PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 11, November 2018, Pages 4723–4734
http://dx.doi.org/10.1090/proc/14097
Article electronically published on August 7, 2018

ASYMPTOTIC PROPERTIES OF BANACH SPACES AND COARSE QUOTIENT MAPS

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(Communicated by Thomas Schlumprecht)

Abstract. We give a quantitative result about asymptotic moduli of Banach spaces under coarse quotient maps. More precisely, we prove that if a Banach space Y is a coarse quotient of a subset of a Banach space X, where the coarse quotient map is coarse Lipschitz, then the (β)-modulus of X is bounded by the modulus of asymptotic uniform smoothness of Y up to some constants. In particular, if the coarse quotient map is a coarse homeomorphism, then the modulus of asymptotic uniform convexity of X is bounded by the modulus of asymptotic uniform smoothness of Y up to some constants.

1. Introduction

The study of asymptotic geometry of Banach spaces dates back to Milman [14], in which he introduced two asymptotic properties that are now known as asymptotic uniform convexity and asymptotic uniform smoothness (cf. [9]). For a Banach space X and $t > 0$, the modulus of asymptotic uniform smoothness of X is defined by

$$\bar{\rho}_X(t) := \sup_{x\in S_X} \inf_{\dim(X/Y)<\infty} \sup_{y\in S_Y} \|x+ty\| - 1,$$

and the modulus of asymptotic uniform convexity of X is defined by

$$\bar{\delta}_X(t) := \inf_{x\in S_X} \sup_{\dim(X/Y)<\infty} \inf_{y\in S_Y} \|x+ty\| - 1.$$

A Banach space X is said to be asymptotically uniformly smooth (AUS for short) if $\lim_{t\to 0} \bar{\rho}_X(t)/t \to 0$ as $t \to 0$, and it is said to be asymptotically uniformly convex (AUC for short) if $\bar{\delta}_X(t) > 0$ for all $0 < t \le 1$.

In close relation to AUC and AUS is Rolewicz's property (β) that was originally defined using the terminology of "drop" [16]; later Kutzarova [12] gave an equivalent definition, according to which one can define a modulus for the property: for a Banach space X and $t \in (0, a]$, where $a \in [1, 2]$ is a number that depends only on the space X, the (β)-modulus of X is defined by

$$\bar{\beta}_X(t) = 1 - \sup\left\{ \inf_{n\ge 1} \left\{ \frac{\|x - x_n\|}{2} \right\} : \ x, x_n \in B_X,\ \mathrm{sep}(\{x_n\}_{n=1}^{\infty}) \ge t \right\}.$$

Received by the editors October 2, 2017, and, in revised form, November 26, 2017, and January 11, 2018.

2010 *Mathematics Subject Classification.* Primary 46B80, 46B06.

The author was supported by the Fundamental Research Funds for the Central Universities, Grant Number 2682017CX060.

Here $\operatorname{sep}(\{x_n\}_{n=1}^\infty)$ denotes the separating constant of the sequence $\{x_n\}_{n=1}^\infty$:

$$\operatorname{sep}(\{x_n\}_{n=1}^\infty) := \inf_{i\neq j} \|x_i - x_j\|.$$

A Banach space X is said to have property (β) if $\bar{\beta}_X(t) > 0$ for all $t > 0$, and for $p \in (1,\infty)$ we say that the (β)-modulus of X has power type p, or X has property (β_p), if there is a constant $C > 0$ independent of t so that $\bar{\beta}_X(t) \geq Ct^p$ for all $t > 0$.

A reflexive Banach space that is simultaneously AUC and AUS has property (β) [11]. Conversely, if a Banach space X has property (β), then it must be reflexive and AUC. More precisely, it was shown in [7] that $\bar{\beta}_X(t) \leq \bar{\delta}_X(2t)$ for all $t \in (0, 1/2]$. However, property (β) does not imply AUS isometrically [11], but every Banach space that has property (β) admits an equivalent AUS norm. A complete renorming argument of property (β) can be found in the recent paper by Dilworth, Kutzarova, Lancien, and Randrianarivony [6].

Bates, Johnson, Lindenstrauss, Preiss, and Schechtman first studied nonlinear quotient maps in the Banach space setting [2]. A map $f : X \to Y$ between two metric spaces X and Y is said to be co-uniformly continuous if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x \in X$,

$$f(B_X(x,\varepsilon)) \supseteq B_Y(f(x),\delta).$$

If δ can be chosen to be ε/C for some constant $C > 0$ that is independent of ε and x, then f is said to be co-Lipschitz. A uniform (resp., Lipschitz) quotient map is a map that is both uniform continuous and co-uniform continuous (resp., Lipschitz and co-Lipschitz), and Y is said to be a uniform (resp., Lipschitz) quotient of X if there exists a uniform (resp., Lipschitz) quotient map from X onto Y.

Lima and Randrianarivony [13] showed that ℓ_q is not a uniform quotient of ℓ_p for $q > p > 1$. Their proof relies on a technical argument called "fork argument". On the other hand, Baudier and Zhang [3] gave a different proof by estimating the ℓ_p-distortion of the countably branching trees. The two proofs are based on similar ideas that use the quantification of property (β). The theorem below, which first appeared in [5], is the quantitative version of the Lima–Randrianarivony result.

Theorem 1.1. *Let X, Y be two Banach spaces. S is a subset of X and $f : S \to Y$ is a uniform quotient map that is Lipschitz for large distances. Then there exists a constant $C > 0$ that depends only on the map f such that for all $0 < t \leq 1$,*

$$\bar{\beta}_X(Ct) \leq \frac{3}{2}\bar{\rho}_Y(t).$$

The main goal of this paper is to give quantitative results of this kind in the coarse category. It should be noted that although property (β) is preserved under uniform quotient maps up to renorming (cf. [7] and [6]), one cannot compare $\bar{\beta}_X$ and $\bar{\beta}_Y$ even if X and Y are uniformly homeomorphic. Indeed, [7] gave an example of two uniformly homeomorphic Banach spaces, one of which has property (β_p), $p \in (1,\infty)$, while the other does not admit any equivalent norm with property (β_p).

Throughout this article all Banach spaces are real and of infinite dimension. For a metric space X, $B_X(x,r)$ denotes the closed ball centered at x with radius r. If X is a Banach space, we denote by B_X and S_X its closed unit ball and unit sphere, respectively.

2. Coarse quotient maps

A map $f : X \to Y$ between two metric spaces X and Y is said to be coarsely continuous if $\omega_f(t) < \infty$ for all $t > 0$, where

$$\omega_f(t) := \sup\{d_Y(f(x), f(y)) : d_X(x, y) \leq t\}$$

is the modulus of continuity of f. If X is unbounded, one can define for every $s > 0$ the Lipschitz constant of f when distances are at least s:

$$\mathrm{Lip}_s(f) := \sup\left\{\frac{d_Y(f(x), f(y))}{d_X(x, y)} : d_X(x, y) \geq s\right\}.$$

Then for all $t \geq 0$ and $s > 0$,

$$\omega_f(t) \leq \max\{\omega_f(s), \mathrm{Lip}_s(f) \cdot t\}.$$

Let

$$\mathrm{Lip}_\infty(f) := \inf_{s>0} \mathrm{Lip}_s(f) = \lim_{s\to\infty} \mathrm{Lip}_s(f).$$

The map f is said to be coarse Lipschitz if $\mathrm{Lip}_\infty(f) < \infty$ or, equivalently, if $\mathrm{Lip}_s(f) < \infty$ for some $s > 0$. If $\mathrm{Lip}_s(f) < \infty$ for all $s > 0$, then we say that f is Lipschitz for large distances. The map f is called a coarse Lipschitz embedding if there exist $d \geq 0$ and $L, C > 0$ such that for all $x, y \in X$,

$$d_X(x, y) \geq d \implies \frac{1}{C} d_X(x, y) \leq d_Y(f(x), f(y)) \leq L d_X(x, y). \tag{2.1}$$

The following notion of coarse quotient was introduced by Zhang [17].

Definition 2.1. Let X, Y be two metric spaces. For a constant $K \geq 0$, a map $f : X \to Y$ is said to be co-coarsely continuous with constant K if for every $d > K$, there exists $\delta = \delta(d) > 0$ such that for all $x \in X$,

$$f(B_X(x, \delta))^K \supseteq B_Y(f(x), d).$$

Here for a subset A of X, A^K denotes the K-neighborhood of A:

$$A^K := \{x \in X : d_X(x, a) \leq K \text{ for some } a \in A\}.$$

If f is both coarsely continuous and co-coarsely continuous (with constant K), then we say f is a coarse quotient map (with constant K). Y is said to be a coarse quotient of X if there exists a coarse quotient map from X to Y.

Recall that a metric space X is said to be metrically convex if for every $x_0, x_1 \in X$ and $0 < \lambda < 1$, there is a point $x_\lambda \in X$ such that

$$d_X(x_0, x_\lambda) = \lambda d_X(x_0, x_1) \quad \text{and} \quad d_X(x_1, x_\lambda) = (1 - \lambda) d_X(x_0, x_1).$$

It is well known that a coarsely continuous map defined on a metrically convex space must be Lipschitz for large distances. Similarly, if the range space of a co-coarsely continuous map with constant K is metrically convex, then the map is "co-Lipschitz for large distances with constant K" in the sense of the lemma below.

Lemma 2.2. *Let X, Y be two metric spaces and assume that Y is metrically convex. If $f : X \to Y$ is a co-coarsely continuous map with constant K, then for every $d > K$, there exists $c = c(d, K) > 0$ such that for all $x \in X$ and $r \geq d$,*

$$f(B_X(x, cr))^K \supseteq B_Y(f(x), r). \tag{2.2}$$

Proof. For $x \in X$ and $r \geq d$, let $n := \lfloor \frac{r}{d-K} \rfloor + 1$. Then for every $y \in B_Y(f(x), r)$, $d_Y(y, f(x)) \leq r < n(d-K)$. By the metric convexity of Y, one can find points $\{u_i\}_{i=0}^n$ in Y with $u_0 = f(x)$ and $u_n = y$ such that $d_Y(u_i, u_{i-1}) < d - K$, $i = 1, \ldots, n$. Since f is co-coarsely continuous with constant K, we have

$$u_1 \in B_Y(f(x), d) \subseteq f(B_X(x, \delta))^K,$$

where $\delta = \delta(d) > 0$ is given by Definition 2.1, so there exists $z_1 \in B_X(x, \delta)$ so that $d_Y(u_1, f(z_1)) \leq K$. This implies, by the triangle inequality, that $u_2 \in B_Y(f(z_1), d)$. Again the co-coarse continuity of f yields another point, $z_2 \in B_X(z_1, \delta)$, that satisfies $d_Y(u_2, f(z_2)) \leq K$. Repeating the procedure n times we get points $\{z_i\}_{i=0}^n$ in X, where $z_0 = x$, with the following properties: $d_X(z_i, z_{i-1}) \leq \delta$ and $d_Y(u_i, f(z_i)) \leq K$, $i = 1, \ldots, n$. It follows that $z_n \in B_X(x, n\delta)$, and hence $y \in f(B_X(x, n\delta))^K$. Note that $n \leq \left(\frac{1}{d-K} + \frac{1}{d}\right) r$; thus (2.2) follows by putting $c = \left(\frac{1}{d-K} + \frac{1}{d}\right) \delta$. □

Remark 2.3. Lemma 2.2 improves Lemma 3.2 in [17], in which $d > 2K$ is required. Also, for $d > K$, it follows from (2.2) that the constant $c = c(d, K) > 0$ satisfies

$$f(B_X(x, cr))^d \supseteq B_Y(f(x), r)$$

for all $x \in X$ and $r > 0$. It means that co-coarsely continuous maps are co-Lipschitz with a slightly larger constant if the range space is metrically convex.

Under the assumption of Lemma 2.2, for $d > K$, let c_d be the infimum of all c that satisfy (2.2) for all $x \in X$ and $r \geq d$. Then $\{c_d\}_{d>K}$ is nonincreasing and bounded below by 0, hence converges. Denote $c_\infty(f) := \inf_{d>K} c_d = \lim_{d\to\infty} c_d$.

Lemma 2.4. *Let X, Y be two metric spaces and assume that Y is metrically convex and unbounded. If $f : X \to Y$ is a coarse quotient map that is coarse Lipschitz, then*

$$\mathrm{Lip}_\infty(f) c_\infty(f) \geq 1.$$

Proof. Let f be a coarse quotient map with constant K. Since $Y = f(X)^K$ is unbounded, it follows that X is also unbounded and $\mathrm{Lip}_s(f) > 0$ for all $s > 0$. Now by Lemma 2.2, for $d > K$, there exists $c = c(d, K) > 0$ such that for all $x \in X$ and $r \geq d$,

$$B_Y(f(x), r) \subseteq f(B_X(x, cr))^K \subseteq B_Y(f(x), \omega_f(cr))^K = B_Y(f(x), \omega_f(cr) + K),$$

and this implies that $r \leq \omega_f(cr) + K$. Since f is coarse Lipschitz, let $s > 0$ be such that $0 < \mathrm{Lip}_s(f) < \infty$. Then for $t \geq s$ one has

$$r \leq \omega_f(cr) + K \leq \max\{\omega_f(t), \mathrm{Lip}_t(f) \cdot cr\} + K.$$

Choose large r so that $\mathrm{Lip}_t(f) \cdot cr > \omega_f(t)$. Then $r \leq \mathrm{Lip}_t(f) \cdot cr + K$, so

$$\mathrm{Lip}_t(f) \cdot c \geq \frac{r - K}{r},$$

and we finish the proof by letting $r \to \infty$ and then $t \to \infty$.

□

3. Quantitative results under coarse quotient maps

Before stating our main theorem, we need the following alternative definition for the modulus of AUS. It may be well known to experts, but we still give a proof here since we could not find one in the literature.

Proposition 3.1. *Let X be a Banach space. Then for all $0 < t \leq 1$*

$$\bar{\rho}_X(t) = \sup_{x \in B_X} \inf_{\dim(X/Y)<\infty} \sup_{y \in S_Y} \|x + ty\| - 1. \tag{3.1}$$

Proof. First we show that for every $x, y \in X$ the function

$$f(\lambda) = \max\{\|\lambda x + y\|, \|\lambda x - y\|\}$$

is nondecreasing on $(0, \infty)$. Let $0 < \lambda_1 < \lambda_2$; we may assume that $\|\lambda_1 x + y\| \geq \|\lambda_1 x - y\|$ and let $x^* \in S_{X^*}$ be such that $x^*(\lambda_1 x + y) = \|\lambda_1 x + y\|$. Then $x^*(x) \geq 0$, since otherwise

$$\|\lambda_1 x + y\| \geq \|\lambda_1 x - y\| \geq (-x^*)(\lambda_1 x - y) > x^*(\lambda_1 x + y) = \|\lambda_1 x + y\|.$$

Therefore, $f(\lambda_1) = x^*(x)\lambda_1 + x^*(y) \leq x^*(x)\lambda_2 + x^*(y) \leq \|\lambda_2 x + y\| \leq f(\lambda_2)$.

Now we prove (3.1). If $x = 0$, then

$$\inf_{\dim(X/Y)<\infty} \sup_{y \in S_Y} \|x + ty\| = t \leq \bar{\rho}_X(t) + 1.$$

For $x \in B_X \backslash \{0\}$ one has

$$\begin{aligned}
\inf_{\dim(X/Y)<\infty} \sup_{y \in S_Y} \|x+ty\| &= \inf_{\dim(X/Y)<\infty} \sup_{y \in S_Y} \max\{\|x + ty\|, \|x - ty\|\} \\
&\leq \inf_{\dim(X/Y)<\infty} \sup_{y \in S_Y} \max\left\{\left\|\frac{x}{\|x\|} + ty\right\|, \left\|\frac{x}{\|x\|} - ty\right\|\right\} \\
&= \inf_{\dim(X/Y)<\infty} \sup_{y \in S_Y} \left\|\frac{x}{\|x\|} + ty\right\| \\
&\leq \bar{\rho}_X(t) + 1;
\end{aligned}$$

thus (3.1) follows. ∎

Theorem 3.2. *Let X, Y be two Banach spaces. S is a subset of X and $f : S \to Y$ is a coarse quotient map that is coarse Lipschitz. Then for all $0 < t \leq 1$,*

$$\bar{\beta}_X\left(\frac{t}{48\mathrm{Lip}_\infty(f)c_\infty(f)}\right) \leq \frac{3}{2}\bar{\rho}_Y(t).$$

Proof. Let $K \geq 0$ be the constant associated with the coarse quotient map f. Since f is coarse Lipschitz, it follows from Lemma 2.4 that $0 < \mathrm{Lip}_\infty(f) < \infty$. Choose $s > 0$ such that $\mathrm{Lip}_s(f) < 2\mathrm{Lip}_\infty(f)$. For $0 < t \leq 1$, one has $0 \leq \bar{\rho}_Y(t) \leq t \leq 1$. Let $\varepsilon > 0$ be small so that

$$\varepsilon < \min\left\{\frac{1}{2}, \frac{2 - \bar{\rho}_Y(t)}{6\mathrm{Lip}_\infty(f) + 2}, c_\infty(f)\right\},$$

and choose a large d that satisfies

$$d > \max\left\{\frac{3K}{\varepsilon}, \frac{12(2K + \omega_f(s))}{t}\right\} \quad \text{and} \quad c_{d/3} < c_\infty(f) + \varepsilon.$$

Since $c_\infty(f) - \varepsilon < c_d$, there exist $z_\varepsilon \in S$ and $R \geq d$ such that

$$B_Y(f(z_\varepsilon), R) \nsubseteq f(B_S(z_\varepsilon, R(c_\infty(f) - \varepsilon)))^K,$$

so there is $y_\varepsilon \in Y$ with $0 < \|y_\varepsilon - f(z_\varepsilon)\| \le R$ such that

$$B_Y(y_\varepsilon, K) \cap f(B_S(z_\varepsilon, R(c_\infty(f) - \varepsilon))) = \emptyset. \tag{3.2}$$

Now cut the line segment $[y_\varepsilon, f(z_\varepsilon)]$ into three equal pieces, namely, let $m, M \in Y$ be such that $m - f(z_\varepsilon) = M - m = y_\varepsilon - M$, then

$$m \in B_Y\left(f(z_\varepsilon), \frac{R}{3}\right) \subseteq f\left(B_S\left(z_\varepsilon, \frac{R}{3}(c_\infty(f) + \varepsilon)\right)\right)^K,$$

so there is $x \in S$ such that

$$\|x - z_\varepsilon\| \le \frac{R}{3}(c_\infty(f) + \varepsilon) \quad \text{and} \quad \|m - f(x)\| \le K.$$

By the definition of $\bar{\rho}_Y(t)$ (Proposition 3.1), there exists a finite-codimensional subspace Z of Y so that

$$\sup_{z \in S_Z} \left\| M - m + \frac{tR}{3} z \right\| < \frac{R}{3}(1 + \bar{\rho}_Y(t) + \varepsilon). \tag{3.3}$$

Set $y_n := M + \frac{tR}{3} e_n$, where $\{e_n\}_{n=1}^\infty$ is a basic sequence in S_Z with basis constant less than 2. Then

$$\|y_n - m\| = \left\| M - m + \frac{tR}{3} e_n \right\| < \frac{R}{3}(1 + \bar{\rho}_Y(t) + \varepsilon),$$

and by the triangle inequality,

$$\|y_n - f(x)\| < \frac{R}{3}(1 + \bar{\rho}_Y(t) + \varepsilon) + K < \frac{R}{3}(1 + \bar{\rho}_Y(t) + 2\varepsilon).$$

Thus

$$\begin{aligned} y_n &\in B_Y\left(f(x), \frac{R(1 + \bar{\rho}_Y(t) + 2\varepsilon)}{3}\right) \\ &\subseteq f\left(B_S\left(x, \frac{R(1 + \bar{\rho}_Y(t) + 2\varepsilon)(c_\infty(f) + \varepsilon)}{3}\right)\right)^K, \end{aligned}$$

so there exists $z_n \in S$ such that

$$\|z_n - x\| \le \frac{R}{3}(1 + \bar{\rho}_Y(t) + 2\varepsilon)(c_\infty(f) + \varepsilon) \quad \text{and} \quad \|y_n - f(z_n)\| \le K.$$

Also, the choice of m, M, and y_n along with inequality (3.3) give us that

$$\|y_\varepsilon - y_n\| = \left\| y_\varepsilon - M - \frac{tR}{3} e_n \right\| = \left\| M - m - \frac{tR}{3} e_n \right\| < \frac{R}{3}(1 + \bar{\rho}_Y(t) + \varepsilon),$$

so again by the triangle inequality,

$$\|y_\varepsilon - f(z_n)\| < \frac{R}{3}(1 + \bar{\rho}_Y(t) + \varepsilon) + K < \frac{R}{3}(1 + \bar{\rho}_Y(t) + 2\varepsilon).$$

Hence

$$\begin{aligned} y_\varepsilon &\in B_Y\left(f(z_n), \frac{R(1 + \bar{\rho}_Y(t) + 2\varepsilon)}{3}\right) \\ &\subseteq f\left(B_S\left(z_n, \frac{R(1 + \bar{\rho}_Y(t) + 2\varepsilon)(c_\infty(f) + \varepsilon)}{3}\right)\right)^K, \end{aligned}$$

and this gives $x_n \in S$ that satisfies

$$\|x_n - z_n\| \le \frac{R}{3}(1 + \bar{\rho}_Y(t) + 2\varepsilon)(c_\infty(f) + \varepsilon) \quad \text{and} \quad \|y_\varepsilon - f(x_n)\| \le K.$$

On the other hand, in view of (3.2), one has $\|z_\varepsilon - x_n\| > R(c_\infty(f)-\varepsilon)$, so

$$\begin{aligned}\|z_\varepsilon - z_n\| &\geq \|z_\varepsilon - x_n\| - \|x_n - z_n\| \\ &> R(c_\infty(f)-\varepsilon) - \frac{R}{3}(1+\bar\rho_Y(t)+2\varepsilon)(c_\infty(f)+\varepsilon) \\ &= \frac{R}{3}(c_\infty(f)+\varepsilon)\left(\frac{3(c_\infty(f)-\varepsilon)}{c_\infty(f)+\varepsilon} - 1 - \bar\rho_Y(t) - 2\varepsilon\right) \\ &\geq \frac{R}{3}(c_\infty(f)+\varepsilon)\left(3\left(1-\frac{2\varepsilon}{c_\infty(f)}\right) - 1 - \bar\rho_Y(t) - 2\varepsilon\right) \\ &\overset{(*)}{\geq} \frac{R}{3}(c_\infty(f)+\varepsilon)(2-\bar\rho_Y(t) - (6\mathrm{Lip}_\infty(f)+2)\varepsilon) > 0,\end{aligned}$$

where the inequality $(*)$ follows from Lemma 2.4 that $\mathrm{Lip}_\infty(f)c_\infty(f) \geq 1$.

Now for $n, k \in \mathbb{N}$ with $n \neq k$,

$$\|y_n - y_k\| = \frac{tR}{3}\|e_n - e_k\| > \frac{tR}{6} > \omega_f(s) + 2K;$$

also note that

$$\begin{aligned}\|y_n - y_k\| &\leq \|y_n - f(z_n)\| + \|f(z_n) - f(z_k)\| + \|y_k - f(z_k)\| \\ &\leq 2K + \omega_f(\|z_n - z_k\|),\end{aligned}$$

so $\omega_f(\|z_n - z_k\|) > \omega_f(s)$, which implies $\|z_n - z_k\| > s$. It then follows that

$$\begin{aligned}\frac{tR}{6} < \|y_n - y_k\| &\leq \|y_n - f(z_n)\| + \|f(z_n) - f(z_k)\| + \|y_k - f(z_k)\| \\ &\leq 2K + \mathrm{Lip}_s(f)\|z_n - z_k\| \\ &< \frac{tR}{12} + 2\mathrm{Lip}_\infty(f)\|z_n - z_k\|,\end{aligned}$$

and hence $\|z_n - z_k\| > tR/(24\mathrm{Lip}_\infty(f))$.

In summary, for $n, k \in \mathbb{N}$ with $n \neq k$ we have the following:

$$\|z_n - z_k\| > \frac{tR}{24\mathrm{Lip}_\infty(f)}, \ \|z_\varepsilon - z_n\| > \frac{R}{3}(c_\infty(f)+\varepsilon)(2-\bar\rho_Y(t)-(6\mathrm{Lip}_\infty(f)+2)\varepsilon),$$

$$\|z_\varepsilon - x\| \leq \frac{R}{3}(c_\infty(f)+\varepsilon), \qquad \|z_n - x\| \leq \frac{R}{3}(c_\infty(f)+\varepsilon)(1+\bar\rho_Y(t)+2\varepsilon).$$

Denote

$$\bar x := \frac{z_\varepsilon - x}{\frac{R}{3}(c_\infty(f)+\varepsilon)(1+\bar\rho_Y(t)+2\varepsilon)} \quad \text{and} \quad \bar x_n := \frac{z_n - x}{\frac{R}{3}(c_\infty(f)+\varepsilon)(1+\bar\rho_Y(t)+2\varepsilon)}.$$

Then $\bar x, \bar x_n \in B_X$, and for $n \neq k$,

$$\|\bar x_n - \bar x_k\| > \frac{tR}{24\mathrm{Lip}_\infty(f)} \cdot \frac{1}{\frac{R}{3}(c_\infty(f)+\varepsilon)(1+\bar\rho_Y(t)+2\varepsilon)} \geq \frac{t}{48\mathrm{Lip}_\infty(f)c_\infty(f)},$$

$$\frac{\|\bar x - \bar x_n\|}{2} > \frac{1}{2}\cdot\frac{2-\bar\rho_Y(t)-(6\mathrm{Lip}_\infty(f)+2)\varepsilon}{1+\bar\rho_Y(t)+2\varepsilon} \geq 1 - \frac{3}{2}\bar\rho_Y(t) - (3\mathrm{Lip}_\infty(f)+3)\varepsilon.$$

Therefore, by the definition of (β)-modulus we obtain

$$\bar\beta_X\left(\frac{t}{48\mathrm{Lip}_\infty(f)c_\infty(f)}\right) \leq \frac{3}{2}\bar\rho_Y(t) + (3\mathrm{Lip}_\infty(f)+3)\varepsilon.$$

The proof is complete by letting $\varepsilon \to 0$. $\square$

It is easy to compute that for $1 < p < \infty$ and $0 < t \leq 1$,

$$\bar{\delta}_{\ell_p}(t) = \bar{\rho}_{\ell_p}(t) = (1+t^p)^{\frac{1}{p}} - 1,$$
$$\bar{\delta}_{c_0}(t) = \bar{\rho}_{c_0}(t) = 0,$$

so it follows from Theorem 4.1 in [6] that ℓ_p has property (β_p), and thus we can recover the main result of [17] as an immediate consequence of Theorem 3.2.

Corollary 3.3.

(i) *ℓ_q is not a coarse quotient of ℓ_p for $1 < p < q < \infty$.*
(ii) *c_0 is not a coarse quotient of any Banach space with property (β).*

4. Quantitative results under coarse homeomorphisms

This section is devoted to a special case of Theorem 3.2 when the coarse quotient map f is a coarse homeomorphism. Recall that a coarsely continuous map $f : X \to Y$ between two metric spaces X and Y is called a coarse homeomorphism if there exists another coarsely continuous map $g : Y \to X$ such that

$$\sup_{x\in X} d_X(g\circ f(x), x) < \infty \quad \text{and} \quad \sup_{y\in Y} d_Y(f\circ g(y), y) < \infty.$$

It was proved in [17] that a coarse homeomorphism is necessarily a coarse quotient map.

The main tool we need is approximate metric midpoint, which was first used by Enflo (unpublished) to show that L_1 is not uniformly homeomorphic to ℓ_1 (see, e.g., [4]). Given two points x, y in a metric space X and $\delta \in (0,1)$, the set of δ-approximate metric midpoints between x and y is defined by

$$\mathrm{Mid}(x,y,\delta) := \left\{ z \in X : \max\{d_X(z,x), d_X(z,y)\} \leq \frac{1+\delta}{2} d_X(x,y) \right\}.$$

The lemma below, which first appeared in [10] (see also Proposition 14.5.5 in [1]), is sometimes known as the "stretching lemma".

Lemma 4.1. *Let $f : X \to Y$ be a coarse Lipschitz map from an unbounded metric space X to a metric space Y. If $\mathrm{Lip}_\infty(f) > 0$, then for any $d > 0$, any $\varepsilon > 0$, and any $0 < \delta < 1$ there exist $x, y \in X$ with $d_X(x,y) \geq d$ such that*

$$f(\mathrm{Mid}(x,y,\delta)) \subseteq \mathrm{Mid}(f(x), f(y), (1+\varepsilon)\delta).$$

The next lemma, which follows immediately froms Propositions 3.4.1 and 3.4.2 in [15], relates the set of approximate metric midpoints in a Banach space with the moduli of AUC and AUS of the space.

Lemma 4.2. *Let X be a Banach space. $x, y \in X$ and $0 < t \leq 1$.*

(i) *For every $\varepsilon > 0$, there exists a finite-codimensional subspace Y of X such that*
$$\frac{x+y}{2} + \frac{t\|x-y\|}{2} B_Y \subseteq \mathrm{Mid}(x,y,\bar{\rho}_X(t)+\varepsilon).$$
(ii) *If $\bar{\delta}_X(t) > 0$, then for every $0 < \varepsilon < 1$, there exists a compact subset K of X such that*
$$\mathrm{Mid}(x,y,(1-\varepsilon)\bar{\delta}_X(t)) \subseteq K + \frac{3t\|x-y\|}{2} B_X.$$

We also need the following easy lemma.

Lemma 4.3. *Let $f : X \to Y$ be a map between Banach spaces X and Y. If there exist $r, s > 0$, a point $x \in X$, an infinite-dimensional subspace Z of X, and a compact subset K of Y such that*

$$f(x + rB_Z) \subseteq K + sB_Y,$$

then the compression modulus φ_f of f satisfies

$$\varphi_f(r) := \inf\{\|f(x) - f(y)\| : \|x - y\| \geq r\} \leq 2s.$$

Proof. Since Z is infinite dimensional, we may choose an r-separating sequence $\{x_n\}_{n=1}^{\infty}$ in $x + rB_Z$ and let $f(x_n) = z_n + y_n$, where $z_n \in K$ and $y_n \in sB_Y$ for every $n \in \mathbb{N}$. For $\varepsilon > 0$, since K is compact, by passing to a subsequence we may assume that $\|z_n - z_m\| < \varepsilon$ for all $m, n \in \mathbb{N}$. Then for $m \neq n$,

$$2s \geq \|y_n - y_m\| \geq \|f(x_n) - f(x_m)\| - \|z_n - z_m\| \geq \varphi_f(r) - \varepsilon,$$

and we are done by letting $\varepsilon \to 0$. □

Theorem 4.4 below is a variant of Theorem 3.5.1 in [15]. Here we present a proof with improved constants.

Theorem 4.4. *Let X and Y be two Banach spaces and let $f : X \to Y$ be a coarse Lipschitz embedding with constants d, L, and C given by* (2.1)*. Then for all $0 < t \leq 1$,*

$$\bar{\delta}_Y\left(\frac{t}{7LC}\right) \leq \bar{\rho}_X(t).$$

Proof. The proof is trivial if $t \in (0, 1]$ satisfies $\bar{\delta}_Y(t/7LC) = 0$, so we may assume that $\bar{\delta}_Y(t/7LC) > 0$. Let $0 < \varepsilon < 1/2$. Note that $1/C \leq \mathrm{Lip}_\infty(f) \leq L$, then one can apply the stretching lemma to find $u, v \in X$ with $\|u - v\| \geq 2d/t$ that satisfy

$$f\left(\mathrm{Mid}\left(u, v, (1 - 2\varepsilon)\bar{\delta}_Y\left(\frac{t}{7LC}\right)\right)\right) \subseteq \mathrm{Mid}\left(f(u), f(v), (1 - \varepsilon)\bar{\delta}_Y\left(\frac{t}{7LC}\right)\right).$$

By Lemma 4.2(ii), there exists a compact set $K \subseteq Y$ such that

$$\begin{aligned}\mathrm{Mid}\left(f(u), f(v), (1 - \varepsilon)\bar{\delta}_Y\left(\frac{t}{7LC}\right)\right) &\subseteq K + \frac{3t}{14LC}\|f(u) - f(v)\|B_Y \\ &\subseteq K + \frac{3t}{14C}\|u - v\|B_Y.\end{aligned}$$

Assume that there exists $\tau > 0$ such that

$$(1 - 2\varepsilon)\bar{\delta}_Y\left(\frac{t}{7LC}\right) > \bar{\rho}_X(t) + \tau.$$

Then by Lemma 4.2(i), there exists a finite-codimensional subspace Z of X such that

$$f\left(\mathrm{Mid}\left(u, v, (1 - 2\varepsilon)\bar{\delta}_Y\left(\frac{t}{7LC}\right)\right)\right) \supseteq f\left(\frac{u + v}{2} + \frac{t\|u - v\|}{2}B_Z\right),$$

so we have

$$f\left(\frac{u + v}{2} + \frac{t\|u - v\|}{2}B_Z\right) \subseteq K + \frac{3t}{14C}\|u - v\|B_Y.$$

Now it follows from Lemma 4.3 that

$$\frac{t\|u - v\|}{2C} \leq \varphi_f\left(\frac{t\|u - v\|}{2}\right) \leq \frac{3t}{7C}\|u - v\|,$$

which is a contradiction. Therefore, we must have

$$(1-2\varepsilon)\bar{\delta}_Y\left(\frac{t}{7LC}\right) \leq \bar{\rho}_X(t).$$

We then finish the proof by letting $\varepsilon \to 0$. ☐

Theorem 4.5. *Let X, Y be two Banach spaces. S is a subset of X and $f : S \to Y$ is a coarse homeomorphism that is coarse Lipschitz. Then for all $0 < t \leq 1$,*

$$\bar{\delta}_X\left(\frac{t}{56\mathrm{Lip}_\infty(f)c_\infty(f)}\right) \leq \bar{\rho}_Y(t).$$

Proof. Let $g : Y \to S$ be a coarsely continuous map that satisfies

$$\sup_{x\in S}\|g\circ f(x) - x\| := M < \infty \quad \text{and} \quad \sup_{y\in Y}\|f\circ g(y) - y\| := K < \infty.$$

We claim that g is a coarse Lipschitz embedding from Y into X. Indeed, it follows from Proposition 2.5 in [17] that f is a coarse quotient map with constant K. Choose $s > 2K$ such that $\mathrm{Lip}_s(f) < 2\mathrm{Lip}_\infty(f)$, and let $d > 6K$ be such that $\varphi_g(d) > s$. For $y_1, y_2 \in Y$ with $\|y_1 - y_2\| \geq d$, one has $\|g(y_1) - g(y_2)\| \geq \varphi_g(d) > s$, and thus

$$\|f\circ g(y_1) - f\circ g(y_2)\| \leq 2\mathrm{Lip}_\infty(f)\|g(y_1) - g(y_2)\|.$$

By the triangle inequality,

$$\|f\circ g(y_1) - f\circ g(y_2)\| \geq \|y_1 - y_2\| - 2K \geq \frac{2}{3}\|y_1 - y_2\|,$$

so we obtain

$$\frac{1}{3\mathrm{Lip}_\infty(f)}\|y_1 - y_2\| \leq \|g(y_1) - g(y_2)\|.$$

On the other hand, we could make d even larger so that $c_{d-K} < 2c_\infty(f)$ and

$$\frac{d}{3}\cdot c_\infty(f) > \omega_g(K) + M.$$

Note that

$$\|y_1 - y_2\| + K \geq r := \|y_1 - f\circ g(y_2)\| \geq \|y_1 - y_2\| - K \geq d - K > K,$$

so it follows from Lemma 2.2 and the definition of c_{d-K} that

$$y_1 \in B_Y(f\circ g(y_2), r) \subseteq f(B_S(g(y_2), 2rc_\infty(f)))^K,$$

and from this we can find $x \in S$ satisfying

$$\|x - g(y_2)\| \leq 2rc_\infty(f) \quad \text{and} \quad \|y_1 - f(x)\| \leq K.$$

Now again by the triangle inequality,

$$\begin{aligned}
\|g(y_1) - g(y_2)\| &\leq \|g(y_1) - g\circ f(x)\| + \|g\circ f(x) - x\| + \|x - g(y_2)\| \\
&\leq \omega_g(K) + M + 2rc_\infty(f) \\
&\leq \frac{c_\infty(f)}{3}\|y_1 - y_2\| + 2c_\infty(f)(\|y_1 - y_2\| + K) \\
&\leq \frac{8c_\infty(f)}{3}\|y_1 - y_2\|.
\end{aligned}$$

Therefore, for sufficiently large d, one has

$$\frac{1}{3\mathrm{Lip}_\infty(f)}\|y_1 - y_2\| \leq \|g(y_1) - g(y_2)\| \leq \frac{8c_\infty(f)}{3}\|y_1 - y_2\|$$

whenever $\|y_1 - y_2\| \geq d$. The desired inequality then follows from Theorem 4.4. □

Remark 4.6. In connection with the modulus of asymptotic uniform convexity, the modulus of asymptotic midpoint uniform convexity of a Banach space X was introduced in [8] as follows:

$$\tilde{\delta}_X(t) := \inf_{x \in S_X} \sup_{\dim(X/Y)<\infty} \inf_{y \in S_Y} \max\{\|x + ty\|, \|x - ty\|\} - 1.$$

A Banach space X is said to be asymptotically midpoint uniformly convex (AMUC for short) if $\tilde{\delta}_X(t) > 0$ for all $0 < t \leq 1$. It was shown in the proof of Theorem 2.1 in [8] that Lemma 4.2(ii) still holds true for $\tilde{\delta}_X(t)$. Therefore, Theorems 4.4 and 4.5 can be strengthened by replacing the AUC modulus with the AMUC modulus.

Corollary 4.7.

(i) ℓ_q *does not coarse Lipschitz embed into* ℓ_p *for* $1 < p < q < \infty$.

(ii) c_0 *does not coarse Lipschitz embed into any AMUC Banach space.*

Acknowledgment

The author would like to thank William B. Johnson for the proof of Proposition 3.1. The author would also like to thank the referee for all the comments and suggestions.

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