PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 11, November 2018, Pages 4693–4705
http://dx.doi.org/10.1090/proc/14005
Article electronically published on August 10, 2018

ON THE OPTIMALLY DEFINED HARDY OPERATOR IN L^p-SPACES

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(Communicated by Thomas Schlumprecht)

ABSTRACT. For each $1 < p < \infty$, the optimal extension of the classical Hardy operator from $L^p(\mathbb{R}^+)$ into itself has been identified by Delgado and Soria. By relaxing the target space to be $L^p_{loc}(\mathbb{R}^+)$ we determine the optimal Hardy operator which maps into this target space.

1. INTRODUCTION AND SUMMARY OF RESULTS

Let $f \in L^1_{loc}(\mathbb{R}^+)$ with $\mathbb{R}^+ := [0,\infty)$. The Cesàro average of f is the function $Cf \in L^0(\mathbb{R}^+)$ defined by

$$Cf : x \mapsto \frac{1}{x}\int_0^x f(y)\,dy, \qquad x \in (0,\infty), \tag{1.1}$$

where $L^0(\mathbb{R}^+)$ is the space of all $\mathbb{C}$-valued, measurable functions defined on $\mathbb{R}^+$. For some authors the linear map $C : f \mapsto Cf$ is called the (infinite range) *Cesàro operator*, [7], [8], [21], and for others it is called the *Hardy operator*, [6], [14], [16]. Evidently C is an integral operator with kernel $K(x,y) := \frac{1}{x}\chi_{(0,x]}(y)$ on $\mathbb{R}^+ \times \mathbb{R}^+$. The boundedness of the operator $C : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$, for $1 < p < \infty$, denoted by C_p for the sake of clarity, is due to G. H. Hardy, [17, p. 240], who showed that its operator norm is $\|C_p\|_{\mathrm{op}} = p'$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Considering C_p as an operator on the (complex) Banach function space (briefly, B.f.s.) $L^p(\mathbb{R}^+)$, it is clear from (1.1) that C_p is a positive operator, i.e., $C_p f \geq 0$ for every function $0 \leq f \in L^p(\mathbb{R}^+)$. The spectrum $\sigma(C_p)$ and the point spectrum $\sigma_{pt}(C_p)$ of C_p are known, [7, Theorem 2], [8], [21, p. 28]; namely,

$$\sigma(C_p) = \Big\{\lambda \in \mathbb{C} : |\lambda - \frac{p'}{2}| = \frac{p'}{2}\Big\} \text{ and } \sigma_{pt}(C_p) = \emptyset. \tag{1.2}$$

At the level of individual functions we see that $C : L^1_{loc}(\mathbb{R}^+) \to L^0(\mathbb{R}^+)$. Fixing the target space to be the B.f.s. $L^p(\mathbb{R}^+) \subseteq L^0(\mathbb{R}^+)$, $1 < p < \infty$, Hardy's inequality shows that C maps $L^p(\mathbb{R}^+) \subseteq L^1_{loc}(\mathbb{R}^+)$ continuously into this target space. Recalling that $L^p(\mathbb{R}^+)$ has *order continuous norm*, briefly o.c.-norm (i.e., order bounded, increasing sequences of $\mathbb{R}$-valued functions in $L^p(\mathbb{R}^+)$ are norm convergent), one may ask if there exists a *largest* B.f.s. (with o.c.-norm) inside $L^1_{loc}(\mathbb{R}^+)$ which contains $L^p(\mathbb{R}^+)$ and which C maps continuously into the target space $L^p(\mathbb{R}^+)$. Clearly

Received by the editors July 28, 2017, and, in revised form, October 30, 2017.
2010 *Mathematics Subject Classification.* Primary 46E30, 47A67; Secondary 46G10, 47B38.
Key words and phrases. Hardy operator, L^p_{loc}-space, optimal domain, vector measure, integral representation.

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this is the B.f.s.

$$[C_p, L^p(\mathbb{R}^+)] := \big\{ f \in L^1_{loc}(\mathbb{R}^+) : C|f| \in L^p(\mathbb{R}^+) \big\}, \tag{1.3}$$

equipped with the norm $\|f\| := \|C|f|\|_{L^p(\mathbb{R}^+)}$. An alternate description of this largest (or optimal) domain space is presented in [14, Proposition 3.4 & Example 3.5]. Since the appearance of [14], several authors have made a detailed analysis of the spaces (1.3) and exposed many of their Banach space properties; see, for example, [2], [3], [4], [19], [22], [23] and the references therein.

This note is inspired by an alternate description of the space (1.3) which is also presented in [14, §3]. Considering the δ-ring of sets

$$\mathcal{R} := \{A \in \mathcal{B}(\mathbb{R}^+) : \mu(A) < \infty \text{ and } \exists\, \epsilon > 0 \text{ with } \mu(A \cap [0, \epsilon]) = 0\},$$

where $\mathcal{B}(\mathbb{R}^+)$ is the σ-algebra of Borel subsets of $\mathbb{R}^+$ and μ is Lebesgue measure, it is shown that the finitely additive set function

$$\nu_p : A \mapsto C_p \chi_A, \qquad A \in \mathcal{R},$$

is actually σ-additive on $\mathcal{R}$, i.e., ν_p is an $L^p(\mathbb{R}^+)$-valued vector measure on $\mathcal{R}$. Via the theory of vector measures defined on *δ-rings* (see [12], [13], [25], [26], [27], and the references therein) it is shown that there is a B.f.s. $L^1(\nu_p)$, consisting of all the ν_p-integrable functions, having o.c.-norm in which the $\mathcal{R}$-simple functions are dense and such that $L^p(\mathbb{R}^+) \subseteq L^1(\nu_p)$ with a continuous inclusion, [14, Proposition 2.3 & Example 3.5]. Moreover, the associated integration map $I_{\nu_p} : L^1(\nu_p) \to L^p(\mathbb{R}^+)$, given by $I_{\nu_p}(f) := \int_{\mathbb{R}^+} f \, d\nu_p$, for $f \in L^1(\nu_p)$, is linear, continuous and provides an *integral representation* of C_p; namely,

$$I_{\nu_p}(f) = C_p f = \int_{\mathbb{R}^+} f d\nu_p, \qquad f \in L^p(\mathbb{R}^+) \subseteq L^1(\nu_p), \tag{1.4}$$

i.e., I_{ν_p} is an $L^p(\mathbb{R}^+)$-valued, linear extension of C_p, [13, Corollary 2.4]. Moreover, $L^1(\nu_p)$ is *optimal* with these properties. So,

$$L^1(\nu_p) = [C_p, L^p(\mathbb{R}^+)], \tag{1.5}$$

[14, Example 3.5]. According to [14, Proposition 2.3], the containment $L^p(\mathbb{R}^+) \subseteq L^1(\nu_p)$, $1 < p < \infty$, is *proper*, i.e., I_{ν_p} *genuinely* extends C_p.

The above description of the optimal domain $[C_p, L^p(\mathbb{R}^+)]$ for C_p via the vector measure ν_p defined on the *δ-ring* $\mathcal{R}$ is forced by the fact that there exist sets $A \in \mathcal{B}(\mathbb{R}^+)$ for which $\chi_A \notin L^p(\mathbb{R}^+)$, and so C_p does not act on χ_A. However, since $\chi_A \in L^1_{loc}(\mathbb{R}^+)$, the function $C\chi_A$ is surely well defined via (1.1) and belongs to $L^0(\mathbb{R}^+)$; actually, it clearly belongs to $L^\infty(\mathbb{R}^+)$ for every $A \in \mathcal{B}(\mathbb{R}^+)$. Since integration with respect to vector measures defined over the more traditional class of *σ-algebras* is better understood and has significant advantages to the more intricate and involved theory based on δ-rings, it seems that a consideration of the Fréchet function space (briefly, F.f.s.) $L^p_{loc}(\mathbb{R}^+) \subseteq L^0(\mathbb{R}^+)$ as the target space for C (instead of the B.f.s. $L^p(\mathbb{R}^+)$) is worthwhile and not without interest. Recall that $L^p_{loc}(\mathbb{R}^+)$, $1 < p < \infty$, is the F.f.s. consisting of all functions $f \in L^0(\mathbb{R}^+)$ satisfying

$$q_j(f) := \Big(\int_0^j |f(x)|^p \, dx \Big)^{1/p} < \infty \qquad \forall j \in \mathbb{N}, \tag{1.6}$$

and is endowed with the metrizable, locally convex topology generated by the fundamental, increasing sequence of seminorms $\{q_j\}_{j=1}^\infty$. It is important to note that each q_j, for $j \in \mathbb{N}$, is a *Riesz seminorm*, meaning that $q_j(f) \le q_j(g)$ whenever

$f, g \in L^p_{loc}(\mathbb{R}^+)$ satisfy $|f| \le |g|$ on $\mathbb{R}^+$ (pointwise μ-a.e.), and that $L^p_{loc}(\mathbb{R}^+)$ is an *ideal* of functions in $L^0(\mathbb{R}^+)$, i.e., $f \in L^0(\mathbb{R}^+)$ and $g \in L^p_{loc}(\mathbb{R}^+)$ with $|f| \le |g|$ implies that $f \in L^p_{loc}(\mathbb{R}^+)$. By the definition of the seminorms (1.6) and Hardy's inequality, [17, p. 240], the positive linear operator $C : L^1_{loc}(\mathbb{R}^+) \to L^0(\mathbb{R}^+)$ maps the F.f.s. $L^p_{loc}(\mathbb{R}^+)$, $1 < p < \infty$, continuously into itself; this operator will be denoted by $C_{[p]}$. An important feature is that the F.f.s. $L^p_{loc}(\mathbb{R}^+)$ has a *Lebesgue topology* (i.e., sequences of $[0, \infty)$-valued functions in $L^p_{loc}(\mathbb{R}^+)$ which decrease to 0 μ-a.e. are convergent to 0 in the topology of $L^p_{loc}(\mathbb{R}^+)$). One may again ask: Is there an *optimal* domain space for $C_{[p]}$ lying within $L^1_{loc}(\mathbb{R}^+)$, having a Lebesgue topology and containing $L^p_{loc}(\mathbb{R}^+)$, to which $C_{[p]} : L^p_{loc}(\mathbb{R}^+) \to L^p_{loc}(\mathbb{R}^+)$ has a linear, continuous, $L^p_{loc}(\mathbb{R}^+)$-valued extension?

For *Banach* function spaces and vector measures defined on σ-*algebras* an existing theory is available; see [9], [30], and the references therein. This theory has recently been extended to the setting of *Fréchet* function spaces, [5], which we will make use of in determining the optimal extension of $C_{[p]} : L^p_{loc}(\mathbb{R}^+) \to L^p_{loc}(\mathbb{R}^+)$, $1 < p < \infty$. A relevant aspect is that the spectrum of $C_{[p]}$ (acting in $L^p_{loc}(\mathbb{R}^+)$) is significantly different from that of C_p (acting in $L^p(\mathbb{R}^+)$); see (1.2). Namely,

$$(1.7) \qquad \sigma_{pt}(C_{[p]}) = \Big\{ \lambda \in \mathbb{C} : |\lambda - \frac{p'}{2}| < \frac{p'}{2} \Big\} \text{ and } \sigma(C_{[p]}) = \overline{\sigma_{pt}(C_{[p]})},$$

[1, Theorem 4.2]. This will generate some new features.

Let us summarize our main results. For fixed $1 < p < \infty$, define

$$(1.8) \qquad \big[C_{[p]}, L^p_{loc}(\mathbb{R}^+)\big] := \big\{ f \in L^1_{loc}(\mathbb{R}^+) : C|f| \in L^p_{loc}(\mathbb{R}^+) \big\}.$$

It turns out that the set function $m_p : \mathcal{B}(\mathbb{R}^+) \to L^p_{loc}(\mathbb{R}^+)$ given by

$$(1.9) \qquad m_p(A) := C\chi_A, \qquad A \in \mathcal{B}(\mathbb{R}^+),$$

is a σ-additive vector measure on the σ-*algebra* $\mathcal{B}(\mathbb{R}^+)$; see the discussion prior to and after Proposition 3.1. Associated with m_p is its space $L^1(m_p)$ of all m_p-integrable functions; it is an F.f.s. only (in particular, $f \in L^1(m_p)$ whenever $f \in L^0(\mathbb{R}^+)$ and $0 \le g \in L^1(m_p)$ satisfy $|f| \le g$) with a Lebesgue topology and contains $L^p_{loc}(\mathbb{R}^+)$ continuously. Moreover, the integration operator $I_{m_p} : L^1(m_p) \to L^p_{loc}(\mathbb{R}^+)$ given by

$$(1.10) \qquad I_{m_p}(f) := \int_{\mathbb{R}^+} f \, dm_p, \qquad f \in L^1(m_p),$$

is linear, continuous and an extension of $C_{[p]}$ from $L^p_{loc}(\mathbb{R}^+)$ to $L^1(m_p)$, which provides an *integral representation* of $C_{[p]}$. In fact, $L^1(m_p)$ is the *largest* F.f.s. over $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+), \mu)$ with a Lebesgue topology and containing $L^p_{loc}(\mathbb{R}^+)$ to which $C_{[p]}$ has a continuous, $L^p_{loc}(\mathbb{R}^+)$-valued, linear extension (namely, I_{m_p}). In addition, it turns out that

$$(1.11) \qquad L^1(m_p) = \big[C_{[p]}, L^p_{loc}(\mathbb{R}^+)\big]$$

and that

$$(1.12) \qquad I_{m_p}(f) = Cf, \qquad f \in L^1(m_p);$$

see Theorem 3.3. Moreover, both the containments

$$(1.13) \qquad L^p_{loc}(\mathbb{R}^+) \subseteq L^1(m_p) \subseteq L^1_{loc}(\mathbb{R}^+)$$

hold (cf. Corollary 3.4) and are *proper* and continuous; see Corollary 3.7 and Proposition 3.8. In view of (1.3), (1.5) and the containment $L^p(\mathbb{R}^+) \subseteq L^p_{loc}(\mathbb{R}^+)$ we have

$$L^1(\nu_p) \subseteq L^1(m_p). \tag{1.14}$$

It is also the case (see Corollary 3.5) that

$$I_{m_p}(f) = I_{\nu_p}(f), \qquad f \in L^1(\nu_p).$$

In particular, $I_{\nu_p}(f) = Cf$, for $f \in L^1(\nu_p)$, which extends (1.4). Moreover, the containment (1.14) is *proper*; see the discussion after Proposition 3.8. A curious feature is that $L^1(\nu_p) \nsubseteq L^p_{loc}(\mathbb{R}^+)$ and conversely, that $L^p_{loc}(\mathbb{R}^+) \nsubseteq L^1(\nu_p)$. The second containment in (1.13), valid for each $p \in (1,\infty)$, is actually stronger; namely,

$$\bigcup_{1<p<\infty} L^1(m_p) \subsetneqq L^1_{loc}(\mathbb{R}^+);$$

see Corollary 3.7. This is a consequence of (1.7), which implies that $L^1(m_p) \subsetneqq L^1(m_q)$ whenever $1 < q < p < \infty$; see Proposition 3.6.

2. Integral extension of the Hardy operator C_p

In addition to $\mathcal{R}$ there are two further δ-rings which are natural candidates for generating an integral representation for $C_p : L^p(\mathbb{R}^+) \longrightarrow L^p(\mathbb{R}^+)$. Indeed, consider the δ-ring of sets $\mathcal{R}_f := \{A \in \mathcal{B}(\mathbb{R}^+) : \mu(A) < \infty\}$, in which case $\mathcal{R} \subsetneqq \mathcal{R}_f$ and

$$\{\chi_A : A \in \mathcal{R}_f\} \subseteq L^1(\nu_p), \qquad p \in (1,\infty). \tag{2.1}$$

To see this note that if $A \in \mathcal{R}_f$, then $\chi_A \in L^p(\mathbb{R}^+)$ and hence, also $C_p\chi_A \in L^p(\mathbb{R}^+)$. According to (1.3) and (1.5) it follows that $\chi_A \in L^1(\nu_p)$. This establishes (2.1). Clearly, the finitely additive set function $\widetilde{\nu}_p : \mathcal{R}_f \to L^p(\mathbb{R}^+)$ defined by

$$\widetilde{\nu}_p(A) := I_{\nu_p}(\chi_A), \qquad A \in \mathcal{R}_f, \tag{2.2}$$

agrees with ν_p on $\mathcal{R} \subseteq \mathcal{R}_f$. The following fact is recorded in [14, Remark 3.3]; we include a proof for the sake of completeness.

Proposition 2.1. *For each $1 < p < \infty$, the set function $\widetilde{\nu}_p : \mathcal{R}_f \to L^p(\mathbb{R}^+)$ is σ-additive.*

Proof. Let $\{A(n)\}_{n=1}^\infty \subseteq \mathcal{R}_f$ be pairwise disjoint sets with $\bigcup_{n=1}^\infty A(n) =: A \in \mathcal{R}_f$. Then $\chi_{A\setminus \bigcup_{i=1}^n A(i)} \downarrow 0$ ν_p-a.e. Since $L^1(\nu_p)$ has o.c.-norm, [12, Section 2], it follows that $\chi_{\bigcup_{i=1}^n A(i)} \to \chi_A$ in $L^1(\nu_p)$ for $n \to \infty$. By continuity of the integration operator $I_{\nu_p} : L^1(\nu_p) \to L^p(\mathbb{R}^+)$, [12, p. 434], the limit

$$\lim_{n\to\infty} \sum_{k=1}^n \widetilde{\nu}_p(A(k)) = \lim_{n\to\infty} I_{\nu_p}\big(\chi_{\bigcup_{i=1}^n A(i)}\big) = I_{\nu_p}(\chi_A) = \widetilde{\nu}_p(A),$$

exists in the norm of $L^p(\mathbb{R}^+)$ and equals $\sum_{k=1}^\infty \widetilde{\nu}_p(A(k))$. □

Another natural δ-ring to consider is $\mathcal{R}_b := \{A \in \mathcal{B}(\mathbb{R}^+) : A \text{ is bounded in } \mathbb{R}^+\}$. Note that $\mathcal{R} \nsubseteq \mathcal{R}_b$ and $\mathcal{R}_b \nsubseteq \mathcal{R}$. Given any δ-ring $\mathcal{S} \subseteq \mathcal{B}(\mathbb{R}^+)$, let $\mathcal{S}^{loc}$ denote the σ-algebra of all sets $A \in \mathcal{B}(\mathbb{R}^+)$ such that $A \cap B \in \mathcal{S}$ for every $B \in \mathcal{S}$. Observe that $\mathcal{R}^{loc} = (\mathcal{R}_b)^{loc} = (\mathcal{R}_f)^{loc} = \mathcal{B}(\mathbb{R}^+)$, even though $\mathcal{R} \subsetneqq \mathcal{R}_f$ and $\mathcal{R}_b \subsetneqq \mathcal{R}_f$. Since $\mathcal{R} \subseteq \mathcal{R}_f$ and $\mathcal{R}^{loc} = (\mathcal{R}_f)^{loc}$, it is clear from [12, p. 433] that the variation measure $|\langle \nu_p, g\rangle|$ (resp., $|\langle \widetilde{\nu}_p, g\rangle|$) of each scalar measure $\langle \nu_p, g\rangle : A \mapsto \langle \nu_p(A), g\rangle$,

$A \in \mathcal{R}$ (resp., $\langle \widetilde{\nu}_p, g\rangle : A \mapsto \langle \widetilde{\nu}_p(A), g\rangle$, $A \in \mathcal{R}_f$), for each g in the dual Banach space $(L^p(\mathbb{R}^+))^* = L^{p'}(\mathbb{R}^+)$, satisfies

$$|\langle \nu_p, g\rangle|(A) \le |\langle \widetilde{\nu}_p, g\rangle|(A), \qquad A \in \mathcal{B}(\mathbb{R}^+). \tag{2.3}$$

Recall that

$$L^1_w(\nu_p) := \big\{ f \in L^0(\mathbb{R}^+) : \int_{\mathbb{R}^+} |f| \, d|\langle \nu_p, g\rangle| < \infty \ \forall g \in L^{p'}(\mathbb{R}^+) \big\},$$

with $L^1_w(\widetilde{\nu}_p)$ defined similarly, [12, p. 434]. Moreover, for $f \in L^1_w(\nu_p)$ (resp., $f \in L^1_w(\widetilde{\nu}_p)$) its norm is defined by

$$\|f\|_{L^1_w(\nu_p)} := \sup\Big\{ \int_{\mathbb{R}^+} |f| \, d|\langle \nu_p, g\rangle| : \|g\|_{L^{p'}(\mathbb{R}^+)} \le 1 \Big\}, \tag{2.4}$$

respectively, by

$$\|f\|_{L^1_w(\widetilde{\nu}_p)} := \sup\Big\{ \int_{\mathbb{R}^+} |f| \, d|\langle \widetilde{\nu}_p, g\rangle| : \|g\|_{L^{p'}(\mathbb{R}^+)} \le 1 \Big\}. \tag{2.5}$$

Since the reflexive Banach space $L^p(\mathbb{R}^+)$ contains no isomorphic copy of c_0, it is known that $L^1(\nu_p) = L^1_w(\nu_p)$ and that $L^1(\widetilde{\nu}_p) = L^1_w(\widetilde{\nu}_p)$, with the norms in $L^1(\nu_p)$ and $L^1(\widetilde{\nu}_p)$ given by (2.4) and (2.5), respectively, [25, Theorem 5.1]. In view of (2.3) it is then clear that $L^1(\widetilde{\nu}_p) \subseteq L^1(\nu_p)$ with $\|f\|_{L^1(\nu_p)} \le \|f\|_{L^1(\widetilde{\nu}_p)}$, for $f \in L^1(\widetilde{\nu}_p)$.

Recall if $\varphi = \sum_{i=1}^n a_i \chi_{A(i)}$ is an $\mathcal{R}$-simple function (resp., $\mathcal{R}_f$-simple function), then $\varphi \in L^1(\nu_p)$ (resp., $\varphi \in L^1(\widetilde{\nu}_p)$) with $\int_A \varphi \, d\nu_p := \sum_{i=1}^n a_i \nu_p(A \cap A(i))$ (resp., $\int_A \varphi \, d\widetilde{\nu}_p := \sum_{i=1}^n a_i \widetilde{\nu}_p(A \cap A(i))$), for each $A \in \mathcal{R}^{loc} = \mathcal{B}(\mathbb{R}^+)$ (resp., each $A \in (\mathcal{R}_f)^{loc} = \mathcal{B}(\mathbb{R}^+)$), [12, p. 434]. In particular, if φ is $\mathcal{R}$-simple, then it is also $\mathcal{R}_f$-simple and $\int_A \varphi \, d\nu_p = \int_A \varphi \, d\widetilde{\nu}_p$, for $A \in \mathcal{B}(\mathbb{R}^+)$. Now, let $f \in L^1(\nu_p)$. According to [12, Proposition 2.3] there is a sequence $\{\varphi_n\}_{n=1}^\infty$ of $\mathcal{R}$-simple functions such that $\varphi_n \to f$ (μ-a.e.) on $\mathbb{R}^+$ and $\{\int_A \varphi_n \, d\nu_p\}_{n=1}^\infty$ converges in $L^p(\mathbb{R}^+)$, for each $A \in \mathcal{R}^{loc}$. Then $\{\varphi_n\}_{n=1}^\infty$ is also a sequence of $\mathcal{R}_f$-simple functions and, via (2.2) and the previous comments, satisfies $\int_A \varphi_n \, d\nu_p = \int_A \varphi_n \, d\widetilde{\nu}_p$, for all $n \in \mathbb{N}$ and all $A \in \mathcal{R}^{loc} = (\mathcal{R}_f)^{loc}$. In particular, $\{\int_A \varphi_n \, d\widetilde{\nu}_p\}_{n=1}^\infty$ converges in $L^p(\mathbb{R}^+)$, for each $A \in (\mathcal{R}_f)^{loc}$. Applying Proposition 2.3 of [12] to $\widetilde{\nu}_p : \mathcal{R}_f \to L^p(\mathbb{R}^+)$ it follows that $f \in L^1(\widetilde{\nu}_p)$. Hence, $L^1(\nu_p) \subseteq L^1(\widetilde{\nu}_p)$. So, we have established the following fact which shows that no extra $\widetilde{\nu}_p$-integrable functions are generated. Of course, $\widetilde{\nu}_p$ is just ν_p but considered as being defined on the *larger* δ-ring $\mathcal{R}_f \supsetneqq \mathcal{R}$.

Proposition 2.2. *For each* $1 < p < \infty$*, we have*

$$L^1(\nu_p) = L^1(\widetilde{\nu}_p) = [C_p, L^p(\mathbb{R}^+)].$$

Let us return to the δ-ring $\mathcal{R}_b$. Since $\mathcal{R}_b \subseteq \mathcal{R}_f$, Proposition 2.1 implies that $\widehat{\nu}_p : \mathcal{R}_b \to L^p(\mathbb{R}^+)$ defined by

$$\widehat{\nu}_p(A) := I_{\nu_p}(\chi_A), \qquad A \in \mathcal{R}_b,$$

is σ-additive and agrees with $\widetilde{\nu}_p$ restricted to $\mathcal{R}_b \subseteq \mathcal{R}_f$. This also follows from Proposition 3.1(a) of [13] because the (non-negative) Hardy kernel $K(x,y) := \frac{1}{x}\chi_{(0,x]}(y)$ on $\mathbb{R}^+ \times \mathbb{R}^+$ is *admissible* in the sense of [13, Section 3], i.e., $\int_A K(x,y)\, dy < \infty$ for each $x \in (0,\infty)$ and $A \in \mathcal{R}_b$. As already noted, $(\mathcal{R}_b)^{loc} = (\mathcal{R}_f)^{loc}$ even though $\mathcal{R}_b \subsetneqq \mathcal{R}_f$. The same argument used to prove Proposition 2.2 can also be applied to the pair of vector measures $\widehat{\nu}_p : \mathcal{R}_b \to L^p(\mathbb{R}^+)$ and $\widetilde{\nu}_p : \mathcal{R}_f \to L^p(\mathbb{R}^+)$

(in place of the pair $\nu_p : \mathcal{R} \to L^p(\mathbb{R}^+)$ and $\widetilde{\nu}_p : \mathcal{R}_f \to L^p(\mathbb{R}^+)$) to establish the following result.

Proposition 2.3. *For each $1 < p < \infty$, we have*

$$L^1(\widehat{\nu}_p) = L^1(\widetilde{\nu}_p) = [C_p, L^p(\mathbb{R}^+)].$$

It is immediate from Propositions 2.2 and 2.3 that

$$L^1(\nu_p) = L^1(\widehat{\nu}_p) = L^1(\widetilde{\nu}_p) = [C_p, L^p(\mathbb{R}^+)], \qquad 1 < p < \infty. \tag{2.6}$$

A pleasant consequence of (2.6) is that the "same" vector measure $A \longmapsto C_p\chi_A$, when interpreted to be defined on any one of the three *distinct* δ-rings $\mathcal{R}, \mathcal{R}_b, \mathcal{R}_f$, generates in each case the *same* optimal domain space $[C_p, L^p(\mathbb{R}^+)]$ and the same integral representation of C_p. It is also worth noting that $\widetilde{\nu}_p$ *cannot* be extended to an $L^p(\mathbb{R}^+)$-valued vector measure defined on the σ-algebra $\mathcal{B}(\mathbb{R}^+)$. Indeed, since $0 \le \chi_{[0,a]} \le C_p\chi_{[0,a]}$ we have

$$\|\tilde{\nu}_p([0,a])\|_{L^p(\mathbb{R}^+)} = \|C_p\chi_{[0,a]}\|_{L^p(\mathbb{R}^+)} \ge \|\chi_{[0,a]}\|_{L^p(\mathbb{R}^+)} = a^{1/p} \quad \forall\, a > 0.$$

Hence, the range $\tilde{\nu}_p(\mathcal{R}_f)$ of $\tilde{\nu}_p$ is an unbounded subset of $L^p(\mathbb{R}^+)$. Since the range of every σ-additive, Banach-space-valued vector measure defined on a σ-algebra is a bounded set, [15, p. 14 Corollary 7], it follows that $\tilde{\nu}_p$ does not have a σ-additive extension to $\mathcal{B}(\mathbb{R}^+)$.

3. Optimal extension of the Hardy operator $C_{[p]}$

With the choice of the B.f.s. $L^p(\mathbb{R}^+) \subseteq L^0(\mathbb{R}^+)$ as the target space for $C : L^1_{loc}(\mathbb{R}^+) \to L^0(\mathbb{R}^+)$ it is not possible to exhibit an integral representation for $C_p : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ via an $L^p(\mathbb{R}^+)$-valued vector measure based on the σ-*algebra* $\mathcal{B}(\mathbb{R}^+) = \mathcal{R}^{loc} = (\mathcal{R}_f)^{loc}$. However, if the F.f.s. $L^p_{loc}(\mathbb{R}^+)$, $1 < p < \infty$, is taken as the target space for $C : L^1_{loc}(\mathbb{R}^+) \to L^0(\mathbb{R}^+)$, then this phenomenon disappears. In order to investigate this situation we summarize some relevant aspects from the theory of integration with respect to Fréchet-space-valued vector measures defined on σ-algebras.

Let X be a (complex) locally convex Fréchet space with $\{q_n\}_{n=1}^{\infty}$ a fundamental (i.e., $\bigcap_{n=1}^{\infty} q_n^{-1}(\{0\}) = \{0\}$), increasing sequence of seminorms generating the locally convex topology of X. The *topological dual space* of X, consisting of all continuous linear functionals on X, is denoted by X^*. Let $m : \Sigma \to X$ be a vector measure defined on a σ-algebra of subsets Σ of a non-empty set Ω; this will always mean that m is σ-*additive* on Σ. For each $n \in \mathbb{N}$, the q_n-*semivariation* of m is the subadditive set function $q_n(m) : \Sigma \to [0, \infty)$ defined by

$$q_n(m)(A) := \sup_{x^* \in U^\circ_{q_n}} |\langle m, x^* \rangle|(A), \qquad A \in \Sigma,$$

where $|\langle m, x^*\rangle|$ is the variation measure of the complex measure $\langle m, x^* \rangle : A \mapsto \langle m(A), x^* \rangle$, $A \in \Sigma$, for each $x^* \in X^*$, and $U^\circ_{q_n} := \{x^* \in X^* : |\langle x, x^* \rangle| \le 1$ for all $x \in U_q\}$ is the polar set of $U_{q_n} := q_n^{-1}([0,1])$. A set $A \in \Sigma$ is called m-*null* if $q_n(m)(A) = 0$, for all $n \in \mathbb{N}$.

A $\mathbb{C}$-valued, Σ-measurable function f on Ω is called m-*integrable* if

(I-1) $\int_\Omega |f|\, d|\langle m, x^* \rangle| < \infty$, for all $x^* \in X^*$, and

(I-2) for each $A \in \Sigma$ there exists a vector $\int_A f\, dm \in X$ satisfying

$$\Big\langle \int_A f\, dm, x^* \Big\rangle = \int_A f\, d\langle m, x^*\rangle, \qquad x^* \in X^*;$$

see [20], [24]. If f only satisfies (I-1), then it is called *scalarly m-integrable.* Two Σ-measurable functions are identified if they are equal m-a.e. The space $L^1_w(m)$ of all (classes of) scalarly m-integrable functions becomes an F.f.s. when equipped with the sequence of Riesz seminorms

$$\widetilde{q}_n(f) := \sup_{x^* \in U^\circ_{q_n}} \int_\Omega |f|\, d|\langle m, x^*\rangle|, \qquad f \in L^1_w(m) \quad \forall\, n \in \mathbb{N}.$$

It turns out that the space $L^1(m)$ consisting of all m-integrable functions is a *closed* subspace of $L^1_w(m)$, is itself an F.f.s. and contains the space $\operatorname{sim}\Sigma$ of all Σ-simple functions as a dense subspace. As standard references we refer to [10], [20], when X is a real Fréchet space (with all Σ-measurable functions in $L^0(\Sigma)$ being $\mathbb{R}$-valued) and to [18] for Fréchet spaces over $\mathbb{C}$. If X does not contain an isomorphic copy of the Banach sequence space c_0, then necessarily $L^1_w(m) = L^1(m)$, [25, Theorem 5.1]. Moreover, $L^1(m)$ always has a Lebesgue topology; this follows from [18, Proposition 1(iv)]. The *integration operator* $I_m : L^1(m) \to X$, defined by

$$I_m(f) := \int_\Omega f\, dm, \qquad f \in L^1(m),$$

is always linear and continuous. This follows from the inequalities

$$\sup_{A\in\Sigma} q_n\Big(\int_A f\, dm\Big) \le \widetilde{q}_n(f) \le 4 \sup_{A\in\Sigma} q_n\Big(\int_A f\, dm\Big), \qquad f \in L^1(m),$$

valid for each $n \in \mathbb{N}$, [24].

Let (Ω, Σ, μ) be a positive, σ-finite measure space and let $X(\mu)$ be an F.f.s. over (Ω, Σ, μ); for the definition see, for example, [10], [11], when $X(\mu)$ is a real space, and [5, Section 2.3] for $X(\mu)$ a complex space. Given a Fréchet space X, an F.f.s. $X(\mu)$ containing $\operatorname{sim}\Sigma$, and a continuous linear operator $T : X(\mu) \to X$, the finitely additive set function $m_T : \Sigma \to X$ defined by $m_T(A) := T(\chi_A)$, for $A \in \Sigma$, is called the vector measure associated with T; it is necessarily σ-additive if $X(\mu)$ has a Lebesgue topology. Whenever m_T is σ-additive the operator T is called *μ-determined* if m_T and μ have the same null sets. The following *Optimal Domain Theorem* occurs in [5, Theorem 3.3.1]. For $X(\mu)$ a B.f.s. over a finite, positive measure space and X a Banach space, this result first occurred in [9]; see also [30, Ch.4].

Proposition 3.1. *Let (Ω, Σ, μ) be a positive, σ-finite measure space. Let X be a Fréchet space, $X(\mu)$ be an F.f.s. over (Ω, Σ, μ) with a Lebesgue topology such that $\operatorname{sim}\Sigma \subseteq X(\mu)$ and let $T : X(\mu) \to X$ be a μ-determined, continuous linear operator. Then $L^1(m_T)$ is the largest amongst all F.f.s.' over (Ω, Σ, μ) having a Lebesgue topology into which $X(\mu)$ is continuously included and to which T admits an X-valued, continuous linear extension. Moreover, such an extension of T is unique and is precisely the integration operator $I_{m_T} : L^1(m_T) \to X$, i.e.,*

$$I_{m_T}(f) = T(f) = \int_\Omega f dm_T, \qquad f \in X(\mu).$$

We will require the above notions and facts for the particular setting of the σ-finite measure space $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+), \mu)$ with $X = X(\mu) = L^p_{loc}(\mathbb{R}^+)$, for any fixed $1 < p < \infty$, and with $T : X(\mu) \to X$ being the Hardy operator $C_{[p]}$. The associated measure m_T is then precisely $m_p : \mathcal{B}(\mathbb{R}^+) \to L^p_{loc}(\mathbb{R}^+)$; see (1.9). Clearly, $\operatorname{sim}\mathcal{B}(\mathbb{R}^+) \subseteq X(\mu)$. Since each B.f.s. $L^p([0,j])$, for $j \in \mathbb{N}$, has o.c.-norm, it is routine to verify that $X(\mu)$ has a Lebesgue topology, [5, Example 2.3.2(iv)]. Consequently, m_p is σ-additive. It only remains to check that m_p and μ have the same null sets. This was already implicitly assumed to be so for ν_p and μ; it is stated to be the case on p. 126 of [14] but, without a proof. It is clear from (1.1), with $f = \chi_A$, that $\mu(A) = 0$ implies $C\chi_B = 0$ in $L^0(\mathbb{R}^+)$ for every $B \in \mathcal{B}(\mathbb{R}^+)$ with $B \subseteq A$ and hence, that A is both ν_p-null and m_p-null. Conversely, suppose that $A \in \mathcal{R}^{loc} = \mathcal{B}(\mathbb{R}^+)$ is ν_p-null (see [12, p. 433] for the definition) or m_p-null. Then, in both cases, $C\chi_A = 0$ (in $L^0(\mathbb{R}^+)$) which implies, via Lemma 3.2 below, that the *continuous* function $C\chi_A : (0,\infty) \to [0,\infty)$ is identically 0 on $(0,\infty)$. So, $\int_0^x \chi_A(t)\,dt = 0$ for every $x > 0$, with $0 \le \chi_A \in L^\infty(\mathbb{R}^+)$, which implies that $\mu(A) = 0$. Hence, both $C_{[p]}$ and C_p are μ-determined.

Lemma 3.2. *Let $f \in L^1_{loc}(\mathbb{R}^+)$. Then the function $Cf : (0,\infty) \to \mathbb{C}$ defined via* (1.1) *is continuous.*

Proof. As $x \mapsto \frac{1}{x}$ is continuous, it suffices to show that $h(x) := \int_0^x f(t)\,dt$ is continuous on $(0,\infty)$; see (1.1). Fix $x > 0$. If $x_n \to x^-$ in $[0,x)$, then

$$|h(x) - h(x_n)| = \Big|\int_{x_n}^x f(t)\,dt\Big| \le \int_0^x \chi_{[x_n,x]}(t)|f(t)|\,dt.$$

But, $\chi_{[x_n,x]}|f| \to 0$ pointwise μ-a.e. on $[0,x]$ with $\chi_{[x_n,x]}|f| \le |f|$ and $|f| \in L^1([0,x])$. Then the dominated convergence theorem yields that $\lim_{n\to\infty} h(x_n) = h(x)$, i.e., h is left-continuous at x. A similar argument shows that h is also right-continuous at x. □

Fix $1 < p < \infty$. Then $L^p(\mathbb{R}^+)$ is a dense subspace of $L^p_{loc}(\mathbb{R}^+)$ and the natural inclusion $L^p(\mathbb{R}^+) \subseteq L^p_{loc}(\mathbb{R}^+)$ is continuous with

$$q_j(f) \le \|f\|_{L^p(\mathbb{R}^+)}, \qquad f \in L^p(\mathbb{R}^+) \qquad \forall\, j \in \mathbb{N}.$$

As a general reference for (locally convex) Fréchet spaces we refer to [28], for example. Since $\{q_j\}_{j=1}^\infty$ (cf. (1.6)) is a fundamental, increasing sequence of seminorms generating the locally convex topology of the Fréchet space $L^p_{loc}(\mathbb{R}^+)$, it follows from [28, Corollary 22.7] that every element of $(L^p_{loc}(\mathbb{R}^+))^*$ has the form

$$f \mapsto \langle f, \xi\rangle := \int_{\mathbb{R}^+} f(t)\xi(t)\,dt, \qquad f \in L^p_{loc}(\mathbb{R}^+),$$

for some unique, *compactly supported* function $\xi \in L^{p'}(\mathbb{R}^+)$, [5, Example 2.3.2(i)]. It follows from (1.1) and this description of $(L^p_{loc}(\mathbb{R}^+))^* \subseteq L^0(\mathbb{R}^+)$ that the dual operator $C^*_{[p]} : (L^p_{loc}(\mathbb{R}^+))^* \to (L^p_{loc}(\mathbb{R}^+))^*$ is given by

$$(3.1) \qquad \big(C^*_{[p]}\xi\big)(x) = \int_x^\infty \frac{\xi(t)}{t}\,dt, \qquad x \in (0,\infty), \qquad \xi \in (L^p_{loc}(\mathbb{R}^+))^*.$$

Suppose $\mathrm{supp}(\xi) \subseteq [0, j]$ for some $j \in \mathbb{N}$. Clearly, $\mathrm{supp}(C^*_{[p]}\xi) \subseteq [0, j]$; see (3.1). It follows from (3.1) that the complex measure $\langle m_p, \xi \rangle$ is given by

$$\langle m_p, \xi\rangle(A) = \int_{A\cap[0,j]} \Big(\int_x^\infty \frac{\xi(t)}{t}\, dt \Big)\, dx, \qquad A \in \mathcal{B}(\mathbb{R}^+).$$

Hence, $\langle m_p, \xi \rangle$ has its support in $[0, j]$, is absolutely continuous with respect to μ and its variation measure $|\langle m_p, \xi\rangle|$ has Radon-Nikodým derivative

$$\frac{d|\langle m_p, \xi\rangle|}{d\mu}(x) = \chi_{_{[0,j]}}(x) \Big| \int_x^\infty \frac{\xi(t)}{t}\, dt \Big|, \qquad x \in \mathbb{R}^+.$$

In view of Proposition 3.1 and the discussion immediately afterwards, the following *optimality result* for the Hardy operator $C_{[p]} : L^p_{loc}(\mathbb{R}^+) \to L^p_{loc}(\mathbb{R}^+)$ is the main result of this section.

Theorem 3.3. *Let $1 < p < \infty$. Then* (1.11) *is valid, that is,*

$$L^1(m_p) = [C_{[p]}, L^p_{loc}(\mathbb{R}^+)].$$

Moreover, the optimal extension (*see* (1.10)) $I_{m_p} : L^1(m_p) \to L^p_{loc}(\mathbb{R}^+)$ *of* $C_{[p]}$ *is given by* (1.12)*; namely,*

$$I_{m_p}(f) = Cf = \int_{\mathbb{R}^+} f\, dm_p, \qquad f \in L^1(m_p).$$

Proof. We first prove (1.12). If $s = \sum_{j=1}^n \alpha_j \chi_{A(j)} \in \mathrm{sim}\, \mathcal{B}(\mathbb{R}^+)$, then

$$I_{m_p}(s) = \sum_{j=1}^n \alpha_j I_{m_p}\big(\chi_{A(j)}\big) = \sum_{j=1}^n \alpha_j C \chi_{A(j)} = Cs. \tag{3.2}$$

Claim. If $0 \le f \in L^1(m_p)$, then $f \in L^1_{loc}(\mathbb{R}^+)$ and $I_{m_p}(f) = Cf$.

To establish the Claim, choose $\mathcal{B}(\mathbb{R}^+)$-simple functions $0 \le s_n \uparrow f$ pointwise on $\mathbb{R}^+$. Using the identity (3.2) and the dominated convergence theorem for m_p, [24, Theorem 2.2], we have (in the topology of $L^p_{loc}(\mathbb{R}^+)$) that

$$I_{m_p}(f) = \lim_{n\to\infty} I_{m_p}(s_n) = \lim_{n\to\infty} Cs_n.$$

Choose a subsequence $\{Cs_{n(k)}\}_{k=1}^\infty$ of $\{Cs_n\}_{n=1}^\infty$ such that $Cs_{n(k)} \to I_{m_p}(f)$ pointwise μ-a.e. on $\mathbb{R}^+$, for $k \to \infty$. Since $I_{m_p}(f) \in L^p_{loc}(\mathbb{R}^+)$, necessarily $|I_{m_p}(f)(x)| < \infty$ (μ-a.e.). On the other hand, the monotone convergence theorem implies, for each fixed $x > 0$, that

$$\lim_{k\to\infty} (Cs_{n(k)})(x) = \frac{1}{x} \lim_{k\to\infty} \int_0^x s_{n(k)}(t)\, dt = \frac{1}{x} \int_0^x f(t)\, dt.$$

Accordingly, for μ-a.e. $x \in \mathbb{R}^+$, we see that $\frac{1}{x}\int_0^x f(t)\, dt$ is finite as it coincides with $I_{m_p}(f)(x)$. Choose a sequence $x_r \uparrow \infty$ in $\mathbb{R}^+$ for which $\int_0^{x_r} f(t)\, dt$ is finite for each $r \in \mathbb{N}$ (and recall that $f \ge 0$) implies that $f \in L^1_{loc}(\mathbb{R}^+)$ and, from the previous discussion, that $Cf = I_{m_p}(f)$ as elements of $L^p_{loc}(\mathbb{R}^+)$. This establishes the Claim.

Since C is linear on $L^1_{loc}(\mathbb{R})$ and I_{m_p} is linear on $L^1(m_p)$ and each $f \in L^1(m_p)$ has a decomposition as a sum of four non-negative functions from $L^1(m_p)$, we can conclude that (1.12) actually holds for all $f \in L^1(m_p)$.

Concerning (1.11), let $f \in L^1(m_p)$. Then also $|f| \in L^1(m_p)$ and hence, by the above Claim we conclude that $C|f| = I_{m_p}(|f|) \in L^p_{loc}(\mathbb{R}^+)$ with $|f| \in L^1_{loc}(\mathbb{R}^+)$. According to (1.8) we see that $f \in [C_{[p]}, L^p_{loc}(\mathbb{R}^+)]$.

Conversely, let $f \in L^1_{loc}(\mathbb{R}^+)$ satisfy $C|f| \in L^p_{loc}(\mathbb{R}^+)$. Choose a sequence $\{t_n\}_{n=1}^\infty \subseteq \operatorname{sim}\mathcal{B}(\mathbb{R}^+)$ with $0 \le t_n \uparrow |f|$ pointwise on $\mathbb{R}^+$. Fix $A \in \mathcal{B}(\mathbb{R}^+)$. For each $x > 0$, it follows from (1.1) that

$$\big|\big(C(|f|\chi_A)\big)(x) - \big(C(t_n\chi_A)\big)(x)\big| \le \frac{1}{x}\int_0^x (|f|(u) - t_n(u))\,du.$$

Since $0 \le t_n \le |f|$ with $t_n \to |f|$ pointwise on $[0,x]$ and the restriction $|f|_{[0,x]} \in L^1([0,x])$, the dominated convergence theorem yields that $\lim_{n\to\infty}(C(t_n\chi_A))(x) = (C(|f|\chi_A))(x)$. Hence, $C(t_n\chi_A) \to C(|f|\chi_A)$ pointwise on $\mathbb{R}^+$ as $n \to \infty$. Moreover,

$$\big|C(|f|\chi_A) - C(\chi_A t_n)\big| \le C(|f|\chi_A) + C(\chi_A t_n) \le 2C|f|, \qquad n \in \mathbb{N},$$

shows that $|C(|f|\chi_A) - C(\chi_A t_n)|^p \le 2^p(C|f|)^p$ on $\mathbb{R}^+$, for all $n \in \mathbb{N}$. But, $C|f| \in L^p_{loc}(\mathbb{R}^+)$ and so $2^p(C|f|^p) \in L^1([0,j])$, for each $j \in \mathbb{N}$. Since $|C(|f|\chi_A) - C(\chi_A t_n)|^p \to 0$ pointwise on $\mathbb{R}^+$ for $n \to \infty$, the dominated convergence theorem yields

$$\lim_{n\to\infty}\int_0^j |C(|f|\chi_A) - C(\chi_A t_n)|^p\,d\mu = 0, \qquad j \in \mathbb{N},$$

that is, $\lim_{n\to\infty} q_j(C(|f|\chi_A) - C(\chi_A t_n)) = 0$ for all $j \in \mathbb{N}$. This implies (see (3.2)) that $\{I_{m_p}(\chi_A t_n)\}_{n=1}^\infty = \{C(\chi_A t_n)\}_{n=1}^\infty$ is a convergent sequence in $L^p_{loc}(\mathbb{R}^+)$. But, $A \in \mathcal{B}(\mathbb{R}^+)$ is arbitrary and so $|f| \in L^1(m_p)$, [24, Theorem 2.4(3)]. Then, also $f \in L^1(m_p)$. Thereby (1.11) is verified. □

Corollary 3.4. *Let $1 < p < \infty$. Then*

$$L^p_{loc}(\mathbb{R}^+) \subseteq L^1(m_p) \subseteq L^1_{loc}(\mathbb{R}^+). \tag{3.3}$$

Proof. The first inclusion in (3.3) follows from $X(\mu) \subseteq L^1(m_T)$; see Proposition 3.1 when applied to the operator $T := C_{[p]}$ with $X(\mu) = X = L^p_{loc}(\mathbb{R}^+)$. The second inclusion in (3.3) follows via the Claim in the proof of Theorem 3.3 (as $f \in L^1(m_p)$ if and only if $0 \le |f| \in L^1(m_p)$). □

Corollary 3.5. *Let $1 < p < \infty$. Then* (1.14) *is valid, that is,*

$$L^1(\nu_p) \subseteq L^1(m_p). \tag{3.4}$$

Moreover,

$$I_{\nu_p}(f) = I_{m_p}(f) = Cf, \qquad f \in L^1(\nu_p), \tag{3.5}$$

with $I_{m_p}(f) \in L^p(\mathbb{R}^+) \subseteq L^p_{loc}(\mathbb{R}^+)$.

Proof. Let $f \in L^1(\nu_p) = [C_p, L^p(\mathbb{R}^+)]$, [14, Example 3.5], in which case $C|f| \in L^p(\mathbb{R}^+)$; see (1.13). Since $L^p(\mathbb{R}^+) \subseteq L^p_{loc}(\mathbb{R}^+)$, we also have $C|f| \in L^p_{loc}(\mathbb{R}^+)$ and so, by (1.11) and (1.12), it follows that $f \in L^1(m_p)$ and $I_{m_p}(f) = Cf$. This establishes (3.4).

It remains to show that $I_{\nu_p}(f) = I_{m_p}(f)$ whenever $f \in L^1(\nu_p)$. According to the discussion immediately prior to Proposition 2.2 there exists a sequence $\{\varphi_n\}_{n=1}^\infty$ of $\mathcal{R}$-simple functions such that $\varphi_n \to f$ pointwise μ-a.e. on $\mathbb{R}^+$ and $\{\int_A \varphi_n\,d\nu_p\}_{n=1}^\infty$ converges in $L^p(\mathbb{R}^+)$, for each $A \in \mathcal{R}^{loc} = \mathcal{B}(\mathbb{R}^+)$. But, $L^p(\mathbb{R}^+) \subseteq L^p_{loc}(\mathbb{R}^+)$ continuously and so $\{\int_A \varphi_n\,d\nu_p\}_{n=1}^\infty$ converges in $L^p_{loc}(\mathbb{R}^+)$, for all $A \in \mathcal{B}(\mathbb{R}^+)$. Via the definition of $\int_A \varphi_n \, d\nu_p$ (see again the discussion immediately prior to Proposition 2.2) it is clear that $\int_A \varphi \, d\nu_p = \int_A \varphi_n \, dm_p$, for $n \in \mathbb{N}$. Hence, $\{\int_A \varphi_n \, dm_p\}_{n=1}^\infty$ converges in $L^p_{loc}(\mathbb{R}^+)$, for each $A \in \mathcal{B}(\mathbb{R}^+)$. It follows that $f \in L^1(m_p)$ and

$$I_{m_p}(f) = \lim_{n\to\infty} \int_{\mathbb{R}^+} \varphi_n \, dm_p = \lim_{n\to\infty} \int_{\mathbb{R}^+} \varphi_n \, d\nu_p = I_{\nu_p}(f) \in L^p(\mathbb{R}^+),$$

[24, Theorem 2.4(3)], with both limits existing in $L^p(\mathbb{R}^+) \subseteq L^p_{loc}(\mathbb{R}^+)$. This establishes (3.5). □

It is time to analyze (1.7). Fix $1 < p < \infty$. Then $\lambda \in \mathbb{C}$ satisfies $|\lambda - \frac{p'}{2}| < \frac{p'}{2}$ (i.e., $\mathrm{Re}(\frac{1}{\lambda}) > \frac{1}{p'} = \frac{(p-1)}{p}$) precisely when $\lambda \in \sigma_{pt}(C_{[p]})$ with

$$f_\lambda(x) := x^{\frac{1}{\lambda}-1}, \qquad x \in \mathbb{R}^+,$$

being an eigenfunction of λ (and belonging to $L^p_{loc}(\mathbb{R}^+)$), [1, Section 4]. Suppose now that $1 < q < p$, in which case $1 < p' < q' < \infty$. Choose any real number $\alpha \in (p', q')$, i.e., $\frac{1}{q'} < \mathrm{Re}(\frac{1}{\alpha}) < \frac{1}{p'}$. Then (1.7) implies that $\alpha \in \sigma_{pt}(C_{[q]}) \setminus \sigma_{pt}(C_{[p]})$ with $f_\alpha \in L^q_{loc}(\mathbb{R}^+) \setminus L^p_{loc}(\mathbb{R}^+)$ satisfying $Cf_\alpha = \alpha f_\alpha$. Hence, also $Cf_\alpha \in L^q_{loc}(\mathbb{R}^+) \setminus L^p_{loc}(\mathbb{R}^+)$. It follows from (1.11) that $f_\alpha \in L^1(m_q) \setminus L^1(m_p)$. Moreover, $L^1(m_p) \subseteq L^1(m_q)$ is clear from (1.8), (1.11) and the inclusion $L^p_{loc}(\mathbb{R}^+) \subseteq L^q_{loc}(\mathbb{R}^+)$. Noting that

$$\|h\|_{L^q([0,j])} \le j^s \|h\|_{L^p([0,j])}, \qquad h \in L^p([0,j]) \qquad \forall\, j \in \mathbb{N},$$

with $s = \frac{1}{q} - \frac{1}{p}$, it follows via the definition of the seminorms $\tilde{q}_n$ for $n \in \mathbb{N}$ that the inclusion $L^1(m_p) \subseteq L^1(m_q)$ is continuous. So, we have verified the following result.

Proposition 3.6. *Let $1 < q < p$. Then , with a continuous inclusion,*

$$L^1(m_p) \subsetneq L^1(m_q).$$

Combining Corollary 3.4 and Proposition 3.6 with a standard fact concerning certain nested sequences of Fréchet spaces (see [29, Lemma 4.15], for example) yields the following fact.

Corollary 3.7. *The proper inclusion $\bigcup_{1<p<\infty} L^1(m_p) \subsetneq L^1_{loc}(\mathbb{R}^+)$ holds.*

The function

$$f(x) := (1-x)^{-1/p}\chi_{(0,1)}(x), \qquad x \in \mathbb{R}^+, \qquad 1 < p < \infty, \tag{3.6}$$

belongs to $L^1(\nu_p) \setminus L^p(\mathbb{R}^+)$; see the proof of Proposition 2.3 in [14]. Since $\int_0^1 |f|^p \, d\mu = \infty$, we see that also $f \notin L^p_{loc}(\mathbb{R}^+)$. Combined with Corollary 3.4 and Corollary 3.5 this yields the following result.

Proposition 3.8. *For each $1 < p < \infty$ it is the case that*

$$L^p_{loc}(\mathbb{R}^+) \subsetneq L^1(m_p).$$

The function (3.6) also shows that

$$L^1(\nu_p) \nsubseteq L^p_{loc}(\mathbb{R}^+), \qquad 1 < p < \infty.$$

On the other hand for each $a > 0$ the function $\chi_{[a,\infty)} \in L^p_{loc}(\mathbb{R}^+)$. Since $(C\chi_{[a,\infty)})(x) = (1 - \frac{a}{x})\chi_{[a,\infty)}(x)$ for $x > 0$ does not belong to $L^p(\mathbb{R}^+)$ it follows that $\chi_{[a,\infty)} \notin L^1(\nu_p) = L^1(\tilde{\nu}_p)$; see (1.3) and (1.5). Hence,

$$L^p_{loc}(\mathbb{R}^+) \nsubseteq L^1(\nu_p), \qquad 1 < p < \infty.$$

So, there is no relationship between the optimal domain of $C_p : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ and the domain of definition of $C_{[p]} : L^p_{loc}(\mathbb{R}^+) \to L^p_{loc}(\mathbb{R}^+)$.

In conclusion we wish to clarify a subtle point; namely, whether the formula (1.4) remains valid for all $f \in L^1(\nu_p)$. That this is actually so is stated in the last paragraph on p. 126 of [14], via the formula

$$Sf = \int f \, d\nu_X, \qquad f \in L^1(\nu_X); \tag{3.7}$$

for our setting here $S := C$ and $X = L^p(\mathbb{R}^+)$, $1 < p < \infty$, is the relevant space. The reference given for the formula (3.7) is Proposition 3.1(b) of [13]. However, in that result the vector measure ν (in our setting $\widehat{\nu}_p$) is defined on $\mathcal{R}_b$ and not on $\mathcal{R}$ as in [14] (in our setting ν_p); as seen previously these two δ-rings are not comparable. However, thanks to (2.6) and Corollary 3.5 we see that (1.4) is indeed valid for all $f \in L^1(\nu_p)$ and not just for $f \in L^p(\mathbb{R}^+)$.

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