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LINEAR REPRESENTATIONS OF 3–MANIFOLD GROUPS OVER RINGS

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Abstract. The fundamental groups of compact 3–manifolds are known to be residually finite. Feng Luo conjectured that a stronger statement is true, by only allowing finite groups of the form $\mathrm{PGL}_2(R)$, where R is some finite commutative ring with identity. We give an equivalent formulation of Luo's conjecture via faithful representations and provide various examples and a counterexample.

1. Introduction

In this paper, by 3–manifold we mean connected 3–manifold. A hyperbolic structure of finite volume on an orientable 3–manifold M gives rise to a developing map $\mathrm{dev}\colon \widetilde{M} \to \mathbb{H}^3$, where $\widetilde{M}$ is the universal cover of M, and an embedding $\mathrm{hol}\colon \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{PSL}_2(\mathbb{C})$, which is the holonomy representation associated to the geometric structure and chosen developing map. For details, see, for example, [20, Chapter 3] or [15, Chapter 8]. In the case that M is triangulated, the decomposition of M into simplices lifts to one of $\widetilde{M}$ which gives rise to a labelling of the 0-skeleton of the lifted triangulation by elements of $\partial\overline{\mathbb{H}}^3 = \mathbb{C}P^1$, and this labelling encodes all the information necessary to construct the holonomy representation; see [21] for the case of torus cusps and [11] for the closed case. In [10], Luo generalises these labellings to labellings over $\mathbb{P}^1(R)$, the projective line over an arbitrary commutative ring with identity R, and constructs representations into $\mathrm{PGL}_2(R)$. An example illustrating the strength of Luo's generalisation is given in §3.8. Luo makes the following conjecture:

Conjecture 1.1 (Luo [10])**.** *If M is a compact 3–manifold and $\gamma \in \pi_1(M) \setminus \{1\}$, there exist a finite commutative ring R with identity and a homomorphism $\pi_1(M) \to \mathrm{PGL}_2(R)$ whose kernel does not contain γ.*

Investigation of this conjecture leads us to consider linearity of groups over arbitrary commutative rings with identity, as opposed to just over fields or, equivalently, over integral domains, as is traditionally done. The strength of allowing zero divisors is illustrated in §§3.7-3.8.

At this point it is helpful to introduce the following definition.

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Definition 1.2. Given a group G, say that G is *residually* PGL_2*–finite* if for any $g \in G \setminus \{1\}$, there exists a finite commutative ring R with identity and a homomorphism $G \to \mathrm{PGL}(2,R)$ whose kernel does not contain g. Define, in an analogous manner, *residually* PSL_2*–finite*, *residually* SL_2*–finite*, and *residually* GL_2*–finite*.

Luo's conjecture thus says that fundamental groups of compact 3–manifolds are residually PGL_2–finite. In §2, we show that for most groups the different notions of residual finiteness are equivalent. More precisely, we show that for a finitely generated group G, we have the following implications:

$$\begin{array}{ccc} \text{residually } \mathrm{SL}_2\text{–finite} & \underset{Z(G)=1}{\overset{\Longrightarrow}{\Longleftarrow}} & \text{residually } \mathrm{GL}_2\text{–finite} \\ Z(G)\ \text{2-t.f.}\ \Downarrow \Uparrow & & \Uparrow \Downarrow\ Z(G)=1 \\ \text{residually } \mathrm{PSL}_2\text{–finite} & \overset{\Longrightarrow}{\Longleftarrow} & \text{residually } \mathrm{PGL}_2\text{–finite} \end{array}$$

where 2-t.f. means 2-torsion-free and $Z(G)$ denotes the centre of G.

In the following let K be one of the symbols SL_2, GL_2, PSL_2, PGL_2. In this paper we investigate different types of groups and check whether they are residually K–finite. If a group is residually K–finite, then one can usually show it by writing a representation. On the other hand, the proof of Proposition 2.6 gives a practical approach to showing that a group is not residually K–finite.

As applications we consider several classes of groups in §3, giving both positive and negative results. For example in Theorem 3.1 we show that the symmetric group S_n in n letters is residually PGL_2–finite if and only if $n < 5$. We also show that some 3–manifold groups are residually K–finite for all choices of K. Our main result though is that Conjecture 1.1 does not hold. More precisely, we prove the following result in §4.

Theorem 1.3. *There exists a closed graph manifold M such that $\pi_1(M)$ is not residually K–finite for $K = \mathrm{SL}_2$, GL_2, PSL_2, PGL_2.*

We conclude this introduction with a short discussion of the linearity of fundamental groups of 3–manifolds. The fact that our counterexample is a closed graph manifold is perhaps not surprising since it is still unknown whether fundamental groups of closed graph manifolds are linear. This raises the following question.

Question 1.4. Does there exist a counterexample to Conjecture 1.1 that is a prime 3–manifold but that is not a graph manifold?

We also recall that Thurston [8, Problem 3.33] asked whether every finitely generated 3–manifold group has a faithful representation in $\mathrm{GL}(4,\mathbb{R})$. Button [4] recently answered this question in the negative. More precisely, he showed that there exists a closed graph manifold M such that $\pi_1(M)$ does not admit a faithful representation into $\mathrm{GL}(4,\mathbb{R})$.

We conclude this introduction with the following questions.

Question 1.5. Does there exist a natural number n such that the fundamental group of every compact 3–manifold admits a faithful representation into $\mathrm{GL}(n,\mathbb{C})$?

Question 1.6. Does there exist a natural number n such that the fundamental group of every compact 3–manifold is residually GL_n–finite?

2. Alternative characterisations

We have the following result of Baumslag, proven in [2, Theorem 5.3, p. 64].

Theorem 2.1. *If a commutative ring R with identity is finitely generated, then it is residually finite; that is, for each non-zero $r \in R$, there exist a finite commutative R' with identity and a ring map $\varphi\colon R \to R'$ which respects the identities, such that $\varphi(r) \neq 0$.*

Let K be one of the symbols SL_2, GL_2, PSL_2, PGL_2. The following is essentially how, in [2, Theorem 5.6, p. 66], Baumslag proved that finitely generated groups of matrices over an arbitrary commutative ring are residually finite. Earlier, in [12], Mal'cev had proved this result in the case of integral domains by a somewhat different method.

Proposition 2.2. *Given a finitely generated group G, an element $g \in G \setminus \{1\}$, and a homomorphism $\rho\colon G \to K(R)$ for a commutative but not necessarily finite ring R such that $\rho(g) \neq 1$, there exists a homomorphism $\rho'\colon G \to K(R')$ for a finite commutative R' such that $\rho'(g) \neq 1$. In particular, if there exists a faithful representation $G \to K(R)$ for some commutative but not necessarily finite R, then G is residually K–finite.*

Proof. Suppose first that K is one of SL_2, GL_2. Let $\{g_1, \ldots, g_k\}$ be a set of generators for G. For $i = 1, \ldots, k$, let $A_i = \rho(g_i)$ and let $\widetilde{R}$ be the ring generated by the entries of $A_1, \ldots, A_k$ as well as the elements $(\det A_1)^{-1}, \ldots, (\det A_k)^{-1}$. In the case that $K = \mathrm{SL}_2$, the inclusion of the determinants is superfluous. Note that $\widetilde{R}$ contains the entries of $A_1^{-1}, \ldots, A_k^{-1}$. As such, we can restrict ρ to attain a faithful representation $\rho'\colon G \to \mathrm{SL}_2(\widetilde{R})$, and because $\widetilde{R}$ is a finitely generated ring, it is residually finite by Theorem 2.1.

(i) If $\rho'(g) = \rho(g)$ has a non-zero off-diagonal entry, say a, and we let $\phi\colon \widetilde{R} \to R'$ be such that $\phi(a) \neq 0$ and R' is finite, then the image of g under the map $G \xrightarrow{\rho'} K(\widetilde{R}) \xrightarrow{\phi_*} K(R')$ is non-trivial. Note that the image matrix under ϕ_* still has determinant 1 or a unit because ϕ preserves 1 and units.

Suppose then that $\rho'(g)$ is diagonal, say $\mathrm{diag}(a, b)$.

(ii) If $a - b \neq 0$, then we can choose $\phi\colon \widetilde{R} \to R'$ such that $\phi(a - b) \neq 0$ and R' is finite. Then the image of g under the map $G \xrightarrow{\rho'} K(\widetilde{R}) \xrightarrow{\phi_*} K(R')$ is non-trivial.

(iii) If $a = b$, say with both equal to $c \neq 1$, and we let $\phi\colon \widetilde{R} \to R'$ be such that $\phi(c - 1) \neq 0$ and R' is finite, then the image of g under the map $G \xrightarrow{\rho'} K(\widetilde{R}) \xrightarrow{\phi_*} K(R')$ is non-trivial.

Now suppose that K is one of PSL_2, PGL_2. Let $A_1, \ldots, A_k$ be matrices that are representatives of $\rho(g_1), \ldots, \rho(g_k)$ respectively and let $\widetilde{R}$ be the ring generated by the entries of $A_1, \ldots, A_k$; note that $\widetilde{R}$ contains the entries of representatives for $\rho(g_1^{-1}), \ldots, \rho(g_k^{-1})$. Let $\iota\colon K(\widetilde{R}) \to K(R)$ denote the obvious embedding which leads to an isomorphism $K(\widetilde{R}) \cong \mathrm{im}(\iota)$ and note that $\mathrm{im}(\rho) \subseteq \mathrm{im}(\iota)$; thus by restriction and composition we get a representation $\rho'\colon G \to K(\widetilde{R})$ such that $\rho'(g) \neq 1$. The remainder of the argument is now precisely as in the cases (i) and (ii) above. □

Thus, if we define "*residually* K" to mean the same thing as residual K–finiteness but with the finiteness requirement on the ring dropped, we have:

Corollary 2.3. *A finitely generated group G is residually K–finite if and only if it is residually K.*

We also have:

Proposition 2.4. *Let G be a group. If G is residually K–finite, then it admits a faithful representation $G \to K(R)$ for some not necessarily finite R.*

Proof. We start out with the following two observations:

(1) K is functorial in the ring; i.e., a ring homomorphism $\varphi\colon S \to S'$ induces a group homomorphism $\varphi_*\colon K(S) \to K(S')$.

(2) If $S_i, i \in I$, is a family of rings, then

$$\prod_{i\in I} K(S_i) \to K\Big(\prod_{i\in I} S_i\Big)\colon (A_i)_{i\in I} \mapsto \prod_{i\in I} A_i,$$

where the product matrix $\Pi_{i\in I} A_i$ is formed entrywise in the direct product of the rings S_i, is well-defined, and is a group isomorphism.

Now we turn to the actual proof of the proposition. For any $g \neq 1$ in G, we have a representation $\rho_g\colon G \to K(R_g)$ where R_g is finite and $\rho_g(g) \neq 1$. Let $R = \prod_{g\neq 1} R_g$. We compose $(\rho_g)_{g\neq 1}\colon G \to \prod_{g\neq 1} K(R_g)$ with the group homomorphism given in (2) and we obtain representation $\rho\colon G \to K(R)$.

We claim that ρ is faithful. Let $g \in G$ be non-trivial. Then the image of $\rho(g)$ under the projection map $K(R) \to K(R_g)$ equals $\rho_g(g)$; hence it is non-trivial. $\square$

Combining Propositions 2.2 and 2.4, we have:

Corollary 2.5. *Suppose G is a finitely generated group. Then G is residually K–finite if and only if it admits a faithful representation $G \to K(R)$ for some not necessarily finite R.*

Thus we can check residual K–finiteness by looking for faithful representations. The next result provides a tautological representation through which all representations factor. In particular, we only need to study this tautological representation.

Proposition 2.6. *Suppose G is a finitely generated group. Then there exist a commutative ring S_K, an ideal $I_K \trianglelefteq S_K$, and a map $\varphi_K\colon G \to K(S_K/I_K)$ such that any representation $G \to K(R)$ factors through φ_K; that is, for each $\rho\colon G \to K(R)$, there exists a mediating map $\psi\colon K(S_K/I_K) \to K(R)$ such that the following diagram commutes:*

$$\begin{array}{ccc} G & \xrightarrow{\ \varphi_K\ } & K(S_K/I_K) \\ & \searrow^{\rho} & \downarrow{\scriptstyle\psi} \\ & & K(R). \end{array}$$

Proof. Let $G = \langle g_1, \ldots, g_n \mid r_i = e, i \in I\rangle$ and suppose first that $K = \mathrm{SL}_2$. Let

$$S_{\mathrm{SL}_2} = \mathbb{Z}[x_{1a}, x_{1b}, x_{1c}, x_{1d}, \ldots, x_{na}, x_{nb}, x_{nc}, x_{nd}]$$

and then define

$$p(g_i) = \begin{pmatrix} x_{ia} & x_{ib} \\ x_{ic} & x_{id} \end{pmatrix}, \quad p(g_i^{-1}) = \begin{pmatrix} x_{id} & -x_{ib} \\ -x_{ic} & x_{ia} \end{pmatrix}.$$

Define also $p(r_i)$ by setting p to be multiplicative and then setting

$$I_{\mathrm{SL}_2} = \langle \{\det p(g_i) - 1\}_i \cup \{(p(r_i) - 1)_{k,l}\}_{i,k,l} \rangle \,.$$

Then we set

$$\varphi_{\mathrm{SL}_2} \colon G \to \mathrm{SL}_2(S_{\mathrm{SL}_2}/I_{\mathrm{SL}_2}) \colon g_i \mapsto \begin{pmatrix} \overline{x_{ia}} & \overline{x_{ib}} \\ \overline{x_{ic}} & \overline{x_{id}} \end{pmatrix},$$

which can be checked to be well-defined. Now, suppose that $\rho\colon G \to \mathrm{SL}_2(R)$ is given. Let $\rho(g_i) = (a^i_{kl})_{kl}$ and define $q\colon S_{\mathrm{SL}_2}/I_{\mathrm{SL}_2} \to R$ by

$$1, x_{1a}, x_{1b}, x_{1c}, x_{1d}, \dots, x_{na}, x_{nb}, x_{nc}, x_{nd} \mapsto 1, a^1_{11}, a^1_{12}, a^1_{21}, a^1_{22}, \dots, a^n_{11}, a^n_{12}, a^n_{21}, a^n_{22}.$$

The map q is well-defined because $a^i_{11}a^i_{22} - a^i_{12}a^i_{21} - 1 = 0$ for each i and because computation of $\rho(r_j)$ and $\rho(s_j)$ will give the required remaining equations defining I. This map q induces a map

$$\psi\colon\ \mathrm{SL}_2(S_{\mathrm{SL}_2}/I_{\mathrm{SL}_2}) \xrightarrow{q_*} \mathrm{SL}_2(R)$$

by applying q to each entry, and one can then verify that $\psi \circ \varphi_{\mathrm{SL}_2} = \rho$ holds.

If $K = \mathrm{GL}_2$, we alter the definitions as follows:

$$S_{\mathrm{GL}_2} = \mathbb{Z}[x_{1a}, x_{1b}, x_{1c}, x_{1d}, \dots, x_{na}, x_{nb}, x_{nc}, x_{nd}, y_1, \dots, y_n],$$

$$p(g_i) = \begin{pmatrix} x_{ia} & x_{ib} \\ x_{ic} & x_{id} \end{pmatrix}, \quad p(g_i^{-1}) = y_i \begin{pmatrix} x_{id} & -x_{ib} \\ -x_{ic} & x_{ia} \end{pmatrix};$$

$p(r_i)$ are defined by setting p to be multiplicative:

$$I_{\mathrm{GL}_2} = \langle \{(\det p(g_i))y_i - 1\}_i \cup \{(p(r_i) - 1)_{k,l}\}_{i,k,l} \rangle \,,$$

$$\varphi_{\mathrm{GL}_2} \colon G \to \mathrm{GL}_2(S_{\mathrm{GL}_2}/I_{\mathrm{GL}_2}) \colon g_i \mapsto \begin{pmatrix} \overline{x_{ia}} & \overline{x_{ib}} \\ \overline{x_{ic}} & \overline{x_{id}} \end{pmatrix};$$

and finally given $\rho\colon G \to \mathrm{GL}_2(R)$ and $\rho(g_i) = (a^i_{kl})_{kl}$, $q\colon S_{\mathrm{GL}_2}/I_{\mathrm{GL}_2} \to R$:

$$1, x_{ia}, x_{ib}, x_{ic}, x_{id}, y_i \mapsto 1, a^i_{11}, a^i_{12}, a^i_{21}, a^i_{22}, (a^i_{11}a^i_{22} - a^i_{12}a^i_{21})^{-1},$$

and $\psi = q_*$.

If $K = \mathrm{PSL}_2$, we alter the definitions as follows:

$$S_{\mathrm{PSL}_2} = \mathbb{Z}[x_{1a}, x_{1b}, x_{1c}, x_{1d}, \dots, x_{na}, x_{nb}, x_{nc}, x_{nd}, \{\lambda_i\}_{i\in I}],$$

$$p(g_i) = \begin{pmatrix} x_{ia} & x_{ib} \\ x_{ic} & x_{id} \end{pmatrix}, \quad p(g_i^{-1}) = \begin{pmatrix} x_{id} & -x_{ib} \\ -x_{ic} & x_{ia} \end{pmatrix};$$

$p(r_i)$ are defined by setting p to be multiplicative:

$$I_{\mathrm{PSL}_2} = \left\langle \{\det p(g_i) - 1\}_i \cup \{\lambda_i^2 - 1\}_i \cup \{(p(r_i) - \lambda_i)_{k,l}\}_{i,k,l} \right\rangle,$$

$$\varphi_{\mathrm{PSL}_2} \colon G \to \mathrm{PSL}_2(S_{\mathrm{PSL}_2}/I_{\mathrm{PSL}_2}) \colon g_i \mapsto \begin{bmatrix} \overline{x_{ia}} & \overline{x_{ib}} \\ \overline{x_{ic}} & \overline{x_{id}} \end{bmatrix};$$

and finally given $\rho\colon G \to \mathrm{PSL}_2(R)$, $\rho(g_i) = [a^i_{kl}]_{kl}$, and that the corresponding representative for $p(r_i)$ is equal to μ_i times the identity matrix, $q\colon S_{\mathrm{PSL}_2}/I_{\mathrm{PSL}_2} \to R\colon 1, x_{ia}, x_{ib}, x_{ic}, x_{id}, y_i \mapsto 1, a^i_{11}, a^i_{12}, a^i_{21}, a^i_{22}, \mu_i$, and $\psi = q_*$.

If $K = \mathrm{PGL}_2$, we alter the definitions as follows:

$$S_{\mathrm{PGL}_2} = \mathbb{Z}[x_{1a}, x_{1b}, x_{1c}, x_{1d}, \dots, x_{na}, x_{nb}, x_{nc}, x_{nd}, y_1, \dots, y_n, \{\lambda_i\}_{i\in I}],$$

$$p(g_i) = \begin{pmatrix} x_{ia} & x_{ib} \\ x_{ic} & x_{id} \end{pmatrix}, \quad p(g_i^{-1}) = \begin{pmatrix} x_{id} & -x_{ib} \\ -x_{ic} & x_{ia} \end{pmatrix};$$

$p(r_i)$ are defined by setting p to be multiplicative:

$$I_{\mathrm{PGL}_2} = \langle \{(\det p(g_i))y_i - 1\}_i \cup \{(p(r_i) - \lambda_i)_{k,l}\}_{i,k,l} \rangle\,,$$

$$\varphi_{\mathrm{PGL}_2} \colon G \to \mathrm{PGL}_2(S_{\mathrm{PGL}_2}/I_{\mathrm{PGL}_2}) \colon g_i \mapsto \begin{bmatrix} \overline{x_{ia}} & \overline{x_{ib}} \\ \overline{x_{ic}} & \overline{x_{id}} \end{bmatrix};$$

and finally given $\rho \colon G \to \mathrm{PGL}_2(R)$, $\rho(g_i) = [a^i_{kl}]_{kl}$, and that the corresponding representative for $p(r_i)$ is equal to μ_i times the identity matrix, $q \colon S_{\mathrm{PGL}_2}/I_{\mathrm{PGL}_2} \to R\colon 1, x_{ia}, x_{ib}, x_{ic}, x_{id}, y_i, \lambda_i \mapsto 1, a^i_{11}, a^i_{12}, a^i_{21}, a^i_{22}, (a^i_{11}a^i_{22} - a^i_{12}a^i_{21})^{-1}, \mu_i$, and $\psi = q_*$. □

Remark 2.7. We could use any other characteristic zero ring instead of $\mathbb{Z}$ for the coefficients in S_K. We will sometimes use $\mathbb{C}$ instead.

Proposition 2.8. *Suppose G is a finitely generated group. Then G is residually K–finite if and only if the map $\varphi_K \colon G \to K(S_K/I_K)$ above is an injection.*

Proof. By Corollary 2.5, if G is residually K–finite, there exists a faithful $\rho \colon G \to K(R)$ for some R so that, as ρ factors through φ_K, φ_K too is an injection. Conversely, if φ_K is faithful, we apply Corollary 2.5 again with φ_K as the injection to conclude that G is residually K–finite. □

Proposition 2.9. *Let G be a finitely generated group. We have the following implications for G:*

$$\begin{array}{ccc} \textit{residually } \mathrm{SL}_2\textit{–finite} & \begin{array}{c}\Longrightarrow \\ \Longleftarrow \\ Z(G)=1\end{array} & \textit{residually } \mathrm{GL}_2\textit{–finite} \\ Z(G)\ \textit{2-t.f.} \Downarrow \Uparrow & & \Uparrow \Downarrow Z(G)=1 \\ \textit{residually } \mathrm{PSL}_2\textit{–finite} & \begin{array}{c}\Longrightarrow \\ \Longleftarrow\end{array} & \textit{residually } \mathrm{PGL}_2\textit{–finite.} \end{array}$$

Hereby recall that 2-t.f. means 2-torsion-free and $Z(G)$ denotes the centre of G.

Note that in passing across these implications, it may be necessary to alter the ring over which the relevant matrix group is considered when one considers the associated faithful representations. We give a simple example. The group $\mathrm{SL}_2(\mathbb{C})$ contains a unique element of order two and hence has no embedding of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Now $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ embeds into $\mathrm{PSL}_2(\mathbb{C})$, but this embedding cannot be lifted to one into $\mathrm{SL}_2(\mathbb{C})$.

Proof. Throughout the proof let G be a finitely generated group. It is clear that residual SL_2–finiteness and residual PSL_2–finiteness imply, respectively, residual GL_2–finiteness and residual PGL_2–finiteness. To see that if G is 2-torsion-free, residual SL_2–finiteness implies residual PSL_2–finiteness, note that via Corollary 2.5 the former gives us a faithful representation into $\mathrm{SL}_2(R)$ for some R and 2-torsion-freeness of $Z(G)$ implies that the image of this representation cannot contain non-identity scalar matrices. A similar proof shows that if G is centreless, residual GL_2–finiteness implies residual PGL_2–finiteness.

Next, we show that residual PGL_2–finiteness implies residual PSL_2–finiteness. Let $\{g_1, \ldots, g_k\}$ be a generating set for G; via Corollary 2.5, we have a faithful $\rho \colon G \to \mathrm{PGL}_2(R)$ for some R. Choose representatives of the generators $\rho(g_1), \ldots,$

$\rho(g_k)$ of $\rho(G)$, let $a_i = \det(\rho(g_i))$, and let $R' = R[x_1,\dots,x_k]/I$ where $I = (x_1^2 - a_1^{-1},\dots,x_k^2 - a_k^{-1})$.

Claim. The obvious map $\epsilon\colon R \to R'$ is injective.

Recall that by definition $R[x_1,\dots,x_k]$ is the free R-module on the monomials $\prod x_i^{n_i}$. We denote by φ the R-module homomorphism $R[x_1,\dots,x_k] \to R$ that is uniquely determined by

$$\prod_{i=1}^m x_i^{n_i} \mapsto \begin{cases} 0, & \text{if one of the } n_i \text{ is not even,} \\ \prod_{i=1}^m a_i^{-n_i/2}, & \text{if all of the } n_i \text{ are even.} \end{cases}$$

We claim that φ vanishes on I. Since φ is R-linear it suffices to show that for any j and any monomial $\prod x_i^{n_i}$ we have $\varphi\big((x_j^2 - a_j^{-1})\prod x_i^{n_i}\big) = 0$. But this follows easily from considering separately the two cases in which the n_i are all even and in which one is not even. It is clear that for any $r \in R$ we have $\varphi(\epsilon(r)) = r$. This shows that ϵ is injective. This concludes the proof of the claim.

It follows from the claim that the $\iota\colon \mathrm{PGL}_2(R) \to \mathrm{PGL}_2(R')$ which applies the previous map $\epsilon R \to R'$ to each entry is injective. This gives us a faithful representation $\iota \circ \rho\colon G \to \mathrm{PGL}_2(R')$. For each i, choosing the same representatives of the $\rho(g_i)$ as earlier we note that the representative $x_i(\iota\circ\rho)(g_i)$ has unit determinant. Thus the image of $\iota\circ\rho$ lies in the copy of $\mathrm{PSL}_2(R')$ inside $\mathrm{PGL}_2(R')$.

Finally we will show that residual PSL_2–finiteness implies residual SL_2–finiteness. This will, using the other implications proven so far, show also that, under the same conditions, residual PGL_2–finiteness implies residual GL_2–finiteness. To show this, we show that, given a representation $\rho\colon G \to \mathrm{PSL}_2(R)$, there exist an R' and a map $\varphi\colon G \to \mathrm{SL}_2(R')$ through which ρ factors. This will complete the proof because if G is residually PSL_2–finite, it admits a faithful $\rho\colon G \to \mathrm{PSL}_2(R)$; this ρ factors through a representation $\rho'\colon G \to \mathrm{SL}_2(R')$ which is then also faithful, and so G is residually SL_2–finite. The construction involved is the same as that for the $K = \mathrm{PSL}_2$ case in the proof of Proposition 2.6. Let $G = \langle g_1,\dots,g_n \mid \{r_i\}_{i\in I}\rangle$, let

$$S = \mathbb{Z}[x_{1a}, x_{1b}, x_{1c}, x_{1d}, \dots, x_{na}, x_{nb}, x_{nc}, x_{nd}, \{\lambda_i\}_{i\in I}],$$

and then define

$$p(g_i) = \begin{pmatrix} x_{ia} & x_{ib} \\ x_{ic} & x_{id} \end{pmatrix}, \quad p(g_i^{-1}) = \begin{pmatrix} x_{id} & -x_{ib} \\ -x_{ic} & x_{ia} \end{pmatrix}.$$

Define also $p(r_i)$ by setting p to be multiplicative and then define

$$I = \left\langle \{\det p(g_i) - 1\}_i \cup \{\lambda_i^2 - 1\}_i \cup \{(p(r_i) - \lambda_i)_{k,l}\}_{i,k,l} \right\rangle.$$

Now set $R' = S/I$ and

$$\varphi\colon G \to \mathrm{SL}_2(R')\colon g_i \mapsto \begin{pmatrix} \overline{x_{ia}} & \overline{x_{ib}} \\ \overline{x_{ic}} & \overline{x_{id}} \end{pmatrix},$$

which, as can be checked, gives a homomorphism. Now, given $\rho\colon G \to \mathrm{PSL}_2(R)$, let $(a^i_{kl})_{kl}$ be representatives for $\rho(g_i)$ and let $\mu_i \in R^\times$ be the element such that the corresponding representative for $\rho(r_i)$ is equal to μ_i times the identity matrix. Note that $\mu_i^2 = 1$. Define $q\colon R' \to R\colon 1, x_{ia}, x_{ib}, x_{ic}, x_{id}, \lambda_i \mapsto 1, a^i_{11}, a^i_{12}, a^i_{21}, a^i_{22}, \mu_i$ and set $\psi = q_*\colon \mathrm{SL}_2(R') \to \mathrm{SL}_2(R) \to \mathrm{PSL}_2(R)$. Then $\rho = \psi\circ\varphi$. □

Corollary 2.10.

(1) *If the fundamental group of a compact* 3*–manifold is residually* PSL_2*–finite or residually* PGL_2*–finite, it is also residually* K*–finite for the other* K*.*
(2) *If* M *is an aspherical* 3*–manifold that is not a Seifert fibred manifold, then all of the above four notions of residually finiteness agree.*

Proof. The first part follows from Proposition 2.9 and the observation that all compact 3–manifold groups are finitely presented; for a proof of this latter fact, see [9], where it is shown that compact topological manifolds have the homotopy type of a finite CW-complex.

The second statement follows again from Proposition 2.9 and the fact that fundamental groups of aspherical 3–manifolds are torsion-free and that the only 3–manifolds with a non-trivial center are Seifert fibred manifolds. We refer to [1, Theorem 2.5.5, (C.3)] for proofs of these two statements. □

3. A trip to the zoo

In this section we examine Luo's conjectured property for various classes of groups, using the results from the previous section.

3.1. Symmetric groups.

Luo conjectured that every compact 3–manifold group is residually PGL_2–finite. As a first observation, recall that every compact 3–manifold group is residually finite. See [7] for the case of Haken manifolds, which can be extended to the general case via geometrisation as discussed in [7] and [19, Theorem 3.3]. Correctness of Luo's conjecture would provide a list of specific finite groups which detect non-triviality. Now, as finite groups embed into symmetric groups, if we had that S_n, the symmetric group on n letters, is residually PGL_2–finite for all n, we would have verified Luo's conjecture. However, we have the following result, which was obtained independently in [13] by the same method.

Theorem 3.1. S_n *is residually* PGL_2*–finite if and only if* $n < 5$*.*

Proof. Given positive integers $n < m$, S_n embeds into S_m (as the stabilizer of the final $m - n$ letters). Thus it suffices to prove that S_4 is residually PSL_2–finite and that S_5 is not. The former follows from the fact that S_4 is isomorphic to $\mathrm{PSL}_2(\mathbb{F}_3)$, where $\mathbb{F}_3$ is the field with 3 elements; see [16, Chapter 8] (the isomorphism arises from the faithful natural action of the latter on the projective line $\mathbb{P}^1(\mathbb{F}_3)$, which has 4 elements). For the latter, we use the following presentation for S_5:

$$\left\langle x_1, x_2, x_3, x_4 \;\middle|\; \begin{array}{ll} x_i^2 = 1 & 1 \le i \le 4, \\ (x_i x_{i+1})^3 = 1 & 1 \le i < 3, \\ (x_i x_j)^2 = 1 & 1 \le i < j - 1 \le 3 \end{array} \right\rangle,$$

where x_i is the transposition $(i\ \ i{+}1)$. This is a particular case of Moore's presentations for the symmetric groups; see [14]. The required result is verified using the characterisation of residual PSL_2–finiteness provided by Proposition 2.8 above. A computation using SageMath [17] certified that S_5 fails residual PSL_2–finiteness in particular for the element $x_1 x_2$. The code can be found in [6]. It follows from Proposition 2.9 that S_5 also fails residual PGL_2–finiteness. □

Remark 3.2. The alternating groups A_5 and A_6 are residually PSL_2–finite. This follows from the existence of isomorphisms $A_5 \cong \mathrm{PSL}_2(\mathbb{F}_4) \cong \mathrm{PSL}_2(\mathbb{F}_5)$ and $A_6 \cong \mathrm{PSL}_2(\mathbb{F}_9)$, where $\mathbb{F}_4$, $\mathbb{F}_5$, and $\mathbb{F}_9$ are the fields with 4, 5, and 9 elements, respectively;

see [16, Chapter 8]. The property of being residually PSL_2–finite is therefore not inherited from finite index subgroups, even in the case of index two.

3.2. **General linear groups.** Note that S_n embeds into $\mathrm{GL}_n(R)$ for any non-zero commutative ring with identity R by mapping each permutation to the corresponding permutation matrix. Thus if we define *residually* PGL_n*–finite* in a manner similar to that in Definition 1.2, we find that S_n is residually PGL_n–finite. To see this, note that the canonical surjection $\mathrm{GL}_n(R) \to \mathrm{PGL}_n(R)$ is injective on the copy of S_n in $\mathrm{GL}_n(R)$. Thus we see that Luo's conjecture holds if we weaken it to allow arbitrary dimension of matrices. On the other hand, the observation that S_n embeds into $\mathrm{GL}_n(R)$, along with the above theorem, also gives the following:

Corollary 3.3. *For any commutative ring with identity R,* $\mathrm{GL}_n(R)$ *is not residually* PGL_2*–finite for $n \geq 5$.*

This raises the following question.

Question 3.4. Let R be a ring, let $n \in \mathbb{N}$ and $k < n$. Is it possible that $\mathrm{GL}_n(R)$ is residually GL_k–finite?

3.3. **Abelian groups.**

Proposition 3.5. *Every finitely generated abelian group G is residually K–finite for all $K = \mathrm{SL}_2$, GL_2, PSL_2, PGL_2.*

Proof. It is easy to see that $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ are residually K–finite for each K via matrices of the form

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

where the ring of entries is $\mathbb{Z}$ in the case of $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ in the case of $\mathbb{Z}/n\mathbb{Z}$. Now, given a finitely generated abelian group G, decompose G via the classification theorem for finitely generated abelian groups and then use the projections onto each factor. □

Thus, given a finitely generated group G and an element $g \in G$ that is not contained in the commutator subgroup $[G,G]$, by passing to the abelianisation, we can construct a finite commutative ring R and a homomorphism $G \to K(R)$ that does not kill g. As such, *it is only elements in the commutator subgroup that we ever need to worry about.*

3.4. **Dihedral groups.** Denote by $D_{2k} = \langle a, b \mid a^k = b^2 = 1, bab = a^{-1} \rangle$ the dihedral group of order $2k$. A faithful representation $D_{2k} \to \mathrm{PSL}_2(\mathbb{C})$ is defined by

$$a \mapsto \pm \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \quad \text{and} \quad b \mapsto \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $\xi = \exp(\pi i/k)$. Whence D_{2k} is residually K–finite for all $K = \mathrm{SL}_2$, GL_2, PSL_2, PGL_2. This family includes examples with 2-torsion in their centre.

3.5. **Finitely generated free groups and surface groups.**

Proposition 3.6. *Finitely generated free groups and the fundamental groups of compact, connected surfaces (possibly with non-empty boundary) are residually K–finite for all $K = \mathrm{SL}_2$, GL_2, PSL_2, PGL_2.*

Proof. The case of finitely generated free groups corresponds to the case of surfaces with non-empty boundary. It therefore suffices to consider surfaces. If the Euler characteristic is non-negative, then the fundamental group is either abelian or it is the fundamental group of a Klein bottle. In the first case, the conclusion follows from Proposition 3.5, and in the second from Proposition 3.8 below.

If the Euler characteristic is negative, then the fundamental group is torsion-free and the holonomy representation for a complete hyperbolic structure gives a faithful representation into $\mathrm{PSL}_2(\mathbb{R})$. Whence we are done upon application of Corollary 2.5 and Proposition 2.9. □

3.6. Hyperbolic manifolds and trivial product geometries.

Proposition 3.7. *If the compact orientable geometric* 3*–manifold* M *is modelled on* $\mathbb{H}^3$*,* $\mathbb{H}^2 \times \mathbb{E}$*, or* $\mathbb{S}^2 \times \mathbb{E}$*, then* $\pi_1 M$ *is residually* SL_2*–,* PSL_2*–,* GL_2*–,* PGL_2*–finite.*

Proof. If M is modelled on $\mathbb{H}^3$, the holonomy representation gives an embedding of $\pi_1(M)$ into $\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2,\mathbb{C})$. Thus $\pi_1(M)$ is residually PSL_2–finite, and Proposition 2.9 completes the result. Note here that another way to see that $\pi_1(M)$ is also residually SL_2–finite is via a well-known result due to Thurston that we can lift the holonomy representation into $\mathrm{PSL}(2,\mathbb{C})$ to one into $\mathrm{SL}(2,\mathbb{C})$; see [18].

If M is modelled on $\mathbb{H}^2 \times \mathbb{E}$, the holonomy representation gives an embedding of $\pi_1(M)$ into $\mathrm{Isom}^+(\mathbb{H}^2 \times \mathbb{E}^1) = \mathrm{Isom}^+(\mathbb{H}^2) \times \mathrm{Isom}^+(\mathbb{E}^1) \cong \mathrm{PSL}(2,\mathbb{R}) \times \mathbb{R}$. Note that $\mathbb{R}$ embeds into $\mathrm{PSL}(2,\mathbb{R})$, exactly via the usual matrix representation for $v \in \mathrm{Isom}^+(\mathbb{E}^1)$ as

$$\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}.$$

Now we post-compose the above embedding with the projections of $\mathrm{PSL}_2(\mathbb{R}) \times \mathbb{R}$ onto either factor and then use Proposition 2.2. This gives us that $\pi_1(M)$ is residually PSL_2–finite, and Corollary 2.10 does the rest of the work.

If M is modelled on $\mathbb{S}^2 \times \mathbb{E}$ and orientable, the holonomy representation gives an embedding of $\pi_1(M)$ into $\mathrm{Isom}^+(\mathbb{S}^2) \times \mathrm{Isom}^+(\mathbb{E}) \cong \mathrm{SO}(3,\mathbb{R}) \times \mathbb{R}$. Again, due to the presence of the projections, we need only worry about the two factors in the product, and we can deal with the $\mathbb{R}$ factor as we did in the previous case. To deal with the $\mathrm{SO}(3,\mathbb{R})$ factor, we recall the well-known double covering $\mathrm{SU}(2,\mathbb{C}) \to \mathrm{SO}(3,\mathbb{R})$ which gives an isomorphism $\mathrm{SO}(3,\mathbb{R}) \cong \mathrm{PSU}(2,\mathbb{C}) \leq \mathrm{PSL}_2(\mathbb{C})$, and so using Proposition 2.2 and Corollary 2.10, we conclude that $\pi_1(M)$ satisfies the conclusion. □

We briefly discuss the remaining five Thurston geometries, which are not treated in this note for the sake of brevity. The fundamental group of a closed, orientable Seifert fibred manifold M fits into an exact sequence

$$\pi_1(S^1) \to \pi_1(M) \to \pi_1^{orb} B \to 1,$$

where B is the (possibly non-orientable) base orbifold associated to the fibration and $\pi_1^{orb} B$ is the orbifold-fundamental group. For a uniform treatment of Seifert fibred manifolds it suffices to establish a result for 2–orbifolds analogous to the one above for surfaces, and hence one only needs to worry about the cyclic subgroup generated by the fibre. In the case of certain spherical manifolds, one encounters central 2–torsion, resulting in an additional technical obstacle. The remaining class of geometric 3–manifolds are solvmanifolds. These are either mapping tori of Anosov automorphisms of the 2–torus or double covered by such a mapping torus (see [1, Theorem 1.8.2]). In particular the fundamental group has a simple description which would be amenable to a direct approach.

3.7. **The integral Heisenberg group and Klein bottle group.** We now consider the integral Heisenberg group $\mathfrak{H}$ and the Klein bottle group $\mathfrak{K}$, which have the well-known presentations $\langle a, b, c \mid [a, b] = c, c \text{ central}\rangle$ and $\langle a, b \mid aba^{-1}b = 1\rangle$ respectively.

Proposition 3.8. *The groups $\mathfrak{H}$ and $\mathfrak{K}$ are residually PSL_2–finite and so residually K–finite for each of $K = \mathrm{SL}_2, \mathrm{GL}_2, \mathrm{PSL}_2, \mathrm{PGL}_2$.*

Proof. The second statement follows from Corollary 2.10(1) once we establish the first. We do this by producing embeddings $\mathfrak{H} \to \mathrm{PSL}_2(R), \mathfrak{K} \to \mathrm{PSL}_2(S)$ and invoking Corollary 2.5. Our ring R is the ring $\mathbb{C}[x, y, z]/I$, where $I := (x(1-y^2)^2, yz - 1)$, and we first define a representation $\rho\colon \mathfrak{H} \to \mathrm{SL}_2(R)$ as follows:

$$a \mapsto \begin{pmatrix} 1 & \overline{x} \\ 0 & 1 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} \overline{y} & 0 \\ 0 & \overline{z} \end{pmatrix}, \quad c \mapsto \begin{pmatrix} 1 & \overline{x(1-y^2)} \\ 0 & 1 \end{pmatrix}.$$

A direct calculation shows that this is a well-defined representation. An element of $\mathfrak{H}$ is a word w in the letters a, b, c. Because c is central, one can write $w = w'c^q$ for some $q \in \mathbb{Z}$ and word w' in a, b. Because $[a, b] = c$ and so $ab = c(ba)$, one can interchange a, b in w' at the cost of introducing a c which can once again be pushed off to the right so that one can write $w = a^m b^n c^p$ for some $m, n, p \in \mathbb{Z}$. By direct computation, one can check that

$$w = a^m b^n c^p \mapsto \begin{pmatrix} \overline{y}^n & \overline{px(1-y^2)}\,\overline{y}^n + \overline{mx}\,\overline{z}^n \\ 0 & \overline{z}^n \end{pmatrix}.$$

Now we note the following, which shows that ρ is faithful:

(H1) We have $x(1-y^2) \notin I$ (this was verified with SageMath; see [6]).

(H2) We claim that $\overline{y}^n \neq \pm 1$ unless $n = 0$. To show this, we need to show that $y^q \pm 1, z^q \pm 1 \notin I$ for any $q \geq 1$. This is clear because, e.g., we have the homomorphism $\mathbb{C}[x, y, z] \to \mathbb{C}\colon 1, x, y, z \mapsto 1, 0, 2, 1/2$ which kills I but none of $y^q \pm 1, z^q \pm 1$ for any $q \geq 1$. Thus, if $n \neq 0$, we have that $\rho(w) \neq \pm\mathrm{id}$.

(H3) Suppose then that $n = 0$, so that $w = a^m c^p$. Then the top-right entry is $\overline{px(1-y^2) + mx}$. Suppose that this is zero. Upon multiplying by $\overline{(1-y^2)}$, we find that $mx(1-y^2) \in I$. By (i), we have that $m = 0$. But then we find that $px(1-y^2) \in I$ and so, again by (i), $p = 0$. Thus w must be the identity in this case, and so ρ is faithful.

Finally, we can projectivise and pass to $\mathrm{PSL}_2(R)$ without losing faithfulness because the above shows that if the diagonal elements are ever ± 1, we must have a non-zero off-diagonal element.

Next, for the Klein bottle group, we take S to be the ring $\mathbb{C}[x, y, z]/J$ where $J := (xy - 1, (x - y)z)$ and we first define a representation $\rho'\colon \mathfrak{K} \to \mathrm{SL}_2(S)$ as follows:

$$a \mapsto \begin{pmatrix} \overline{x} & 0 \\ 0 & \overline{y} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 1 & \overline{z} \\ 0 & 1 \end{pmatrix}.$$

Again, it is straightforward to verify that this is well-defined. An element of $\mathfrak{K}$ is a word w in the letters a, b. Because $ab = b^{-1}a$, we can commute a and b at the cost of altering exponents. Thus one can write $w = a^m b^n$ for some $m, n \in \mathbb{Z}$. By direct computation, one can check that

$$w = a^m b^n \mapsto \begin{pmatrix} \overline{x}^m & \overline{x}^m \overline{nz} \\ 0 & \overline{y}^m \end{pmatrix}.$$

Now we note the following, which shows that ρ' is faithful:

(K1) We claim that $\overline{x}^m \neq \pm 1$ unless $m = 0$. To show this, we need to show that $x^q \pm 1, y^q \pm 1 \notin J$ for any $q \geq 1$. This is clear because, e.g., we have the homomorphism $\mathbb{C}[x, y, z] \to \mathbb{C}\colon 1, x, y, z \mapsto 1, 2, 1/2, 0$ which kills J but none of $x^q \pm 1, y^q \pm 1$ for any $q \geq 1$. Thus, if $m \neq 0$, we have that $\rho'(w) \neq \pm \mathrm{id}$.

(K2) Suppose then that $m = 0$, so that $w = b^n$. Then the top-right entry is $\overline{nz}$. We claim that this is non-zero unless $n = 0$. To show this, we claim that $z \notin J$, which follows because, e.g., we have the homomorphism $\mathbb{C}[x, y, z] \to \mathbb{C}\colon 1, x, y, z \mapsto 1, 1, 1, 2$ which kills J but not z. Thus, if the image is to be trivial, w must be the identity, and so ρ' is faithful.

Finally, again, we can projectivise and pass to $\mathrm{PSL}_2(S)$ without losing faithfulness because the above shows that if the diagonal elements are ever ± 1, we must have a non-zero off-diagonal element. □

Note that the combination of (H1) and (H2) in the proof shows that R is not an integral domain. Similarly, as the homomorphism in (K1) also shows that $x - y \notin J$, the combination of (K1) and (K2) shows that S is also not an integral domain. In fact, no such faithful representations $\mathfrak{H} \to \mathrm{SL}_2(R')$ or $\mathfrak{K} \to \mathrm{SL}_2(S')$ exist for any integral domains R', S', or, equivalently, for any fields. To see this, given any field $\mathbb{F}$, set the following:

$$D_{\mathbb{F}} := \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \middle| \, x \in \mathbb{F} \setminus \{0\} \right\}, \qquad U_{\mathbb{F}} := \left\{ \begin{pmatrix} \varepsilon & x \\ 0 & \varepsilon \end{pmatrix} \middle| \, \varepsilon \in \{-1, 1\} \text{ and } x \in \mathbb{F} \right\}.$$

We have the following lemma, which can be proved via straightforward calculations:

Lemma 3.9. *Let $\mathbb{F}$ be a field. Let $A, B \in \mathrm{SL}_2(\mathbb{F})$ be commuting matrices. Then:*

(i) *If $A \in D_{\mathbb{F}}$ with $A \neq \pm \mathrm{id}$, then $B \in D_{\mathbb{F}}$.*

(ii) *If $A \in U_{\mathbb{F}}$ with $A \neq \pm \mathrm{id}$, then $B \in U_{\mathbb{F}}$.*

Proposition 3.10. *Let $\mathbb{F}$ be a field and consider representations $\rho\colon \mathfrak{H} \to \mathrm{SL}_2(\mathbb{F})$ and $\sigma\colon \mathfrak{K} \to \mathrm{SL}_2(\mathbb{F})$. If $\rho(c) \neq \pm \mathrm{id}$, then one of the following occurs:*

(i) *ρ factors through the abelianisation of $\mathfrak{H}$.*

(ii) *$\rho(a) = \pm \mathrm{id}$ or $\rho(b) = \pm \mathrm{id}$.*

Similarly, if $\sigma(b) \neq \pm \mathrm{id}$, then one of the following occurs:

(iii) *σ factors through the abelianisation of $\mathfrak{K}$.*

(iv) *$\sigma(a^2) = \pm \mathrm{id}$.*

In particular, $\mathfrak{H}$ and $\mathfrak{K}$ do not admit faithful representations into $\mathrm{SL}_2(\mathbb{F})$ for any field $\mathbb{F}$.

Proof. Let $\rho\colon \mathfrak{H} \to \mathrm{SL}_2(\mathbb{F})$ be a representation with $\rho(b) \neq \pm \mathrm{id}$ and $\rho(a), \rho(b) \neq \pm \mathrm{id}$. Let $\overline{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$ and then postcompose with the inclusion $\mathrm{SL}_2(\mathbb{F}) \to \mathrm{SL}_2(\overline{\mathbb{F}})$. We have some $A \in \mathrm{SL}_2(\overline{\mathbb{F}})$ such that under the composite

$$\rho'\colon \mathfrak{K} \xrightarrow{\rho} \mathrm{SL}_2(\mathbb{F}) \xrightarrow{\subseteq} \mathrm{SL}_2(\overline{\mathbb{F}}) \xrightarrow{A(-)A^{-1}} \mathrm{SL}_2(\overline{\mathbb{F}})$$

we have $\rho'(c) \in D_{\overline{\mathbb{F}}}$ or $\rho'(c) \in U_{\overline{\mathbb{F}}}$. We first consider the case that $\rho'(c) \in D_{\overline{\mathbb{F}}}$. Since in $\mathfrak{H}$, c commutes with a and b, it follows from Lemma 3.9 and the assumption that $\rho(c) \neq \pm\mathrm{id}$ that $\rho'(a), \rho'(b) \in D_{\overline{\mathbb{F}}}$. This implies that ρ' factors through the abelianisation of $\mathfrak{H}$, and, upon inverting the conjugation, it follows that ρ itself then factors through the abelianisation of $\mathfrak{H}$. The case that $\rho'(c) \in U_{\overline{\mathbb{F}}}$ can be treated in an analogous manner.

In the case of $\mathfrak{K}$, note that a^2 and b commute. Let $\sigma\colon \mathfrak{K} \to \mathrm{SL}_2(\mathbb{F})$ be a representation with $\sigma(b) \neq \pm\mathrm{id}$ and $\sigma(a^2) \neq \pm\mathrm{id}$. Again by passing to an algebraic closure and conjugating, we can assume that $\sigma(b) \in D_{\overline{\mathbb{F}}}$ or $\sigma(b) \in U_{\overline{\mathbb{F}}}$ and need only consider the first case. It follows from Lemma 3.9 and the assumption that $\sigma(b) \neq \pm\mathrm{id}$ that $\sigma(a^2) \in D_{\overline{\mathbb{F}}}$. Since $\sigma(a)$ also commutes with $\sigma(a^2)$ and $\sigma(a^2) \neq \pm\mathrm{id}$, we have $\sigma(a) \in D_{\overline{\mathbb{F}}}$. This implies that σ factors through the abelianisation of $\mathfrak{K}$. $\square$

3.8. **Quaternionic space.** This example illustrates the use of Luo's construction to obtain positive results using triangulations. Figure 1 below depicts an oriented triangulation of quaternionic space S^3/Q_8 from Regina, [3], where it is identified as "SFS [S2: (2,1) (2,1) (2,-1)]: #1" in "Closed Census (Orientable)". The orientations on the simplices here are $v_i \to v_{i+1}$ and $v'_i \to v'_{i+1}$. The action of Q_8 on S^3 is the natural one after identifying S^3 with $\{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$ and Q_8 with a subgroup of $\mathrm{SL}_2(\mathbb{C})$ via

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The face-pairings, specified via the vertices v_i, v'_i, are as follows: $\varphi_1\colon v_0, v_1, v_2 \mapsto v'_3, v'_0, v'_1$; $\varphi_2\colon v_0, v_1, v_3 \mapsto v'_1, v'_2, v'_0$; $\varphi_3\colon v_0, v_2, v_3 \mapsto v'_2, v'_0, v'_3$; $\varphi_4\colon v_1, v_2, v_3 \mapsto v'_3, v'_2, v'_1$. From these one can compute the edge cycles, i.e., the cyclic sequences of edges identified to one another; these are indicated by colour in Figure 1. We search for solutions to Thurston's equations by labelling the edges of our triangulation by arbitrary elements of some ring R as in Figure 1.

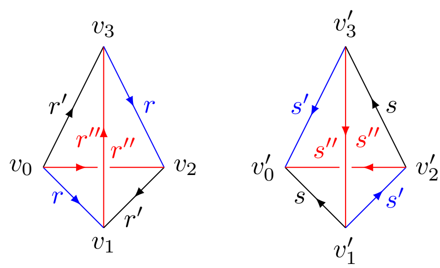

FIGURE 1. Triangulation of quaternionic space S^3/Q_8

Following the edge cycles, the gluing equations are then: $r^2(s')^2 = 1, (r')^2 s^2 = 1, (r'')^2(s'')^2 = 1$. Combining these with the parameter relations, one finds that the parameter relations together with $r^2(s')^2 = 1, 2(rs'-1) = 0$ are necessary and sufficient conditions on the labels. Thus in the case of a ring in which 2 is *not* a zero divisor, we have $s' = r^{-1}$, and the following, for $r \neq 0, 1$, are all the solutions to Thurston's equations:

$$(1) \qquad (r, r', r'', s, s', s'') = \left(r, \frac{1}{1-r}, \frac{r-1}{r}, 1-r, \frac{1}{r}, \frac{r}{r-1}\right).$$

Allowing ourselves extra flexibility, there are other possible solutions. For example, setting $R = \mathbb{F}_4[x]/(x^2)$, where $\mathbb{F}_4 = \{0, 1, a, b\}$ is the field with four elements and $r = a$, $s' = b + x$, gives a solution to Thurston's equations, namely,

$$(r, r', r'', s, s', s'') = (a, a, a, b + bx, b + x, b + ax). \tag{2}$$

Now we compute the associated holonomy representations. First, consider the generic solution (1) over $\mathbb{C}$, for $r = z$. As in [10], we first build a corresponding solution to the homogeneous Thurston equations and then the holonomy representation ρ, choosing $q_0 = \{\{v_0, v_1\}, \{v_2, v_3\}\}$ and $q_0' = \{\{v_0', v_1'\}, \{v_2', v_3'\}\}$ as the initial normal quadrilaterals and $[1, 0]^t$, $[0, 1]^t$, $[1, 1]^t$ as the initial labels for v_0, v_1, v_2. Doing so, we find that the resulting labels for v_3, v_0', v_1', v_2', and v_3' are, respectively, $[1, z]^t$, $[0, 1]^t$, $[1, 1]^t$, $[1, z]^t$, $[1, 0]^t$. The images of the generators φ_2, φ_3, φ_4, or more precisely the elements of $\pi_1(S^3/Q_8) \cong Q_8$ which they represent, are then as follows:

$$\rho(\varphi_2) = \begin{bmatrix} z & -1 \\ z & -z \end{bmatrix}, \quad \rho(\varphi_3) = \begin{bmatrix} 1 & -1 \\ z & -1 \end{bmatrix}, \quad \rho(\varphi_4) = \begin{bmatrix} 0 & 1 \\ z & 0 \end{bmatrix}.$$

It is clear that $\rho(\varphi_2), \rho(\varphi_3), \rho(\varphi_4)$ are pairwise distinct and one can check that any two of these (in either order) multiply to give the third. Thus, for any $z \neq 0, 1$, the image of the holonomy representation is the Klein four group.

Consider now the solution (2) over $\mathbb{F}_4[x]/(x^2)$. We will see that we can achieve a larger image by not working over $\mathbb{C}$ and using this labelling. Repeating the above procedure, we find that $v_0, v_1, v_2, v_3, v_0', v_1', v_2', v_3'$ receive the labels $[1, 0]^t$, $[0, 1]^t$, $[1, 1]^t$, $[1, a]^t$, $[0, 1]^t$, $[1, 1]^t$, $[1, a + bx]^t$, $[1, 0]^t$, respectively. The associated holonomy representation ρ' is generated by

$$\rho'(\varphi_2) = \begin{bmatrix} a & 1 \\ a & a + bx \end{bmatrix}, \ \rho'(\varphi_3) = \begin{bmatrix} 1 & 1 \\ a + bx & 1 + ax \end{bmatrix}, \ \rho'(\varphi_4) = \begin{bmatrix} x & 1 + x \\ a + bx & 0 \end{bmatrix}.$$

It can now be checked that

$$\rho'(\varphi_2)^2 = \rho'(\varphi_3)^2 = \rho'(\varphi_4)^2 = \begin{bmatrix} 1 & bx \\ x & 1 \end{bmatrix},$$

and if we denote this common square J, that $J^2 = 1$ and $\rho'(\varphi_2)\rho'(\varphi_3)\rho'(\varphi_4) = J$. It follows that this holonomy representation is faithful with image isomorphic to Q_8, where an explicit isomorphism is given by $J, \rho'(\varphi_2), \rho'(\varphi_3), \rho'(\varphi_4) \mapsto -1, i, j, k$.

4. A counterexample to Luo's conjecture

Let M be the $(4, 1)$-Dehn filling, using the knot theoretic framing, of the figure-8 knot complement. We will show that M is a counterexample to Luo's conjecture. In SnapPy [5] one can construct a triangulation of M, and this triangulation can then be imported into Regina [3]. Regina then gives the following presentation for $\Gamma = \pi_1(M)$:

$$\Gamma = \langle a, b \mid a^{-1}b^2a^{-3}b^2 = 1, ba^{-2}ba^{-2}b^3a^{-2} = 1\rangle.$$

We rewrite this presentation by making the substitutions $a \rightsquigarrow b^{-1}$, $b \rightsquigarrow a^{-1}$ and set $c = b^2a^{-2}$. This leads to the following presentation:

$$\Gamma = \langle a, b, c \mid ca^2 = b^2, c^{-1}b = bc, ac^{-1}a^{-1} = cac\rangle.$$

This can be rewritten as

$$\langle a, b, c \mid c = b^2 a^{-2}, \underbrace{1 = bcb^{-1}c}_{\text{Klein bottle}}, \overbrace{a^2 = (ac)^3}^{\text{trefoil complement}} \rangle,$$

which highlights the presence of a trefoil knot complement and a Klein bottle. In fact, M can be constructed as the identification of a trefoil knot complement and a twisted I-bundle over a Klein bottle. Letting $\Gamma_1 = \langle u, v \mid u^3 = v^2 \rangle$ and $\Gamma_2 = \langle j, k \mid jkj^{-1}k = 1 \rangle$, the peripheral subgroups $\langle v^{-1}u, u^3 \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$ and $\langle k, j^2 \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$ are glued via the identifications $v^{-1}u \leftrightarrow k$, $u^3 \leftrightarrow k^{-1}j^2$.

We return to the second presentation for Γ above and construct the universal representation of Proposition 2.6. This gives $\varphi_{\mathrm{SL}_2} \colon \Gamma \to \mathrm{SL}_2(S_{\mathrm{SL}_2}/I_{\mathrm{SL}_2})$, where $S_{\mathrm{SL}_2} = \mathbb{Z}[i, j, k, l, p, q, r, s, w, x, y, z]$,

$$a \mapsto \begin{pmatrix} i & j \\ k & l \end{pmatrix}, \qquad b \mapsto \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \qquad c \mapsto \begin{pmatrix} w & x \\ y & z \end{pmatrix},$$

and the ideal I_{SL_2} is generated by $il - kj - 1, ps - rq - 1, wz - yx - 1$, and 12 equations arising from the relations. We find that b^4 has image

$$\begin{pmatrix} (p^2 + qr)^2 + qr(p+s)^2 & q(p+s)(p^2 + 2qr + s^2) \\ r(p+s)(p^2 + 2qr + s^2) & qr(p+s)^2 + (qr + s^2)^2 \end{pmatrix} =: \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}.$$

It can be verified via SageMath [17] that $f_1 - 1, f_2, f_3, f_4 - 1 \in I_{\mathrm{SL}_2}$ so that $\varphi_{\mathrm{SL}_2}(b^4) = 1$ and hence φ_{SL_2} is not injective. See [6]. Thus b^4 is killed in any representation $\Gamma \to \mathrm{SL}_2(R)$, Γ is not residually SL_2–finite and so, using the argument of the proof of Proposition 2.9, is also not residually PSL_2–finite. Note that by §3.7, the obstruction here is not produced by the presence of the Klein bottle subgroup.

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