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PERIODIC ORBIT ANALYSIS FOR THE DELAYED FILIPPOV SYSTEM

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ABSTRACT. In this paper, a general class of the delayed differential equation with a discontinuous right-hand side is considered. Under the extended Filippov differential inclusions framework, some new criteria are obtained to guarantee the existence of a periodic solution by employing Kakutani's fixed point theorem of set-valued maps and matrix theory. Then, we apply these criteria to the time-delayed neural networks with discontinuous neuron activations. Our analysis method and theoretical results are of great significance in the design of time-delayed neural network circuits with discontinuous neuron activation under a periodic environment.

1. INTRODUCTION

In practice, discontinuities arise naturally and are often caused by control actions of many interesting engineering tasks. For instance, the discontinuous feedback controllers are used to realize the stabilization or synchronization, neural network circuits are implemented by memristor possessing a discontinuous switching property, thermostats implement on-off or discontinuous controllers to regulate room temperature, etc. [7, 10, 14, 25, 29]. On the other hand, time-delays are inevitable because of the finite processing time of signals and the energy propagating with a finite speed [13]. Actually, many practical dynamical systems exhibit time-delay phenomenon and so possess memory feature. That is, the future state of the system not only depends upon the current state but also upon the past state. Generally speaking, these practical dynamical systems are usually described by the time-delayed differential equations with discontinuous right-hand sides when both the time-delays and discontinuities exist in such dynamical systems. However, the traditional time-delayed differential equation theory is invalid to deal with the solutions of discontinuous dynamical systems with time-delays. That is because the existence of a continuously differentiable solution is not guaranteed in a discontinuous vector field. Fortunately, in 1964, Filippov developed the theory of differential inclusion to handle an ordinary differential equation whose right-hand side was only required to be Lebesgue measurable in the time and state variables [9]. By constructing the Filippov set-valued map (i.e., Filippov-regularization method), the solution of

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ordinary differential equations could be transformed into a solution of differential inclusion. After that, the time-delayed differential inclusion in the sense of Filippov (that is, time-delayed Filippov system) was further developed [2, 3, 11, 12, 15, 23]. In 1981, a systematic introduction for the solution sets of time-delayed differential inclusion was given by Haddad [12]. In 1984, Aubin and Cellina presented the properties of the set of solution trajectories for time-delayed differential inclusion [2]. In [3] and [23], Benchohra and Lupulescu investigated the existence of the solutions for convex and nonconvex time-delayed differential inclusion, respectively. In [15], Hong discussed the existence of functional differential inclusion with infinite delay. However, most of the previous results on time-delayed differential inclusion were mainly concerned with the fundamental questions of solutions (e.g., the local and global existence of the Filippov solution). To the best of our knowledge, the results on periodic orbit analysis for the time-delayed Filippov system are still few.

In many real applications, the study of periodic orbit is of great significance. For example, in the field of neural networks, the periodic orbit analysis is an important step for understanding the function of the human brain and further enables us to simulate the human brain under periodic environment. Up to now, much attention has been paid to analyzing periodic orbit problems of differential inclusions [5, 6, 8, 17–20, 26, 28, 30]. The authors of [17] obtained the existence of periodic solutions for nonconvex differential inclusions based on degree theory arguments. Also, the existence of periodic solutions for nonconvex differential inclusions was proved by using a continuous selection theorem in [26] and [18]. By constructing the topological degree for the Poincaré maps, the authors of [5] studied the periodic solution problem of differential inclusion under a guiding potential condition. In [8], some fixed point theorems of discontinuous set-valued operators were improved and further applied to solve the periodic boundary value problem for differential inclusion. In [20], by generalizing Halanay's criterion, Yoshizawa's theorem, Krasnosel'skii's theorem and Mawhin's coincidence degree theorem, the existence problem of periodic solutions for differential inclusion was investigated. In [6, 19, 30], the periodic problem of differential inclusions was dealt with by using the Leray-Schauder alternative principle. In [28], the existence of periodic solutions for nonlinear differential inclusions with multi-valued perturbations was investigated based on Schauder's fixed point theorem and Kakutani's fixed point theorem. However, all these papers mentioned above did not consider the case of time-delay. For this reason, we consider the existence of a periodic solution for time-delayed differential inclusion in the sense of Filippov. In this paper, our approach is different from those of [1–3, 10, 21, 26] and our conclusions hold under more general conditions.

Notation. Let $\mathbb{R}$ be the set of real numbers and let $\mathbb{R}^n$ denote the n-dimensional Euclidean space. For $x \in \mathbb{R}^n$, $\|x\|$ represents any vector norm of x. For the matrix $Q = (q_{ij})_{n\times n}$, Q^{-1} represents the inverse of Q, and E_n stands for the identity matrix of size n. $\mathbf{0}$ denotes a zero matrix or zero vector. A vector or matrix $U \geq \mathbf{0}$ means that all entries of U are greater than or equal to zero, and $U > \mathbf{0}$ can be defined similarly. For given vectors or matrices U and V, $U \geq V$ (or $U > V$) means that $U - V \geq \mathbf{0}$ (or $U - V > \mathbf{0}$). Let $L^1([0,\xi),\mathbb{R}^n), \xi \leq +\infty$ denote the Banach space of the Lebesgue integrable functions $g : [0,\xi) :\to \mathbb{R}^n$ equipped with the norm

$\int_0^\xi \|g(t)\|dt$. For any continuous ω-periodic function $h(t)$ defined on $\mathbb{R}$, we denote

$$\overline{h} = \frac{1}{\omega}\int_0^\omega h(t)\mathrm{d}t,\ \widehat{h} = \frac{1}{\omega}\int_0^\omega |h(t)|\mathrm{d}t,\ h^M = \sup_{t\in[0,\omega]} |h(t)|,\ h^L = \inf_{t\in[0,\omega]} |h(t)|.$$

2. Preliminaries

In this section, we give some useful definitions and lemmas on set-valued analysis, delayed differential inclusions and matrix theory, which will be needed in the development. For more details, the readers may consult [1,2,4,7,9,16,21,22,24,27,31].

Let $\mathbb{R}^n (n \geq 1)$ have inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. For given $X \subseteq \mathbb{R}^n$, we introduce some useful notation as follows:

$$\mathscr{P}_0(X) = \{\mathbb{A} \subset X : \mathbb{A} \neq \emptyset\},\ \mathscr{P}(X) = \mathscr{P}_0(X) \cup \{\emptyset\},$$
$$\mathscr{P}_{kc}(X) = \{\mathbb{A} \subset X :\ \text{nonempty compact and convex}\}.$$

For convenience, we sometimes denote $2^X = \mathscr{P}_0(X)$. For $X \subseteq \mathbb{R}^n$, we say the map $x \mapsto F(x)$ is a set-valued map from $X \hookrightarrow \mathscr{P}(\mathbb{R}^n)$ if to each point x of the set X there corresponds a nonempty set $F(x) \subset \mathbb{R}^n$. We say a set-valued map F with nonempty values is upper semi-continuous (USC) at $x_0 \in X$ if for any open set $\mathbb{N}$ containing $F(x_0)$ there exists a neighborhood $\mathbb{M}$ of x_0 such that $F(\mathbb{M}) \subset \mathbb{N}$.

Now we introduce the concept of the Filippov solution by constructing the Filippov set-valued map [9]. Let $\tau > 0$ denote a given real number and let $C_\tau = C([-\tau,0],\mathbb{R}^n)$ represent the Banach space of continuous functions $\phi = (\phi_1,\phi_2,\ldots,\phi_n)^{\mathrm{T}}$ mapping the interval $[-\tau,0]$ into $\mathbb{R}^n$ with the supremum norm $\|\phi\|_{C_\tau} = \sum_{i=1}^n |\phi_i|_0$, where $|\phi_i|_0 = \sup_{-\tau\leq s\leq 0} |\phi_i(s)|$. If for $\xi \in (0,+\infty]$, $x(t) : [-\tau,\xi) \to \mathbb{R}^n$ is continuous, then $x_t \in C_\tau$ is defined by $x_t(\theta) = x(t+\theta)$, $-\tau \leq \theta \leq 0$ for any $t \in [0,\xi)$. Consider the following nonautonomous delayed differential equation:

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -b_i(t)x_i(t) + f_i(t,x_t), \tag{2.1}$$

where t denotes time; $x_i(t)$ is the state variable; $x_t(\cdot)$ represents the history of the state from time $t-\tau$, up to the present time t; $\mathrm{d}x_i(t)/\mathrm{d}t$ is the time derivative of $x_i(t)$; $b_i : \mathbb{R} \to \mathbb{R}$ is continuous; $f_i : \mathbb{R} \times C_\tau \to \mathbb{R}$ is measurable and essentially locally bounded. In this case, $f_i(t,x_t)$ is allowed to be discontinuous. In addition, for fixed $\omega > 0$, $b_i(t) = b_i(t+\omega)$ and $f_i(t,\phi) = f_i(t+\omega,\phi)$ for $\phi \in C_\tau$.

The equation (2.1) can be transformed into the following vector form:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -B(t)x(t) + f(t,x_t), \tag{2.2}$$

where $x(t) = (x_1(t),x_2(t),\ldots,x_n(t))^{\mathrm{T}}$, $f(t,x_t) = (f_1(t,x_t),f_2(t,x_t),\ldots,f_n(t,x_t))^{\mathrm{T}}$, and the matrix $B(t) = \mathrm{diag}(b_1(t),b_1(t),\ldots,b_n(t))$.

Construct the Filippov set-valued map $F = (F_1,F_2,\ldots,F_n)^{\mathrm{T}} : \mathbb{R} \times C_\tau \to 2^{\mathbb{R}^n}$:

$$F(t,x_t) = \bigcap_{\varrho>0}\ \bigcap_{\mathrm{meas}(\mathscr{N})=0} \overline{\mathrm{co}}[f(t,\mathcal{B}(x_t,\varrho)\setminus\mathscr{N})]. \tag{2.3}$$

Here $\mathrm{meas}(\mathscr{N})$ stands for the Lebesgue measure of set $\mathscr{N}$; intersection is taken over all sets $\mathscr{N}$ of Lebesgue measure zero and over all $\varrho > 0$; $\mathcal{B}(x_t,\varrho) := \{x_t' \in C_\tau \mid \|x_t' - x_t\|_{C_\tau} < \varrho\}$; $\overline{\mathrm{co}}[\mathbb{E}]$ denotes the closure of the convex hull of some set $\mathbb{E}$.

Definition 2.1. We say the function $x(t)$ defined on a nondegenerate interval $\mathbb{I} \subseteq \mathbb{R}$ is a Filippov solution for delayed differential equation (2.1) or (2.2) if it is absolutely continuous on any compact subinterval $[t_1, t_2]$ of $\mathbb{I}$, and for a.e. $t \in \mathbb{I}$, $x(t)$ satisfies the following delayed differential inclusion:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} \in -B(t)x(t) + F(t, x_t). \tag{2.4}$$

The initial condition associated with system (2.1) or (2.2) is given as

$$x_i(s) = \phi_i(s),\ s \in [-\tau, 0],\ i = 1, 2, \ldots, n,$$

where $\phi(s) = (\phi_1(s), \phi_2(s), \ldots, \phi_n(s))^{\mathrm{T}} \in C([-\tau, 0], \mathbb{R}^n)$. Because $f(t, x_t)$ is essentially locally bounded, it is easy to check that the set-valued function $F : \mathbb{R} \times C_\tau \to 2^{\mathbb{R}^n}$ is USC with nonempty, compact, convex values and locally bounded.

Lemma 2.2 (Kakutani's fixed point theorem [2])**.** *If Ω is a compact convex subset of a Banach space X and the set-valued map $\varphi : \Omega \to \mathscr{P}_{kc}(\Omega)$ is an upper semicontinuous (USC) convex compact map, then φ has a fixed point in Ω, that is to say, there exists $x \in \Omega$ such that $x \in \varphi(x)$.*

For convenience, set $\mathbb{I}^\omega = [0, \omega]$. Let $F(t, x_t) = (F_1(t, x_t), F_2(t, x_t), \ldots, F_n(t, x_t))^{\mathrm{T}}$ be a set-valued function and let $L^1(\mathbb{I}^\omega, \mathbb{R}^n)$ represent the Banach space of all functions $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)^{\mathrm{T}} : \mathbb{I}^\omega \to \mathbb{R}^n$ which are Lebesgue integrable. Let us define a set-valued operator

$$\mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_n)^{\mathrm{T}} : X \to L^1(\mathbb{I}^\omega, \mathbb{R}^n) \tag{2.5}$$

by letting

$$\mathscr{F}_i(x) = \left\{\gamma_i \in L^1(\mathbb{I}^\omega, \mathbb{R}) : \gamma_i(t) \in F_i(t, x_t) \text{ for a.e. } t \in \mathbb{I}^\omega\right\},\ i = 1, 2, \ldots, n.$$

Definition 2.3 (See [16, 27])**.** We say a set-valued function $F : \mathbb{I}^\omega \times X \to 2^X$ is L^1-Carathéodory if

(i) $t \to F(t, z)$ is measurable with respect to t for each $z \in X$;
(ii) $t \to F(t, z)$ is USC with respect to z for a.e. $t \in \mathbb{I}^\omega$;
(iii) for every real number $\shortparallel > 0$ there exists a function $\hbar_{\shortparallel} \in L^1(\mathbb{I}^\omega, \mathbb{R})$ such that $\left|\left|\left|F(t, z)\right|\right|\right| \le \hbar_{\shortparallel}(t)$ for a.e. $t \in \mathbb{I}^\omega$ and any $z \in X$ with $\|z\| \le \shortparallel$, where $\left|\left|\left|F(t, z)\right|\right|\right| = \sup\{\|\gamma\| : \gamma \in F(t, z)\}$.

If only the assumptions (i) and (ii) hold, then F is said to be Carathéodory.

Lemma 2.4 (See [22])**.** *Suppose that* $\operatorname{diam}(X) < \infty$ *and $F : \mathbb{I}^\omega \times X \to 2^X$ is L^1-Carathéodory; then the set $\mathscr{F}(x)$ is nonempty for every fixed $x \in X$.*

Lemma 2.5 (See [16, 24, 27])**.** *For given compact real interval $\mathbb{I}^\omega = [0, \omega]$, if F is a Carathéodory set-valued map with $\mathscr{F}(x) \neq \emptyset$ for every fixed $x \in X$ and $\mathcal{L} : L^1(\mathbb{I}^\omega, \mathbb{R}^n) \to C(\mathbb{I}^\omega)$ is a continuous linear mapping, then the operator $\mathcal{L} \circ \mathscr{F} : C(\mathbb{I}^\omega) \to 2^{C(\mathbb{I}^\omega)}$ is a closed graph operator in $C(\mathbb{I}^\omega) \times C(\mathbb{I}^\omega)$.*

Definition 2.6. We say a real invertible $n \times n$ matrix $Q = (q_{ij})_{n \times n}$ is an M-matrix, if $q_{ij} \le 0$ for all $i, j = 1, 2, \ldots, n, i \neq j$ and $Q^{-1} \ge \mathbf{0}$.

Lemma 2.7 (See [21])**.** *Let $Q = (q_{ij})_{n \times n}$ be an $n \times n$ matrix with nonpositive off-diagonal elements, then Q is a nonsingular M-matrix if and only if one of the following statements holds:*

(i) *all of the principal minors of Q are positive;*

(ii) *there exists a* $\eta = (\eta_1, \eta_2, \ldots, \eta_n)^{\mathrm{T}} > (0, 0, \ldots, 0)^{\mathrm{T}}$ *such that* $Q\eta > \mathbf{0}$*;*
(iii) *all diagonal elements of* Q *are positive and there exists a diagonal matrix* $\urcorner = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ *with* $\lambda_i > 0 (i = 1, 2, \ldots, n)$ *such that* $Q\urcorner$ *is strictly diagonally dominant, that is to say,* $q_{ii}\lambda_i > \sum_{j \neq i} |q_{ij}|\lambda_j,\ i = 1, 2, \ldots, n.$

Lemma 2.8 (See [4])**.** *Let* $Q = (q_{ij})_{n\times n}$ *be an* $n \times n$ *matrix with* $Q \geq \mathbf{0}$ *and let* $\rho(Q)$ *be the spectral radius of* Q*. If* $\rho(Q) < 1$*, then* $E_n - Q$ *is an M-matrix.*

Lemma 2.9 (See [4])**.** *Let* $U = (u_{ij})_{n\times n}$ *and* $V = (v_{ij})_{n\times n}$ *be nonnegative matrixes with* $U \geq V$*; then the spectral radius satisfies* $\rho(U) \geq \rho(V)$*.*

3. Main results

In this section, by using extended Filippov framework, Kakutanni's fixed point theorem of set-valued maps and matrix theory, we prove the main results on the existence of periodic orbits for the delayed differential inclusion (2.4). First, let

$$C_\omega = \{x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^{\mathrm{T}} \in C(\mathbb{R}, \mathbb{R}^n) : x(t + \omega) = x(t),\ \forall t \in \mathbb{R}\}.$$

Define the supremum norm

$$\|x\|_{C_\omega} = \sum_{i=1}^{n} |x_i|_0,\ |x_i|_0 = \sup_{t\in[0,\omega]} |x_i(t)|,\ i = 1, 2, \ldots, n.$$

Then C_ω is a Banach space endowed with the above norm $\|\cdot\|_{C_\omega}$. Based on Definition 2.1, if $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^{\mathrm{T}} \in C_\omega$ is a Filippov solution of delayed system (2.1) or (2.2), then we can get from (2.4) that

$$\text{(3.1)}\qquad \frac{\mathrm{d}}{\mathrm{d}t}\left[x_i(t)\exp\left\{\int_0^t b_i(s)\mathrm{d}s\right\}\right] \in \exp\left\{\int_0^t b_i(s)\mathrm{d}s\right\} F_i(t, x_t),\ \text{for a.e. } t \geq 0,$$

where $i = 1, 2, \ldots, n$, $F_i(t, x_t)$ is the ith component of Filippov set-valued map $F(t, x_t)$ given by (2.3).

By integrating both sides of delayed differential inclusion (3.1) over the interval $[t, t + \omega]$, we can obtain the following nonlinear integral inclusions:

$$\text{(3.2)}\qquad x_i(t) \in \int_t^{t+\omega} G_i(t, s)F_i(s, x_s)\mathrm{d}s,\ \text{for } t \geq 0,\ i = 1, 2, \ldots, n,$$

where $G_i(t, s)$ denotes Green's function and it is described by

$$\text{(3.3)}\qquad G_i(t, s) = \frac{1}{1 - \exp\{-\omega\bar{b}_i\}}\exp\left\{-\int_s^{t+\omega} b_i(\sigma)\mathrm{d}\sigma\right\},\ \text{for } t \leq s \leq t + \omega.$$

Clearly, the denominator of Green's function $G_i(t, s)$ is not zero, and $G_i(t, s) = G_i(t + \omega, s + \omega)$ for all $(t, s) \in \mathbb{R}^2$. If $x(t)$ is an ω-periodic Filippov solution of system (2.1) or (2.2), then it is easy to find that each ω-periodic Filippov solution of system (2.1) or (2.2) is an ω-periodic solution of integral inclusions (3.2) and the converse is also true. Hence, the existence of an ω-periodic solution for system (2.1) or (2.2) in the sense of Filippov is equivalent to the existence of an ω-periodic solution for integral inclusions (3.2). For $t \leq s \leq t + \omega$ and $i = 1, 2, \ldots, n$, we can obtain from (3.3) that

$$\text{(3.4)}\qquad G_i(t, s) \leq |G_i(t, s)| \leq \frac{\exp\{\omega\widehat{b_i}\}}{|1 - \exp\{-\omega\bar{b}_i\}|} \triangleq G_i^{\max},$$

where $\overline{b}_i = \frac{1}{\omega}\int_0^\omega b_i(t)\mathrm{d}t,\ \widehat{b}_i = \frac{1}{\omega}\int_0^\omega |b_i(t)|\mathrm{d}t$. For convenience, let us set $\Psi(u,v)$ by

$$\Psi(u,v) = \frac{1}{1-\exp\{-\omega\overline{v}\}} \sup_{t\in[0,\omega]} \int_0^\omega u(s+t)\cdot \exp\left\{-\int_s^\omega v(\sigma+t)\mathrm{d}\sigma\right\}\mathrm{d}s,$$

where $u(t)$ and $v(t)$ are any ω-periodic functions on $\mathbb{R}$.

Theorem 3.1. *Suppose that the following conditions are satisfied:*

($\mathscr{H}1$) *For each $i = 1,2,\ldots,n$, $\overline{b}_i = \frac{1}{\omega}\int_0^\omega b_i(s)\mathrm{d}s > 0$.*

($\mathscr{H}2$) *There exist constants $\mathcal{R}_j > 0$ and nonnegative continuous ω-periodic functions $\alpha_{ij}(t)$ and $\beta_i(t) (i,j = 1,2,\ldots,n)$ such that*

$$\sup_{\gamma_i\in F_i(t,\phi)} |\gamma_i| \le \sum_{j=1}^n \alpha_{ij}(t)\mathcal{R}_j + \beta_i(t) \text{ for } \phi\in C_\tau \text{ with } |\phi_j|_0 \le \mathcal{R}_j,\ j=1,2,\ldots,n,$$

and $(E_n-\Theta)(\mathcal{R}_1,\mathcal{R}_2,\ldots,\mathcal{R}_n)^{\mathrm{T}} > (\breve{\beta}_1,\breve{\beta}_2,\ldots,\breve{\beta}_n)^{\mathrm{T}}$; here $\Theta = (\Psi(\alpha_{ij},b_i))_{n\times n}$ and $\breve{\beta}_i = \Psi(\beta_i,b_i)$ for $i=1,2,\ldots,n$.

Then the delayed system (2.1) *or* (2.2) *has at least one ω-periodic solution.*

Proof. Let us choose a compact convex subset $\Omega\subset C_\omega$ defined by

$$\Omega = \left\{x(t) = (x_1(t),x_2(t),\ldots,x_n(t))^{\mathrm{T}} \in C_\omega : |x_i|_0 \le \mathcal{R}_i,\ i=1,2,\ldots,n\right\}. \tag{3.5}$$

For any $x\in C_\omega$, let us define a set-valued map $\varphi : C_\omega \to \mathscr{P}_{kc}(C_\omega)$ given by

$$\varphi(x)(t) = (\varphi_1(x)(t),\varphi_2(x)(t),\ldots,\varphi_n(x)(t))^{\mathrm{T}}, \tag{3.6}$$

where

$$\varphi_i(x)(t) = \int_t^{t+\omega} G_i(t,s)F_i(s,x_s)\mathrm{d}s,\ i=1,2,\ldots,n. \tag{3.7}$$

Recalling the formulas (3.2), (3.6) and (3.7), it is not difficult to verify that $x^*(t) = (x_1^*(t),x_2^*(t),\ldots,x_n^*(t))^{\mathrm{T}} \in C_\omega$ is an ω-periodic solution of delayed system (2.1) or (2.2) provided that $x^*(t)$ is a fixed point of the set-valued map φ in Ω. Actually, if $x^*(t)\in C_\omega$ is a fixed point of the set-valued map φ in Ω, then $x^*(t)\in\varphi(x^*)(t) = (\varphi_1(x^*)(t),\varphi_2(x^*)(t),\ldots,\varphi_n(x^*)(t))^{\mathrm{T}}$, where $\varphi_i(x^*)(t) = \int_t^{t+\omega} G_i(t,s)F_i(s,x_s^*)\mathrm{d}s$, $i=1,2,\ldots,n$. That is, $x^*(t)$ is a solution of integral inclusion (3.2). It is noted that $x^*(t)\in C_\omega = \{x(t) = (x_1(t),x_2(t),\ldots,x_n(t))^{\mathrm{T}} \in C(\mathbb{R},\mathbb{R}^n) : x(t+\omega) = x(t),\ \forall t\in\mathbb{R}\}$. This means that $x^*(t+\omega) = x^*(t)$, i.e., $x^*(t)$ is an ω-periodic function. Thus, $x^*(t)$ is an ω-periodic solution of integral inclusion (3.2), and so $x^*(t)$ is an ω-periodic solution of system (2.1) or (2.2). □

In the following, we will solve the fixed point problem by using Kakutanni's fixed point theorem (see Lemma 2.2). The discussion will be divided into four steps.

Step 1. Let us prove that the set-valued map φ maps Ω into $\mathscr{P}_{kc}(\Omega)$, i.e., $\varphi(x)\in\mathscr{P}_{kc}(\Omega)$ for each fixed $x\in\Omega$.

To see this, let $x = (x_1,x_2,\ldots,x_n)^{\mathrm{T}}\in\Omega$ and $\zeta = (\zeta_1,\zeta_2,\ldots,\zeta_n)^{\mathrm{T}}\in\varphi(x)$. Then there exists measurable function $\gamma = (\gamma_1,\gamma_2,\ldots,\gamma_n)^{\mathrm{T}} : [-\tau,\xi)\to\mathbb{R}^n$ such that $\gamma_i(t)\in F_i(t,x_t)(i=1,2,\ldots,n)$ for a.e. $t\in[-\tau,\xi)$ and

$$\zeta_i(t) = \int_t^{t+\omega} G_i(t,s)\gamma_i(s)\mathrm{d}s \in \int_t^{t+\omega} G_i(t,s)F_i(s,x_s)\mathrm{d}s = \varphi_i(x)(t). \tag{3.8}$$

Notice that

$$\begin{aligned}\zeta_i(t) &= \int_t^{t+\omega} G_i(t,s)\gamma_i(s)\mathrm{d}s \\ &= \frac{1}{1-\exp\{-\omega\bar{b}_i\}}\int_t^{t+\omega}\gamma_i(s)\exp\left\{-\int_s^{t+\omega} b_i(\sigma)\mathrm{d}\sigma\right\}\mathrm{d}s \\ &= \frac{1}{1-\exp\{-\omega\bar{b}_i\}}\int_0^{\omega}\gamma_i(s+t)\exp\left\{-\int_s^{\omega} b_i(\sigma+t)\mathrm{d}\sigma\right\}\mathrm{d}s. \end{aligned}\tag{3.9}$$

Therefore, for any $t \in [0,\omega]$, $x \in C_\omega$ with $|x_i|_0 \le \mathcal{R}_i$, $i = 1,2,\ldots,n$, we can obtain from the condition ($\mathscr{H}2$) and (3.9) that

$$\begin{aligned}|\zeta_i(t)| &\le \frac{1}{1-\exp\{-\omega\bar{b}_i\}}\int_0^{\omega}|\gamma_i(s+t)|\exp\left\{-\int_s^{\omega} b_i(\sigma+t)\mathrm{d}\sigma\right\}\mathrm{d}s \\ &\le \frac{1}{1-\exp\{-\omega\bar{b}_i\}}\int_0^{\omega}(\sum_{j=1}^n \alpha_{ij}(s+t)\mathcal{R}_j+\beta_i(s+t))\exp\{-\int_s^{\omega} b_i(\sigma+t)\mathrm{d}\sigma\}\mathrm{d}s \\ &\le \sum_{j=1}^n \mathcal{R}_j\frac{1}{1-\exp\{-\omega\bar{b}_i\}}\sup_{t\in[0,\omega]}\int_0^{\omega}\alpha_{ij}(s+t)\exp\left\{-\int_s^{\omega} b_i(\sigma+t)\mathrm{d}\sigma\right\}\mathrm{d}s \\ &\quad + \frac{1}{1-\exp\{-\omega\bar{b}_i\}}\sup_{t\in[0,\omega]}\int_0^{\omega}\beta_i(s+t)\exp\left\{-\int_s^{\omega} b_i(\sigma+t)\mathrm{d}\sigma\right\}\mathrm{d}s \\ &= \sum_{j=1}^n \Psi(\alpha_{ij},b_i)\mathcal{R}_j + \breve{\beta}_i < \mathcal{R}_i,\ i = 1,2,\ldots,n. \end{aligned}\tag{3.10}$$

Hence, for any $x \in \Omega$ and $\zeta \in \varphi(x)$, we have $\zeta \in \Omega$. Consequently, $\varphi(x) \in \mathscr{P}_{kc}(\Omega)$ for each fixed $x \in \Omega$, that is to say, $\varphi : \Omega \to \mathscr{P}_{kc}(\Omega)$.

Step 2. We will prove that the set-valued map $\varphi(x)$ is convex for each $x \in \Omega$.

In fact, for any $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \Omega$, let $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n)^{\mathrm{T}} \in \varphi(x)$ and $\zeta^* = (\zeta_1^*, \zeta_2^*, \ldots, \zeta_n^*)^{\mathrm{T}} \in \varphi(x)$. Then there exists measurable function $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)^{\mathrm{T}} : [-\tau, \xi) \to \mathbb{R}^n$ such that $\gamma_i(t) \in F_i(t, x_t)(i = 1, 2, \ldots, n)$ for a.e. $t \in [-\tau, \xi)$ and (3.8) holds. Meanwhile, there also exists measurable function $\gamma^* = (\gamma_1^*, \gamma_2^*, \ldots, \gamma_n^*)^{\mathrm{T}} : [-\tau, \xi) \to \mathbb{R}^n$ such that $\gamma_i^*(t) \in F_i(t, x_t)(i = 1, 2, \ldots, n)$ for a.e. $t \in [-\tau, \xi)$ and

$$\zeta_i^*(t) = \int_t^{t+\omega} G_i(t,s)\gamma_i^*(s)\mathrm{d}s \in \int_t^{t+\omega} G_i(t,s)F_i(s,x_s)\mathrm{d}s = \varphi_i(x)(t). \tag{3.11}$$

From (2.3), it can be seen that $F_i(s, x_s)$ is convex. That is, for $0 \le \lambda \le 1$, $\lambda\gamma_i(s) + (1-\lambda)\gamma_i^*(s) \in F_i(s, x_s)$ for a.e. $s \ge 0$ and all $i = 1, 2, \ldots, n$. Therefore, for all $t \in [0, \omega]$, we have

$$\begin{aligned}\lambda\zeta_i(t) + (1-\lambda)\zeta_i^*(t) &= \int_t^{t+\omega} G_i(t,s)[\lambda\gamma_i(s) + (1-\lambda)\gamma_i^*(s)]\mathrm{d}s \\ &\in \int_t^{t+\omega} G_i(t,s)F_i(s,x_s)\mathrm{d}s = \varphi_i(x),\ i = 1,2,\ldots,n,\end{aligned}$$

and so

$$\lambda\zeta(t) + (1-\lambda)\zeta^*(t) \in \varphi(x).$$

This means that $\varphi(x)$ is a convex set in Ω for each $x \in \Omega$.

Step 3. We will prove that the set-valued map $\varphi : \Omega \to \mathscr{P}_{kc}(\Omega)$ is compact.

According to the Ascoli-Arzela theorem, it is sufficient to prove that $\varphi(\Omega)$ is a uniformly bounded and equi-continuous set. First, we show that $\varphi(\Omega)$ is a uniformly bounded set. To see this, let $x = (x_1, x_2, \dots, x_n)^{\mathrm{T}} \in \Omega$ and $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n)^{\mathrm{T}} \in \varphi(x)$ be arbitrary. Then there exists measurable function $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)^{\mathrm{T}} : [-\tau, \xi) \to \mathbb{R}^n$ such that $\gamma_i(t) \in F_i(t, x_t)(i = 1, 2, \dots, n)$ for a.e. $t \in [-\tau, \xi)$ and (3.8) still holds. Clearly, for any $x \in \Omega$, it follows from (3.10) that

$$\sum_{i=1}^{n} |\zeta_i(t)| < \sum_{i=1}^{n} \mathcal{R}_i \triangleq \mathcal{R}^{\mathrm{sum}}, \tag{3.12}$$

which implies

$$\|\zeta(t)\|_{C_\omega} = \sum_{i=1}^{n} \sup_{t\in[0,\omega]} |\zeta_i(t)| < \mathcal{R}^{\mathrm{sum}}, \ \forall\, x \in \Omega. \tag{3.13}$$

This shows that $\varphi(\Omega)$ is a uniformly bounded set for all $x \in \Omega$.

Next, we will show that $\varphi(\Omega)$ is an equi-continuous set. For this purpose, let $t, t^* \in [0, \omega]$; then for any $\zeta \in \varphi(\Omega)$ and every $i = 1, 2, \dots, n$, we can obtain from (3.4) and (3.8) that

$$\begin{aligned}
|\zeta_i(t) - \zeta_i(t^*)| &= \left| \int_t^{t+\omega} G_i(t,s)\gamma_i(s)\mathrm{d}s - \int_{t^*}^{t^*+\omega} G_i(t^*,s)\gamma_i(s)\mathrm{d}s \right| \\
&\le \left| \int_t^{t+\omega} G_i(t,s)\gamma_i(s)\mathrm{d}s - \int_t^{t+\omega} G_i(t^*,s)\gamma_i(s)\mathrm{d}s \right| \\
&\quad + \left| \int_t^{t+\omega} G_i(t^*,s)\gamma_i(s)\mathrm{d}s - \int_{t^*}^{t^*+\omega} G_i(t^*,s)\gamma_i(s)\mathrm{d}s \right| \\
&\le \left| \int_t^{t+\omega} \left[G_i(t,s) - G_i(t^*,s)\right] \gamma_i(s)\mathrm{d}s \right| \\
&\quad + \left| \int_{t^*}^{t} G_i(t^*,s)\gamma_i(s)\mathrm{d}s \right| + \left| \int_{t^*+\omega}^{t+\omega} G_i(t^*,s)\gamma_i(s)\mathrm{d}s \right| \\
&\le \max_{t\le s\le t+\omega} \{|G_i(t,s) - G_i(t^*,s)|\} \int_0^{\omega} |\gamma_i(s)|\mathrm{d}s \\
&\quad + G_i^{\max} \left| \int_{t^*}^{t} |\gamma_i(s)|\mathrm{d}s \right| + G_i^{\max} \left| \int_{t^*+\omega}^{t+\omega} |\gamma_i(s)|\mathrm{d}s \right|, \ \forall\, x \in \Omega.
\end{aligned} \tag{3.14}$$

Obviously, for any $x \in \Omega$ and each $i = 1, 2, \dots, n$, it follows from $(\mathscr{H}2)$ that

(3.15)

$$|\gamma_i(t)| \le \sup_{\gamma_i(t)\in F_i(t,\phi)} |\gamma_i(t)| \le \sum_{j=1}^{n} \alpha_{ij}(t)\mathcal{R}_j + \beta_i(t) \le \sum_{j=1}^{n} \alpha_{ij}^M \mathcal{R}_j + \beta_i^M \triangleq \mathscr{S}_i.$$

From (3.14) and (3.15), we can obtain that

$$|\zeta_i(t) - \zeta_i(t^*)| \le \max_{t\le s\le t+\omega} \{|G_i(t,s) - G_i(t^*,s)|\}\, \omega\mathscr{S}_i + 2G_i^{\max}\mathscr{S}_i|t - t^*|, \ \forall\, x \in \Omega.$$

As $t \to t^*$, the right-hand side of the above inequality tends to zero. Thus, we have $\|\zeta(t) - \zeta(t^*)\| \to 0$ as $t \to t^*$, where $\|\cdot\|$ denotes any vector norm. This shows that $\varphi(\Omega)$ is an equi-continuous set in C_ω.

Step 4. We will prove that the set-valued map $\varphi : \Omega \to \mathscr{P}_{kc}(\Omega)$ is upper semi-continuous (USC).

It should be pointed out that the upper semi-continuity (USC) of a set-valued map is equivalent to the condition of being a closed graph operator when the map has nonempty compact values. Therefore, we need only to prove that the set-valued map φ is a closed graph operator. Actually, according to Definition 2.3, we can see that $F(t,x_t) = (F_1(t,x_t), F_2(t,x_t), \ldots, F_n(t,x_t))^{\mathrm{T}}$ defined in (2.3) is an L^1-Carathéodory set-valued map. Recalling the set-valued operator $\mathscr{F}$ defined by (3.2), we can obtain from Lemma 2.4 that $\mathscr{F}(x) \neq \emptyset$ for each fixed $x \in C_\omega$. Now, let us define a continuous linear operator $\mathcal{L} : L^1(\mathbb{I}^\omega, \mathbb{R}^n) \to C(\mathbb{I}^\omega)$ given by

$$\mathcal{L}\gamma(t) = \begin{pmatrix} \int_t^{t+\omega} G_1(t,s)\gamma_1(s)ds \\ \int_t^{t+\omega} G_2(t,s)\gamma_2(s)ds \\ \vdots \\ \int_t^{t+\omega} G_n(t,s)\gamma_n(s)ds \end{pmatrix}, \ t \in \mathbb{I}^\omega.$$

By Lemma 2.5, it follows that $\varphi = \mathcal{L} \circ \mathscr{F}$ is a closed graph operator. That is, we have proven that the set-valued map φ is USC.

By now, we have proven that all the requirements of Lemma 2.2 are satisfied; then the set-valued map $\varphi : \Omega \to \mathscr{P}_{kc}(\Omega)$ has at least one fixed point $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^{\mathrm{T}} \in \Omega$ such that $x^*(t) \in \varphi(x^*)(t)$. Therefore, there exists at least one ω-periodic solution of delayed system (2.1) or (2.2). The proof is complete.

Corollary 3.2. *Suppose that ($\mathscr{H}1$) is satisfied and assume further that:*

($\mathscr{H}3$) *There exist nonnegative continuous ω-periodic functions $\alpha_{ij}(t)$ and $\beta_i(t)$ $(i,j = 1,2,\ldots,n)$ such that for any $\phi = (\phi_1, \phi_2, \ldots, \phi_n)^{\mathrm{T}} \in C_\tau$,*

$$\sup_{\gamma_i \in F_i(t,\phi)} |\gamma_i| \leq \sum_{j=1}^{n} \alpha_{ij}(t)|\phi_j|_0 + \beta_i(t), \ i = 1,2,\ldots,n.$$

($\mathscr{H}4$) *$E_n - \Theta$ is an M-matrix, where $\Theta = (\Psi(\alpha_{ij}, b_i))_{n\times n}$.*

Then the delayed system (2.1) *or* (2.2) *has at least one ω-periodic solution.*

Proof. Since $E_n - \Theta$ is an M-matrix, we can deduce from Lemma 2.7 that there exists a vector $\eta = (\eta_1, \eta_2, \ldots, \eta_n)^{\mathrm{T}} > (0,0,\ldots,0)^{\mathrm{T}}$ such that

$$\vartheta = (\vartheta_1, \vartheta_2, \ldots, \vartheta_n)^{\mathrm{T}} = (E_n - \Theta)\eta > \mathbf{0}.$$

Let $\breve{\beta}_i = \Psi(\beta_i, b_i)$ for $i = 1,2,\ldots,n$. We can select a sufficiently large constant $\mathscr{O} > 0$ such that $\mathscr{O}\vartheta_i > \breve{\beta}_i$ for each $i = 1,2,\ldots,n$. Let us denote $\mathcal{R}_i = \mathscr{O}\eta_i$, $i = 1,2,\ldots,n$. Then we have

$$(E_n - \Theta)(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_n)^{\mathrm{T}} > (\breve{\beta}_1, \breve{\beta}_2, \ldots, \breve{\beta}_n)^{\mathrm{T}}.$$

On the other hand, we can obtain from ($\mathscr{H}3$) that

$$\sup_{\gamma_i \in F_i(t,\phi)} |\gamma_i| \leq \sum_{j=1}^{n} \alpha_{ij}(t)\mathcal{R}_j + \beta_i(t) \text{ for } \phi \in C_\tau \text{ with } |\phi_j|_0 \leq \mathcal{R}_j, \ j = 1,2,\ldots,n.$$

This means that the condition ($\mathscr{H}2$) is satisfied. According to Corollary 3.2, the delayed system (2.1) or (2.2) has at least one ω-periodic solution. The proof is complete. □

According to Lemma 2.8 and Corollary 3.2, we can obtain the Corollary 3.3.

Corollary 3.3. *Suppose that ($\mathscr{H}1$) and ($\mathscr{H}3$) are satisfied and assume further that $\rho(\Theta) < 1$, where $\Theta = (\Psi(\alpha_{ij}, b_i))_{n\times n}$. Then the system* (2.1) *or* (2.2) *has at least one ω-periodic solution.*

Corollary 3.4. *Suppose that ($\mathscr{H}1$) and ($\mathscr{H}3$) are satisfied and $b_i^L \geq 0$ for each $i = 1, 2, \ldots, n$. Assume further that $\rho(W) < 1$, where $W = \left(\frac{\omega\alpha_{ij}^M}{1-\exp\{-\omega\bar{b}_i\}}\right)_{n\times n}$. Then the delayed system* (2.1) *or* (2.2) *has at least one ω-periodic solution.*

Proof. Since $b_i^L \geq 0$ for each $i = 1, 2, \ldots, n$, we can obtain

$$\begin{aligned}\Psi(\alpha_{ij}, b_i) &= \frac{1}{1-\exp\{-\omega\bar{b}_i\}} \sup_{t\in[0,\omega]} \int_0^\omega \alpha_{ij}(s+t)\cdot\exp\left\{-\int_s^\omega b_i(\sigma+t)\mathrm{d}\sigma\right\}\mathrm{d}s \\ &\leq \frac{1}{1-\exp\{-\omega\bar{b}_i\}} \sup_{t\in[0,\omega]} \int_0^\omega \alpha_{ij}(s+t)\mathrm{d}s \\ &\leq \frac{\omega\alpha_{ij}^M}{1-\exp\{-\omega\bar{b}_i\}}.\end{aligned}$$

Let $W = \left(\frac{\omega\alpha_{ij}^M}{1-\exp\{-\omega\bar{b}_i\}}\right)_{n\times n}$. Obviously, we have $0 \leq \Theta \leq W$. By virtue of Lemma 2.9, we can get that $\rho(\Theta) \leq \rho(W)$. According to Corollary 3.3, the delayed system (2.1) or (2.2) has at least one ω-periodic solution. The proof is complete. □

Corollary 3.5. *Suppose that ($\mathscr{H}1$) and ($\mathscr{H}3$) are satisfied and $b_i^L > 0$ for each $i = 1, 2, \ldots, n$. Assume further that $\rho(Z) < 1$, where $Z = \left(\frac{\alpha_{ij}^M}{b_i^L}\right)_{n\times n}$. Then the system* (2.1) *or* (2.2) *has at least one ω-periodic solution.*

Proof. Since $b_i(t)$ is an ω-periodic function and $b_i^L > 0$, it follows that

$$\begin{aligned}&\Psi(\alpha_{ij}, b_i) \\ &= \frac{1}{1-\exp\{-\omega\bar{b}_i\}} \sup_{t\in[0,\omega]} \int_0^\omega \alpha_{ij}(s+t)\cdot\exp\left\{-\int_s^\omega b_i(\sigma+t)\mathrm{d}\sigma\right\}\mathrm{d}s \\ &\leq \frac{1}{1-\exp\{-\omega\bar{b}_i\}}\cdot\frac{\alpha_{ij}^M}{b_i^L}\cdot\sup_{t\in[0,\omega]} \int_0^\omega b_i(s+t)\cdot\exp\left\{-\int_s^\omega b_i(\sigma+t)\mathrm{d}\sigma\right\}\mathrm{d}s \\ &= \frac{1}{1-\exp\{-\omega\bar{b}_i\}}\cdot\frac{\alpha_{ij}^M}{b_i^L}\cdot\sup_{t\in[0,\omega]}\left[1-\exp\left\{-\int_0^\omega b_i(\sigma+t)\mathrm{d}\sigma\right\}\right] \\ &= \frac{1}{1-\exp\{-\omega\bar{b}_i\}}\cdot\frac{\alpha_{ij}^M}{b_i^L}\cdot\sup_{t\in[0,\omega]}\left[1-\exp\left\{-\int_0^\omega b_i(\theta)\mathrm{d}\theta\right\}\right] \\ &= \frac{1}{1-\exp\{-\omega\bar{b}_i\}}\cdot\frac{\alpha_{ij}^M}{b_i^L}\cdot\left(1-\exp\{-\omega\bar{b}_i\}\right) \\ &= \frac{\alpha_{ij}^M}{b_i^L}.\end{aligned}$$

Let $Z = \left(\frac{\alpha_{ij}^M}{b_i^L}\right)_{n\times n}$. It is clear that $0 \leq \Theta \leq Z$. By virtue of Lemma 2.9, we can obtain that $\rho(\Theta) \leq \rho(Z)$. According to Corollary 3.3, the delayed system (2.1) or (2.2) has at least one ω-periodic solution. The proof is complete. □

4. Applications to neural networks

Consider a class of neural networks described by the delayed differential equations with discontinuous right-hand sides as follows:

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -\, b_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)g_j(x_j(t)) + \sum_{j=1}^{n} c_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) + J_i(t), \tag{4.1}$$

where $i = 1, 2, \ldots, n$, $x_i(t)$ stands for the state variable of the potential of the ith neuron at time t; $b_i(t)$ denotes the self-inhibition of the ith neuron at time t; $a_{ij}(t)$ and $c_{ij}(t)$ are the connection weights between the jth unit and the ith unit at time t, respectively; $g_j(\cdot)$ represents the activation function of the jth neuron; $J_i(t)$ denotes neuron input at time t; $\tau_{ij}(t)$ stands for the time-varying transmission delay at time t and is a continuous ω-periodic function satisfying $0 \le \tau_{ij}(t) \le \tau$, $\tau = \max\limits_{1\le i,j\le n}\{\tau_{ij}^M\}$. We always assume that $b_i(t), a_{ij}(t), c_{ij}(t), J_i(t)$ are continuous ω-periodic functions on $\mathbb{R}$ and $\int_0^\omega b_i(s)\mathrm{d}s > 0$. The discontinuous activation functions in (4.1) are assumed to satisfy the following conditions:

($\mathscr{H}$5) $g_i : \mathbb{R} \to \mathbb{R}$ is continuous except on a countable set of isolated points $\{\rho_k^i\}$, where there exist finite right and left limits, $g_i^+(\rho_k^i)$ and $g_i^-(\rho_k^i)$, respectively. Moreover, g_i has at most a finite number of discontinuities on any compact interval of $\mathbb{R}$.

($\mathscr{H}$6) There exist nonnegative constants ℓ_i and p_i such that

$$\sup_{\mathscr{I}_i\in\overline{\mathrm{co}}[g_i(x_i)]} |\mathscr{I}_i| \le \ell_i|x_i| + p_i, \forall x_i \in \mathbb{R},$$

where, for $\theta \in \mathbb{R}$, $\overline{\mathrm{co}}[g_i(\theta)] = \left[\min\{g_i^-(\theta), g_i^+(\theta)\}, \max\{g_i^-(\theta), g_i^+(\theta)\}\right]$.

Let $x(t)$ denote a solution of system (4.1) with given initial condition $\phi(s) = (\phi_1(s), \phi_2(s), \ldots, \phi_n(s))^{\mathrm{T}} \in C([-\tau, 0], \mathbb{R}^n)$. By constructing the Filippov set-valued map, it is easy to see that if $x(t)$ is a Filippov solution of (4.1), then it is a solution of the following delayed differential inclusion (4.2):

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} \in & -\, b_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)\overline{\mathrm{co}}[g_j(x_j(t))] \\ & + \sum_{j=1}^{n} c_{ij}(t)\overline{\mathrm{co}}[g_j(x_j(t-\tau_{ij}(t)))] + J_i(t) \\ \triangleq & -b_i(t)x_i(t) + F_i(t, x_t), \text{ for a.e. } t \ge 0.\end{aligned} \tag{4.2}$$

Clearly, the set-valued map $F = (F_1, F_2, \ldots, F_n)^{\mathrm{T}}$ is USC with nonempty, compact, convex values and locally bounded. So, it is measurable. By measurable selections Theorem [10], there exists a measurable function $\mathscr{I} = (\mathscr{I}_1, \mathscr{I}_2, \ldots, \mathscr{I}_n)^{\mathrm{T}} : [-\tau, +\infty) \to \mathbb{R}^n$ such that $\mathscr{I}_j(t) \in \overline{\mathrm{co}}[g_j(x_j(t))]$ for a.e. $t \in [-\tau, +\infty)$ and

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = & -b_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)\mathscr{I}_j(t) \\ & + \sum_{j=1}^{n} c_{ij}(t)\mathscr{I}_j(t-\tau_{ij}(t)) + J_i(t), \text{ a.e. } t \ge 0.\end{aligned}$$

Theorem 4.1. *Suppose that ($\mathscr{H}$5) and ($\mathscr{H}$6) are satisfied and assume further that*
($\mathscr{H}$7) $E_n - \Theta$ *is an M-matrix, where* $\Theta = (\Psi((|a_{ij}| + |c_{ij}|)\ell_j, b_i))_{n\times n}$.
Then the system (4.1) *has at least one* ω*-periodic solution.*

Proof. From (4.2), the set-valued map $F = (F_1, F_2, \dots, F_n)^{\mathrm{T}}$ is given as

$$F_i(t, x_t) = \sum_{j=1}^{n} a_{ij}(t)\overline{\mathrm{co}}[g_j(x_j(t))] + \sum_{j=1}^{n} c_{ij}(t)\overline{\mathrm{co}}[g_j(x_j(t - \tau_{ij}(t)))] + J_i(t),$$

where $i = 1, 2, \dots, n$. That is, for any $\phi = (\phi_1, \phi_2, \dots, \phi_n)^{\mathrm{T}} \in C_\tau$,

$$F_i(t, \phi) = \sum_{j=1}^{n} a_{ij}(t)\overline{\mathrm{co}}[g_j(\phi_j(0))] + \sum_{j=1}^{n} c_{ij}(t)\overline{\mathrm{co}}[g_j(\phi_j(-\tau_{ij}(t)))] + J_i(t).$$

It follows from ($\mathscr{H}$6) that

$$\sup_{\gamma_i \in F_i(t,\phi)} |\gamma_i| \le \sum_{j=1}^{n} \ell_j(|a_{ij}(t)| + |c_{ij}(t)|)|\phi_j|_0 + \sum_{j=1}^{n} p_j(|a_{ij}(t)| + |c_{ij}(t)|) + J_i(t).$$

Let $\alpha_{ij}(t) = \ell_j(|a_{ij}(t)| + |c_{ij}(t)|)$ and $\beta_i(t) = \sum_{j=1}^{n} p_j(|a_{ij}(t)| + |c_{ij}(t)|) + J_i(t)$. This implies that the conditions ($\mathscr{H}$3) and ($\mathscr{H}$4) hold. According to Lemma 2.8, the system (4.1) has at least one ω-periodic solution. The proof is complete. □

Similar to corollaries 3.3-3.5, we can obtain the following corollaries.

Corollary 4.2. *Suppose that ($\mathscr{H}$5) and ($\mathscr{H}$6) are satisfied and assume further that* $\rho(\Theta) < 1$, *where* $\Theta = (\Psi((|a_{ij}| + |c_{ij}|)\ell_j, b_i))_{n\times n}$. *Then the system* (4.1) *has at least one* ω*-periodic solution.*

Corollary 4.3. *Suppose that ($\mathscr{H}$5) and ($\mathscr{H}$6) are satisfied and* $b_i^L \ge 0$ *for each* $i = 1, 2, \dots, n$. *Assume further that* $\rho(W) < 1$, *where* $W = \left(\frac{(a_{ij}^M + c_{ij}^M)\ell_j\omega}{1-\exp\{-\omega\bar{b}_i\}}\right)_{n\times n}$. *Then the system* (4.1) *has at least one* ω*-periodic solution.*

Corollary 4.4. *Suppose that ($\mathscr{H}$5) and ($\mathscr{H}$6) are satisfied and* $b_i^L > 0$ *for each* $i = 1, 2, \dots, n$. *Assume further that* $\rho(Z) < 1$, *where* $Z = \left(\frac{(a_{ij}^M + c_{ij}^M)\ell_j}{b_i^L}\right)_{n\times n}$. *Then the system* (4.1) *has at least one* ω*-periodic solution.*

Example 4.5. Consider the neural network system (4.1) with time-delay $\tau_{ij}(t) = 1$.

Case 1. Take $n = 2$, $b_1(t) = 3 + \cos 4t$, $b_2(t) = 3 + \sin 4t$, $a_{11}(t) = a_{22}(t) = -0.4$, $c_{11}(t) = c_{22}(t) = -0.3$, $a_{21}(t) = a_{12}(t) = 0.4$ $c_{21}(t) = c_{12}(t) = 0.3$, $J_1(t) = \sin 4t$ and $J_2(t) = \cos 4t$. The discontinuous activation functions are given as

$$g_i(x_i) = \begin{cases} 0.4\tanh(x_i) - 0.8, & x_i \ge 0, \\ 0.4\tanh(x_i) + 0.8, & x_i < 0, \end{cases} \qquad i = 1, 2.$$

Clearly, the discontinuous activation functions satisfy ($\mathscr{H}$5) and are nonmonotonic. Furthermore, 0 is a discontinuous point of the activation function $g_i(\cdot)$ and $\overline{\mathrm{co}}[g_i(0)] = [g_i^+(0), g_i^-(0)] = [-0.8, 0.8]$. It is not difficult to check that the linear growth

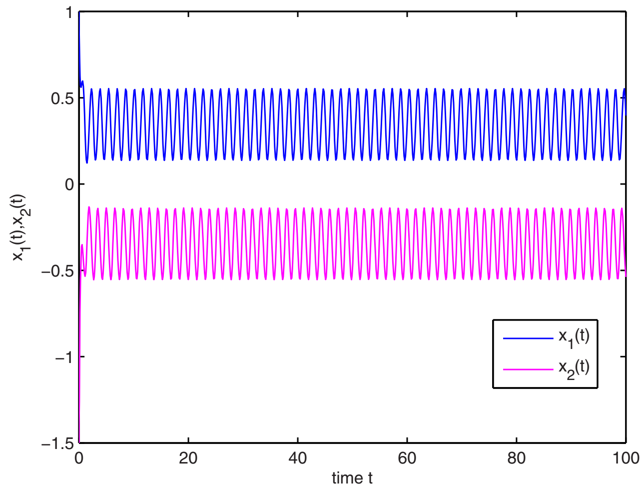

FIGURE 1. Time-domain behaviors of $x_1(t)$ and $x_2(t)$ for Case (1) of system (4.1).

condition ($\mathscr{H}6$) is satisfied by letting $\ell_1 = \ell_2 = 0.4$ and $p_1 = p_2 = 0.8$. By simple computation, we have

$$Z = \left(\frac{(a_{ij}^M + c_{ij}^M)\ell_j}{b_i^L} \right)_{2\times 2} = \begin{pmatrix} 0.14 & 0.14 \\ 0.14 & 0.14 \end{pmatrix}.$$

Moreover, $\rho(Z) = 0.28 < 1$. By Corollary 4.4, we can conclude that the system (4.1) has at least one $\frac{\pi}{2}$-periodic solution. Consider the initial condition of system (4.1): $\phi(t) = (1, -1.5)^{\mathrm{T}}$ for $t \in [-1, 0]$. The numerical simulations are shown in Figure 1 which also confirms the existence of a periodic solution for system (4.1).

Case 2. Take $n = 3$, $b_1(t) = b_2(t) = b_3(t) = 2 + \sin 3t$, $a_{11}(t) = a_{22}(t) = a_{33}(t) = 0.5$, $c_{11}(t) = c_{22}(t) = c_{33}(t) = 0.4$, $a_{ij}(t) = c_{ij}(t) = 0 (i \neq j)$, $J_1(t) = \sin 3t$, $J_2(t) = J_3(t) = \cos 3t$. The discontinuous activation functions are described by

$$g_i(x_i) = \begin{cases} x_i + 0.5, & x_i \geq 0, \\ x_i - 0.5, & x_i < 0, \end{cases} \qquad i = 1, 2, 3.$$

Obviously, the discontinuous activation function $g_i(x_i)$ satisfies ($\mathscr{H}5$) and ($\mathscr{H}6$) with $\ell_1 = \ell_2 = \ell_3 = 1$ and $p_1 = p_2 = p_3 = 0.5$. We can easily calculate that

$$Z = \left(\frac{(a_{ij}^M + c_{ij}^M)\ell_j}{b_i^L} \right)_{3\times 3} = \begin{pmatrix} 0.9 & 0 & 0 \\ 0 & 0.9 & 0 \\ 0 & 0 & 0.9 \end{pmatrix}.$$

Clearly, $\rho(Z) = 0.9 < 1$. Therefore, all the conditions in Corollary 4.4 are satisfied and imply that the system (4.1) has at least one $\frac{2\pi}{3}$-periodic solution. Consider the initial condition of system (4.1): $\phi(t) = (2, -3, 1)^{\mathrm{T}}$ for $t \in [-1, 0]$. The numerical simulation results are shown in Figure 2, which also confirms the validity of the theoretical result concerning the existence of periodic solution.

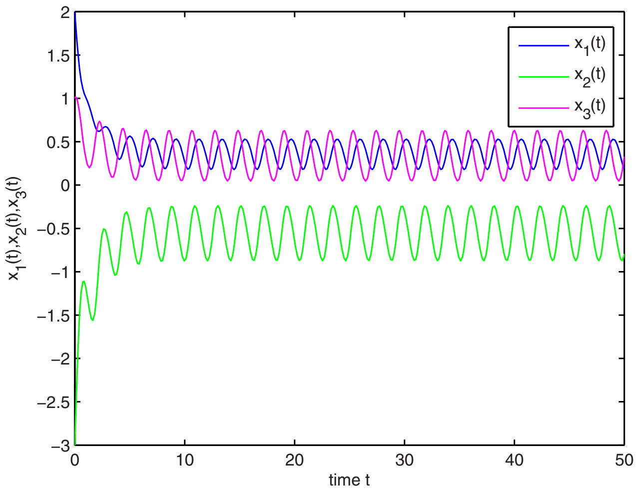

FIGURE 2. Time-domain behaviors of $x_1(t)$, $x_2(t)$, $x_3(t)$ for Case (2) of system (4.1).

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