. Since $\pi_1 \circ \iota_\Omega \sim \pi_2 \circ \iota_\Omega$, it follows from Theorem 3.4 that $\Omega \subset K^2$ is a Farber subcomplex, so we conclude the proof. □

Remark 4.6. Notice that the proof of the theorem above reduces to prove that any categorical subcomplex of K^2 is a Farber subcomplex. In particular, this applies to strongly collapsible subcomplexes.

Corollary 4.7. *The abstract simplicial complex K is strongly collapsible if and only if* $\mathrm{TC}(K) = 0$.

Proof. By definition, K being strongly collapsible is equivalent to $\operatorname{scat} K = 0$. Moreover, as a direct consequence of Theorem 5.5 in [8], we obtained that $\operatorname{scat} K^2 + 1 \leq (\operatorname{scat} K + 1)^2$ and thus, the categorical product of strongly collapsible complexes is strongly collapsible. Then $\mathrm{TC}(K) = 0$. The converse is immediate from the inequality $\mathrm{TC}(K) \geq \operatorname{scat} K$. $\square$

Corollary 4.8. *The diagonal $\Delta\colon K \to K^2$ admits a* global *homotopic section* (*in the sense of contiguity class, that is, there exists $\sigma\colon K^2 \to K$ such that $\Delta_K \circ \sigma \sim 1_K$*) *if and only if the complex K is strongly collapsible.*

Example 4.9. Consider the complex $K = \partial\Delta^2$ given by the simplices

$$K = \{\emptyset, \{a\}, \{b\}, \{c\}, \{b, c\}, \{a, c\}, \{a, b\}\},$$

whose geometric realization (a triangulated circle) is represented in Figure 3. Let us prove that $\mathrm{TC}(K) = 2$.

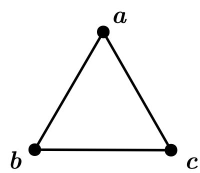

FIGURE 3

Figure 4 contains an explicit covering of K^2 by three Farber subcomplexes; then $TC(K) \leq 2$. In fact, they are strongly collapsible, because each one is the product of two strongly collapsible complexes (by repeating the argument used in the proof of Corollary 4.7). So Remark 4.6 applies and they are Farber subcomplexes.

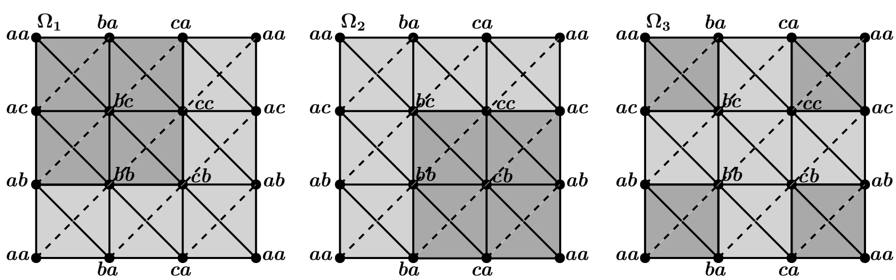

FIGURE 4

We shall prove now that two subcomplexes are not enough, so $\mathrm{TC}(K) = 2$. In fact, suppose that $K^2 = \Omega_1 \cup \Omega_2$ is a covering by two subcomplexes. Since K^2 has nine maximal simplices (see Figure 5) then one of the subcomplexes, say Ω_1, contains at least five of them.

Now there are nine horizontal edges, so two of the maximal simplices in Ω_1, say τ_1 and τ_2, must have one common horizontal edge. Finally, for each vertex $v_0 \in K$, let $i_0\colon K \to K$ be the map $i_0(v) = (v_0, v)$. From Proposition 2.8, the fact that Ω_1 is a Farber subcomplex implies that the subcomplex

$$(i_0)^{-1}(\Omega_1) = (\{v_0\} \times K) \cap \Omega_1 \subset K$$

is categorical in K, in particular it is not K (because K is not strongly collapsible). That means that Ω_1 cannot contain three consecutive vertical edges. Then none of the maximal simplices P, Q, R in Figure 5 can be contained in Ω_1. But Ω_2 is also a Farber subcomplex, so it cannot contain them as well, because by using the map $i_1(v) = (v, v_0)$ one proves that Ω_2 cannot contain three consecutive horizontal edges.

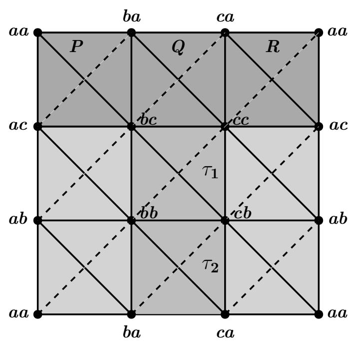

FIGURE 5

5. GEOMETRIC REALIZATION

Let $|K|$ be the geometric realization of the simplicial complex K. We can compute the usual topological complexity $\mathrm{TC}(|K|)$ of the topological space $|K|$ and compare it with the discrete (simplicial) complexity $\mathrm{TC}(K)$.

We need a preliminary result. It is known that $|K^2|$ is not always homeomorphic to the topological product $|K| \times |K|$, but they have the same homotopy type, as proved in Kozlov [12, Prop. 15.23]. The proof is based on the so-called "nerve theorem". However, an explicit formula is required, in order to guarantee the following lemma.

Lemma 5.1. *There exists a homotopy equivalence $u \colon |K| \times |K| \to |K^2|$ satisfying that the projections $p_1, p_2 \colon |K| \times |K| \to |K|$ and $\pi_1, \pi_2 \colon K^2 \to K$ verify (up to homotopy) that $|\pi_i| \circ u = p_i$, for $i = 1, 2$ (see Figure 6).*

$$\begin{tikzcd}
|K| \times |K| \arrow[r, shift left, "u"] \arrow[dr, "p_i"'] & |K^2| \arrow[l, shift left, "v"] \arrow[d, "|\pi_i|"] \\
& |K|
\end{tikzcd}$$

FIGURE 6

Proof. There is a homeomorphism $|K \times K| = |K| \times |K|$ which is induced by the projections [11, p. 538]. On the other hand, the homotopy equivalence $|K \times K| \simeq |K^2|$ is the geometric realization of the simplicial map $K \times K \to K^2$ induced by the natural inclusion map $\sigma_1 \times \sigma_2 \to \sigma_1 \amalg \sigma_2$, for each pair of simplices $\sigma_1, \sigma_2 \in K$ (see [12, Prop. 15.23] and [11, Prop. 4G.2]). $\square$

Theorem 5.2. $\mathrm{TC}(|K|) \leq \mathrm{TC}(K)$.

Proof. Let $\mathrm{TC}(K) \leq n$ and let $K^2 = \Omega_0 \cup \cdots \cup \Omega_n$ be a Farber covering. For each one of the Farber subcomplexes $\Omega = \Omega_j$ let $i_\Omega \subset K^2$ be the inclusion. By construction of the geometric realization we have that $|i_\Omega|$ is the inclusion $i_{|\Omega|} \colon |\Omega| \subset |K^2|$. By hypothesis, the maps $\pi_1 \circ i_\Omega$ and $\pi_2 \circ i_\Omega$ are in the same contiguity class (Theorem 3.4). By applying the functor $|\cdot|$ of geometric realization, and taking into account that contiguous maps induce homotopic continuous maps (see [14]), we have that $|\pi_1| \circ i_{|\Omega|} = |\pi_1 \circ i_\Omega|$ is homotopic to $|\pi_2| \circ i_{|\Omega|}$.

Consider the closed subspace $F = u^{-1}(|\Omega|) \subset |K| \times |K|$. Then the map

$$p_1 \circ i_F = |\pi_1| \circ u \circ i_F = |\pi_1| \circ i_{|\Omega|}$$

is homotopic to $p_2 \circ i_F$. Consider the closed covering $F_0 \cup \cdots \cup F_n$ of $|K| \times |K|$. This implies $\mathrm{TC}(|K|) \leq n$. $\square$

Remark 5.3. The inequality in the latter theorem is still true for all subdivisions of K, because the geometric realizations are homeomorphic, $|\operatorname{sd} K| \cong |K|$. However, it may happen that $\mathrm{TC}(K)$ differs from $\mathrm{TC}(\operatorname{sd} K)$, reflecting some particular property of the combinatorial structure.

Example 5.4. From Corollary 4.7, it follows that $\mathrm{TC}(K) = \mathrm{TC}(|K|) = 0$ if $|K|$ is a tree.

Example 5.5. Consider the wedge of two triangulated circles, that is, the complex $K = \partial\Delta^2 \vee \partial\Delta^2$ given by

$$K = \{\emptyset, \{a\}, \{b\}, \{c\}, \{b'\}, \{c'\}, \{b, c\}, \{a, c\}, \{a, b\}, \{b', c'\}, \{a, c'\}, \{a, b'\}\},$$

whose geometric realization is represented in Figure 7.

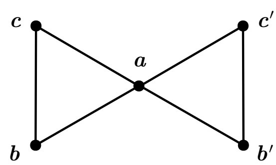

FIGURE 7

Since $|K|$ is a graph with more than one cycle, it follows that $\mathrm{TC}(|K|) = 2$. Then, by Theorem 5.2, we obtain $\mathrm{TC}(K) \geq 2$. We shall exhibit three Farber subcomplexes covering K^2 (see Figure 8), so $\mathrm{TC}(K) = 2 = \mathrm{TC}(|K|)$.

If we consider all the maximal simplices of K^2 in the same horizontal line, we obtain a complex L which is not strong collapsible and which contains four tetrahedrons with a common edge, say τ_1, τ_2, τ_3, and τ_4 (see Figures 9 and 10). Moreover, a strong collapsible subcomplex of L cannot contain more than two tetrahedrons τ_i, nor can two be in the same cycle.

Let Ω be a Farber subcomplex of K^2. For each vertex v_0 of K, let $i_0 \colon K \to K^2$ be the map $i_0(v) = (v_0, v)$. From Proposition 2.8, the subcomplex

$$(i_0)^{-1}(\Omega) = (\{v_0\} \times K) \cap \Omega \subset K$$

is categorical in K, in particular it is not K nor does it contain any cycle of K. Then $\Omega \cap L$ contains at most four tetrahedrons, say for instance all of the tetrahedrons in L excepting τ_2 and τ_4. Analogously, if we consider a subcomplex L' that contains all

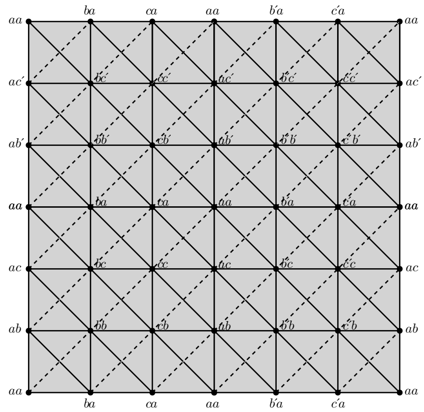

FIGURE 8

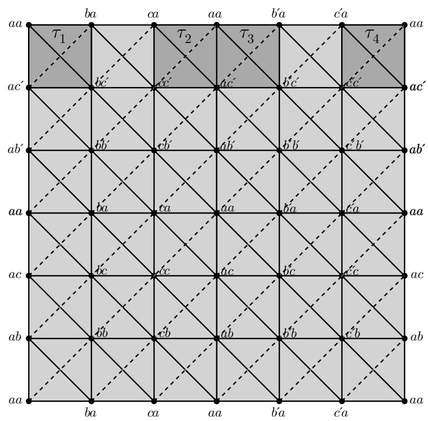

FIGURE 9

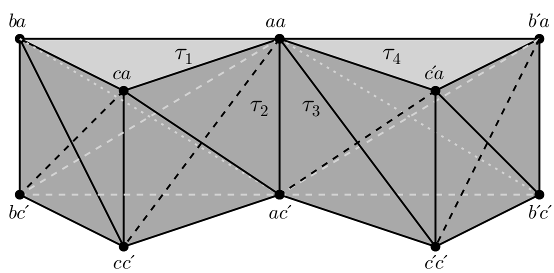

FIGURE 10

the maximal simplices of K^2 in the same vertical line, we obtain that $\Omega \cap L$ contains at most four tetrahedrons and no more than two have a common vertical edge. So we can get a covering of K^2 by three Farber subcomplexes, $K^2 = \Omega_1 \cup \Omega_2 \cup \Omega_3$, by considering the subcomplexes Ω_1 and Ω_2 of Figure 11 and Ω_3 the complement in K of $\Omega_1 \cup \Omega_2$.

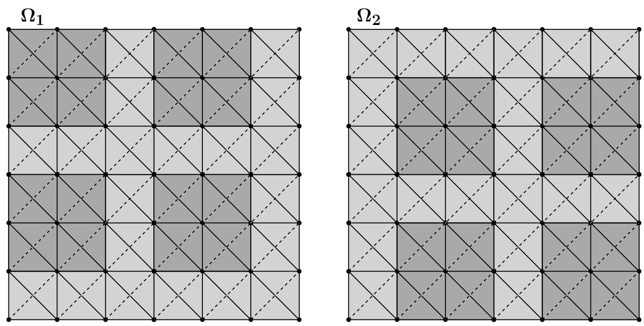

FIGURE 11

Finally, we have the following result about connected graphs if we follow similar reasonings to that of Examples 4.9 and 5.5. Being more precise, if we consider wedges of any finite number of circles, where each circle is triangulated with more than three edges, we can obtain a covering by three Farber subcomplexes constructing them as unions of tetrahedrons which satisfy that no more than two have a common vertical/horizontal edge.

Theorem 5.6. *The discrete topological complexity of the simplicial complex K, whose geometric realization $|K|$ is a wedge of any finite number of circles, satisfies:*

(1) $\mathrm{TC}(K) = 2 > \mathrm{TC}(|K|) = 1$ *if* $|K| = S^1$ *is a circle, and*
(2) $\mathrm{TC}(K) = \mathrm{TC}(|K|) = 2$ *if* $|K| = \vee^{b_1} S^1$*, when* $b_1 \geq 2$*, where* b_1 *is the first Betti number of* $|K|$*.*

ACKNOWLEDGMENTS

We thank Jesús González and Nick Scoville for useful conversations. Corollary 2.7 was pointed out to the second author by John Oprea and inspired our definition of discrete topological complexity.

REFERENCES

[1] Seth Aaronson and Nicholas A. Scoville, *Lusternik-Schnirelmann category for simplicial complexes*, Illinois J. Math. **57** (2013), no. 3, 743–753. MR3275736

[2] Dimitris Askitis, *A strong collapse increasing the geometric simplicial Lusternik-Schnirelmann category*, arXiv:1710.09794v1 (2017), 1–7.

[3] Jonathan A. Barmak, *Algebraic topology of finite topological spaces and applications*, Lecture Notes in Mathematics, vol. 2032, Springer, Heidelberg, 2011. MR3024764

[4] Jonathan Ariel Barmak and Elias Gabriel Minian, *Strong homotopy types, nerves and collapses*, Discrete Comput. Geom. **47** (2012), no. 2, 301–328. MR2872540

[5] Octav Cornea, Gregory Lupton, John Oprea, and Daniel Tanré, *Lusternik-Schnirelmann category*, Mathematical Surveys and Monographs, vol. 103, American Mathematical Society, Providence, RI, 2003. MR1990857

[6] Michael Farber, *Topological complexity of motion planning*, Discrete Comput. Geom. **29** (2003), no. 2, 211–221. MR1957228

[7] Michael Farber, *Invitation to topological robotics*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2008. MR2455573

[8] D. Fernández-Ternero, E. Macías-Virgós, E. Minuz, and J. A. Vilches, *Simplicial Lusternik-Schnirelmann category*, Publ. Mat. to appear, arXiv:1605.01322 (2016), 1–25.

[9] D. Fernández-Ternero, E. Macías-Virgós, and J. A. Vilches, *Lusternik-Schnirelmann category of simplicial complexes and finite spaces*, Topology Appl. **194** (2015), 37–50. MR3404603

[10] Jesús González, *Simplicial complexity: piecewise linear motion planning in robotics*, New York J. Math. **24** (2018), 279–292. MR3778506

[11] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR1867354

[12] Dmitry Kozlov, *Combinatorial algebraic topology*, Algorithms and Computation in Mathematics, vol. 21, Springer, Berlin, 2008. MR2361455

[13] Nicholas A. Scoville and Willie Swei, *On the Lusternik-Schnirelmann category of a simplicial map*, Topology Appl. **216** (2017), 116–128. MR3584127

[14] Edwin H. Spanier, *Algebraic topology*, McGraw-Hill Book Co., New York-Toronto, Ont.-London, 1966. MR0210112

[15] Kohei Tanaka, *Lusternik-Schnirelmann category for cell complexes and posets*, Illinois J. Math. **59** (2015), no. 3, 623–636. MR3554225

[16] A. S. Švarc, *The genus of a fiber space*, Amer. Math. Soc. Transl. Ser.2 **55** (1966), 49–140.

Departamento de Geometría y Topología, Universidad de Sevilla, Spain
Email address: `desamfer@us.es`

Departamento de Matemáticas, Universidade de Santiago de Compostela, Spain
Email address: `quique.macias@usc.es`

Department of Mathematics, Aarhus University, Denmark
Email address: `minuz@math.au.dk`

Departamento de Geometría y Topología, Universidad de Sevilla, Spain
Email address: `vilches@us.es`