PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 11, November 2018, Pages 4741–4753
http://dx.doi.org/10.1090/proc/14099
Article electronically published on August 10, 2018

L^∞-REGULARITY FOR A WIDE CLASS OF PARABOLIC SYSTEMS WITH GENERAL GROWTH

TERESA ISERNIA

(Communicated by Joachim Krieger)

ABSTRACT. We prove the local boundedness of weak solutions for the following non-linear second order parabolic systems:

$$u_t - \operatorname{div}\left(\frac{\varphi'(|Du|)}{|Du|}Du\right) = 0 \text{ in } \Omega_T := \Omega \times (-T, 0),$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and φ is a given N-function. The proof of this result is based on a Moser-type iteration argument.

1. INTRODUCTION

The aim of this work is the study of the local regularity of φ-caloric functions. For φ-caloric functions we mean local weak solutions $u : \Omega_T \to \mathbb{R}^N$ of the following parabolic system:

$$u_t - \operatorname{div}\left(\frac{\varphi'(|Du|)}{|Du|}Du\right) = 0 \ \text{ in } \Omega_T := \Omega \times (-T, 0), \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, $N > 1$, $T > 0$, and $\varphi : [0,\infty) \to [0,\infty)$ is a given N-function that satisfies the natural condition $\varphi(s) \sim s\,\varphi'(s)$ uniformly in $s \geq 0$ (see Section 2). In the model case $\varphi(s) = s^p$ and $p > 1$, (1.1) gives the evolutionary p-Laplacian. Therefore, the system (1.1) can be seen as a generalization of the p-Laplacian parabolic system.

In the last fifty years the study of the regularity of weak solutions for vectorial problems has received great attention from many mathematicians. This is due to the fact that, in contrast with the scalar case, we cannot expect that a weak solution to a non-linear system is a classical solution; see for instance [8, 13].

In what follows, we recall some well-known regularity results related to the investigation of elliptic systems. In the fundamental paper [21], Uhlenbeck proved the everywhere $C^{1,\alpha}$-regularity for local minimizers of a p-growth functional with $p \geq 2$. Later on, a large number of generalizations were made. The case $1 < p < 2$ was studied in [1], where also the dependence of the functional from x and u was investigated. Lipschitz regularity for systems or functionals with general growth conditions have been considered by many authors. The papers [15, 16] of Marcellini have been the starting point; in these papers the author established the Lipschitz

Received by the editors July 24, 2017, and, in revised form, January 16, 2018.
2010 *Mathematics Subject Classification.* Primary 35B65, 49N60, 35K40, 46E30.
Key words and phrases. φ-caloric functions, parabolic systems, Moser iteration, general growth.

continuity of weak solutions in the case of $p-q$- growth. In [17], the local Lipschitz continuity of minimizers of the integral functional

$$\mathcal{F}(u) = \int_\Omega f(Du)\,dx \tag{1.2}$$

has been proved assuming that the integrand function $f(Du)$ is of the form $g(|Du|)$, where $g : [0,+\infty) \to \mathbb{R}$ is a convex function such that $\frac{g'(s)}{s}$ is increasing in $(0,+\infty)$. In [7] the authors dealt with functionals of the type (1.2) with $f(Du) = g(x, |Du|)$ and $g(x,\cdot)$ an N-function: considering the problem in the Orlicz-Musielak spaces, under the main assumption of the uniform Δ_2-condition on g, the authors proved the local boundedness of weak solutions. Marcellini & Papi [18] obtained the Lipschitz regularity for minimizers of functionals of the type (1.2) with growth conditions general enough to embrace linear and exponential ones. In [12] the authors established the $C^{1,\alpha}$-regularity for local minimizers of (1.2) in the case $f(\xi) = \varphi(\xi)$, where φ is a given N-function that satisfies the Δ_2-condition together with its conjugate, giving a unified approach to superquadratic and subquadratic p-growth, besides considering more general functions than powers. Subsequently in [4] the authors proved the local boundedness of the gradient of local minimizers under weaker assumptions on φ.

Regarding the evolutionary framework, many authors are interested in the Hölder regularity of weak solutions for parabolic systems with p-growth. In [9, 10] has been established the Hölder continuity of the gradient of solutions to parabolic systems in the case $p > \frac{2n}{n+2}$, while Wiegner [22] obtained, independently, the same result for $p \geq 2$. Choe [6] proved that if $u \in L^q_{\mathrm{loc}}$ with $q > \frac{N(2-p)}{p}$, then it is possible to establish the Hölder continuity of the spatial gradient of solutions for all $p \in (1,\infty)$.

Compared to the elliptic case, there are only a few regularity results for parabolic problems with non-standard growth. For instance, You [23] extended the fundamental results of DiBenedetto & Friedman in the case of systems with a vector field of Uhlenbeck-type satisfying suitable Orlicz-type growth conditions. More precisely, the author derived L^∞-regularity for solutions of the following non-linear parabolic system:

$$u_t - \operatorname{div}(g(|Du|)Du) = 0 \ \text{ in } \ \Omega_T \tag{1.3}$$

assuming that for every $\xi > 0$, $g(\xi)$ satisfies

$$c_1 \xi^{p-2} \leq g(\xi) \leq c_2(\xi^{p-2} - \xi^{q-2}),$$

where c_1 and c_2 are fixed constants and p, q are related by $1 < p \leq q < p+1 < \infty$. Lieberman [14] established the Hölder continuity for the gradient of solutions to (1.3) assuming that g is a $C^1(0,\infty)$ and positive function satisfying, for all $s \geq 0$,

$$\delta - 1 \leq \frac{s g'(s)}{g(s)} \leq g_0 - 1,$$

with $\delta \in (0, 1]$, $g_0 \geq 1$. Recently, in [5] the authors obtained the spatial second order Caccioppoli estimate for a local weak solution to (1.1) in both the cases of symmetric gradient and full gradient. Subsequently, by using the De Giorgi iteration technique, Diening et al. [11] proved the boundedness of the spatial gradient of solutions to (1.1) assuming that φ satisfies the Δ_2-condition together with its conjugate.

Motivated by the above papers, in this work we aim to show the local boundedness of weak solutions to (1.1). First, we recall the notion of weak solution employed in this paper.

Definition 1.1. A function $u \in C(-T,0;L^2(\Omega)) \cap L^\varphi(-T,0;W^{1,\varphi}(\Omega,\mathbb{R}^N))$ is a weak solution for (1.1) if

$$\int_{\Omega_T} u\phi_t - \Big\langle \frac{\varphi'(|Du|)}{|Du|} Du, D\phi \Big\rangle \, dz = 0$$

is satisfied for all testing functions $\phi \in C_c^\infty(\Omega_T,\mathbb{R}^N)$.

Remark 1.1. In the parabolic setting, a standard difficulty in using test function arguments involving the solution is that we start with solutions having only weak regularity properties with respect to the time variable t (i.e., they are not assumed to be weakly differentiable). In the following, we shall argue on a formal level, that is, arguing as the solutions are differentiable with respect to time. The argument can be made rigorous in a standard way via Steklov averages as, for instance, in [5, 11].

According to the above definition, our main result can be stated as follows:

Theorem 1.1. *Let φ be an N-function satisfying the Δ_2-condition. Let u be a weak solution to* (1.1)*. Then $u \in L^\infty_{\mathrm{loc}}(\Omega_T,\mathbb{R}^N)$. Moreover, for every $\mathcal{Q}_{R_0} \Subset \Omega_T$ the following a priori estimate holds with the constant c depending on n and on the characteristic of φ:*

$$\sup_{\mathcal{Q}_{\frac{R_0}{2}}(z_0)} \varphi(|u|) \le c \left(\int_{\mathcal{Q}_{R_0}(z_0)} \varphi^{\gamma_0+1}(|u|)\, dz \right)^{\frac{1}{\gamma_0}} + c.$$

The proof of Theorem 1.1 is obtained by using a Moser-type iteration argument [19] conveniently adapted for parabolic systems with general growth conditions. We would like to point out that the local regularity of solutions of (1.1) represents a fundamental step to deduce the Hölder regularity of φ-caloric functions when we assume that φ is Δ_2 with its conjugate. Indeed, once we know that u is bounded, one can use the difference quotient method, and arguing as in Theorem 3.1 in [23], we can deduce that $Du \in L^2(\Omega_T)$. At this point, a recent result obtained by Diening et al. in [11], guarantees the Lipschitz regularity for the spatial gradient of weak solutions to (1.1). Thus, by combining this with the Hölder regularity result due to Lieberman [14] for the gradient of weak solutions of parabolic systems, we can infer that $u \in C^{1,\alpha}$. To our knowledge, the result presented here is new in literature.

Before we conclude this Introduction, we emphasize that the existence of a weak solution to (1.1) is guaranteed by the recent results established by Bögelein at al. [2, 3] in which the authors obtained, via a variational approach, some relevant existence results for parabolic systems under very general growth conditions.

1.1. **Outline of the paper.** In Section 2, we collect some preliminary notions about the N-functions and in Section 3 we give the proof of the main result.

2. Assumptions and definitions

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain; in the following Ω_T will denote the parabolic cylinder $\Omega \times (-T, 0)$, where $T > 0$. If $z \in \Omega_T$, we denote $z = (x, t)$ with $x \in \Omega$ and $t \in (-T, 0)$. In what follows c will be a general positive constant, possibly varying from line to line; we will emphasize dependencies on parameters by using parenthesis.

The writing $Du(x,t) \equiv D_x u(x,t)$ denotes the differentiation with respect to the spatial variable x while u_t stands for the differentiation with respect to the time.

With $x_0 \in \mathbb{R}^n$, we set

$$\mathcal{B}_r(x_0) \equiv \mathcal{B}(x_0, r) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$$

the open ball of $\mathbb{R}^n$ with radius $r > 0$ and center x_0. When dealing with parabolic regularity, the geometry of cylinders plays an important role. We shall use a parabolic cylinder with vertex (x_0, t_0) and width $r > 0$ given by

$$\mathcal{Q}_r(x_0, t_0) := \mathcal{B}(x_0, r) \times (t_0 - r^2, t_0).$$

Given a cylinder $\mathcal{Q} = \mathcal{B} \times (s, t)$, its parabolic boundary is

$$\partial_{\mathcal{P}} \mathcal{Q} := (\mathcal{B} \times \{s\}) \cup (\partial \mathcal{B} \times [s, t]).$$

The integral average of a function u on $\mathcal{U} \subset \mathbb{R}^{n+1}$ measurable subset with positive measure is given by

$$(u)_{\mathcal{U}} = \fint_{\mathcal{U}} u(x)dz := \frac{1}{|\mathcal{U}|} \int_{\mathcal{U}} u(x)dz,$$

where $|\mathcal{U}|$ is the $(n+1)$-Lebesgue measure of $\mathcal{U}$. The parabolic metric is defined as usual by

$$\mathrm{dist}_{\mathcal{P}}(z, z_0) := \sqrt{|x - x_0|^2 + |t - t_0|}$$

whenever $z = (x, t), z_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$.

2.1. N-functions.

The following definitions and results are standard in the context of N-functions (see [20]).

We shall say that two real functions φ_1 and φ_2 are *equivalent* and write $\varphi_1 \sim \varphi_2$ if there exist positive constants c_1, c_2 such that $c_1 \varphi_1(s) \le \varphi_2(s) \le c_2 \varphi_1(s)$ if $s \ge 0$.

Definition 2.1. A real function $\varphi : [0, \infty) \to [0, \infty)$ is said to be an N-function if $\varphi(0) = 0$ and there exists a right continuous non-decreasing derivative φ' satisfying $\varphi'(0) = 0$, $\varphi'(s) > 0$ for $s > 0$ and $\lim\limits_{s \to \infty} \varphi'(s) = \infty$. In particular, φ is convex.

The assumption widely used in order to study regularity for systems with Orlicz growth is the following:

Definition 2.2. We say that φ satisfies the Δ_2-condition (we shall write $\varphi \in \Delta_2$) if there exists a constant $c > 0$ such that

$$\varphi(2s) \le c\,\varphi(s) \quad \text{for all } s \ge 0.$$

We denote the smallest possible constant by $\Delta_2(\varphi)$.

Since $\varphi(s) \le \varphi(2s)$ the Δ_2-condition implies $\varphi(2s) \sim \varphi(s)$. Moreover, if φ is a function satisfying the Δ_2-condition, then $\varphi(s) \sim \varphi(as)$ uniformly in $s \ge 0$ for any fixed $a > 1$. Let us also note that if φ satisfies the Δ_2-condition, then any N-function which is equivalent to φ satisfies this condition, too.

From now on, we assume that φ is an N-function satisfying the Δ_2-condition.

Since $\varphi'(\cdot)$ is non-decreasing, for all $s_1, s_2 > 0$ we have

$$\varphi'(s_1)s_2 \le \varphi'(s_1)s_1 + \varphi'(s_2)s_2. \tag{2.1}$$

Moreover, by the Δ_2-condition it follows that there exists $m > 1$ such that

$$\varphi'(s)s \le m\,\varphi(s) \ \text{ for every } s > 0, \tag{2.2}$$

$$\varphi(\lambda s) \le \lambda^m\,\varphi(s) \ \text{ for every } s > 0 \text{ and } \lambda > 1. \tag{2.3}$$

Let us observe that by the convexity of φ and by the Δ_2-condition it follow that

$$\varphi(s) \sim s\,\varphi'(s) \tag{2.4}$$

uniformly in $s \ge 0$.

Finally, we say that a measurable function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ belongs to the Orlicz-Lebesgue space $L^\varphi(\Omega)$ if it satisfies

$$\int_\Omega \varphi(|u|)\,dx < \infty.$$

The space $L^\varphi(\Omega)$ is a Banach space if endowed with the Luxembourg norm

$$\|u\|_{L^\varphi(\Omega)} := \inf\left\{\lambda > 0 : \int_\Omega \varphi\left(\frac{|u|}{\lambda}\right)\,dx \le 1\right\}.$$

If u and Du belong to $L^\varphi(\Omega)$, we say that $u \in W^{1,\varphi}(\Omega)$. We denote by $W_0^{1,\varphi}(\Omega)$ the closure of $C_c^\infty(\Omega)$ functions with respect to the norm

$$\|u\|_{W^{1,\varphi}(\Omega)} := \|u\|_{L^\varphi(\Omega)} + \|Du\|_{L^\varphi(\Omega)}.$$

3. L^∞-estimate for u

This section is devoted to the proof of the main theorem. The following lemma is a key ingredient in the proof of the L^∞-estimate for the system (1.1).

Let us define, for $\gamma \ge 0$, the following auxiliary function:

$$\psi_\gamma(\xi) = \int_0^\xi \tau\,T_k^\gamma(\varphi(\tau))\,d\tau. \tag{3.1}$$

Then ψ_γ satisfies the following:

Lemma 3.1. *Let φ be an N-function such that $\varphi \in \Delta_2$ and let $\gamma \ge 0$. Then there exists a positive constant C, depending only on the characteristic of φ, such that*

$$\psi_\gamma(\xi) \sim C\xi^2 T_k^\gamma(\varphi(\xi)).$$

Proof. Let $\varphi(\tau) \in [0, k]$; then $T_k^\gamma(\varphi(\tau)) = \varphi^\gamma(\tau)$ for any $\gamma \ge 0$. Therefore, integrating by parts and by using the fact that $\varphi(s) \sim s\,\varphi'(s)$ we can see that

$$\begin{aligned}\int_0^\xi \tau\,\varphi^\gamma(\tau)\,d\tau &= \left[\frac{\tau^2}{2}\,\varphi^\gamma(\tau)\right]_0^\xi - \frac{\gamma}{2}\int_0^\xi \tau^2\,\varphi^{\gamma-1}(\tau)\,\varphi'(\tau)\,d\tau \\ &\le \frac{\xi^2}{2}\,\varphi^\gamma(\xi) - \frac{\gamma}{2}c_1\int_0^\xi \tau\,\varphi^\gamma(\tau)d\tau\end{aligned}$$

from which

$$\int_0^\xi \tau\,\varphi^\gamma(\tau)\,d\tau \le \frac{1}{2+\gamma c_1}\xi^2\,\varphi^\gamma(\xi) = \frac{1}{2+\gamma c_1}\xi^2\,T_k^\gamma(\varphi(\xi)). \tag{3.2}$$

Analogously, by applying (2.4) we have

$$\int_0^\xi \tau\,\varphi^\gamma(\tau)\,d\tau = \left[\frac{\tau^2}{2}\,\varphi^\gamma(\tau)\right]_0^\xi - \frac{\gamma}{2}\int_0^\xi \tau^2\,\varphi^{\gamma-1}(\tau)\,\varphi'(\tau)\,d\tau$$
$$\geq \frac{\xi^2}{2}\,\varphi^\gamma(\xi) - \frac{\gamma}{2}c_2\int_0^\xi \tau\,\varphi^\gamma(\tau)d\tau$$

and we deduce

$$\int_0^\xi \tau\,\varphi^\gamma(\tau)\,d\tau \geq \frac{1}{2+\gamma c_2}\xi^2\,\varphi^\gamma(\xi) = \frac{1}{2+\gamma c_2}\xi^2\,T_k^\gamma(\varphi(\xi)). \tag{3.3}$$

Putting together (3.2) and (3.3) we have the assertion.

Now, let $\varphi(\tau) > k$. Then $T_k^\gamma(\varphi(\tau)) = k^\gamma$, and we can infer

$$\psi_\gamma(\xi) = k^\gamma\int_0^\xi \tau\,d\tau = C\xi^2\,k^\gamma,$$

from which follows the thesis. □

Now we are in the position to give the proof of the main result.

Proof of Theorem 1.1. Let $0 < \rho < R$ and $z_0 = (x_0, t_0)$. Let $\chi \in C_c^1(\mathcal{B}_R(x_0))$ be a cut-off function in space such that

$$\begin{cases} 0 \leq \chi(x) \leq 1, & \\ \chi(x) = 1, & \text{in } \mathcal{B}_\rho(x_0), \\ |D\chi| \leq \frac{c}{R-\rho}, & \end{cases} \tag{3.4}$$

and let $\eta_\varepsilon \in C^1(\mathbb{R})$ be a cut-off function in time such that, with $\varepsilon > 0$ being arbitrary

$$\begin{cases} \eta_\varepsilon = 1 & \text{on } (t_0-\rho^2,\tau), \\ \eta_\varepsilon = 0 & \text{on } (-T, t_0-R^2)\cup(\tau+\varepsilon,0), \\ 0 \leq \eta_\varepsilon(t) \leq 1 & \text{on } \mathbb{R}, \\ (\eta_\varepsilon)_t = -\frac{1}{\varepsilon} & \text{on } (\tau,\tau+\varepsilon), \\ |(\eta_\varepsilon)_t| \leq \frac{C}{R-\rho} & \text{on } (t_0-R^2, t_0-\rho^2), \end{cases} \tag{3.5}$$

where $\tau \in (t_0-\rho^2, t_0)$ such that $\tau+\varepsilon < t_0$. We take as a test function in (1.1)

$$g(x,t) = u\,T_k^\gamma(\varphi(|u|))\,\eta_\varepsilon^2(t)\chi^{p_0}(x),$$

with $p_0 > 1$. We note that the following computations are formal concerning the use of the time derivative u_t. However, they can be made rigorous by the use of a mollification procedure as, for instance, by Steklov averages with respect to time (see [5, 11] for more details).

Thus we have

$$
\begin{aligned}
0 = \int_{\mathcal{Q}_R(z_0)} & - \big\{ [T_k^\gamma(\varphi(|u|))]_t |u|^2 \eta_\varepsilon^2\, \chi^{p_0} \\
& \qquad + T_k^\gamma(\varphi(|u|)) u\, u_t\, \eta_\varepsilon^2\, \chi^{p_0} + T_k^\gamma(\varphi(|u|))|u|^2 (\eta_\varepsilon^2)_t\, \chi^{p_0} \big\}\, dz \\
& + \int_{\mathcal{Q}_R(z_0)} \frac{\varphi'(|Du|)}{|Du|} T_k^\gamma(\varphi(|u|))\eta_\varepsilon^2\, \chi^{p_0} \langle Du, Du\rangle\, dz \\
& + \int_{\mathcal{Q}_R(z_0)} \frac{\varphi'(|Du|)}{|Du|} \eta_\varepsilon^2\, \chi^{p_0} \langle Du, u\, D[T_k^\gamma(\varphi(|u|))]\rangle\, dz \\
& + p_0 \int_{\mathcal{Q}_R(z_0)} \frac{\varphi'(|Du|)}{|Du|} T_k^\gamma(\varphi(|u|))\eta_\varepsilon^2\, \chi^{p_0-1} \langle u\, Du, D\chi\rangle\, dz \\
& = I + II + III + IV.
\end{aligned}
\tag{3.6}
$$

Let us consider I. Integrating by parts we obtain

$$
\begin{aligned}
& \int_{\mathcal{Q}_R(z_0)} -[T_k^\gamma(\varphi(|u|))]_t |u|^2 \eta_\varepsilon^2\, \chi^{p_0}\, dz \\
& \qquad = \int_{\mathcal{Q}_R(z_0)} [T_k^\gamma(\varphi(|u|)) 2u\, u_t\, \eta_\varepsilon^2\, \chi^{p_0} + T_k^\gamma(\varphi(|u|))|u|^2 (\eta_\varepsilon^2)_t\, \chi^{p_0}]\, dz
\end{aligned}
$$

so, taking into account the definition of η_ε, I becomes

$$
\begin{aligned}
I &= \int_{\mathcal{Q}_R(z_0)} T_k^\gamma(\varphi(|u|)) u\, u_t\, \eta_\varepsilon^2\, \chi^{p_0}\, dz \\
&= \int_{\mathcal{Q}_R(z_0)} \frac{\partial}{\partial t}(\psi_\gamma(|u|)\, \eta_\varepsilon^2\, \chi^{p_0})\, dz - \int_{\mathcal{Q}_R(z_0)} \psi_\gamma(|u|)(\eta_\varepsilon^2)_t\, \chi^{p_0}\, dz \\
&= -\int_{\mathcal{B}_R(x_0)} \int_{t_0-R^2}^{t_0-\rho^2} \psi_\gamma(|u|)(\eta_\varepsilon^2)_t\, \chi^{p_0}\, dz - \int_{\mathcal{B}_R(x_0)} \int_\tau^{\tau+\varepsilon} \psi_\gamma(|u|)(\eta_\varepsilon^2)_t\, \chi^{p_0}\, dz \\
&= -\int_{\mathcal{B}_R(x_0)} \int_{t_0-R^2}^{t_0-\rho^2} \psi_\gamma(|u|) 2\eta_\varepsilon (\eta_\varepsilon)_t\, \chi^{p_0}\, dz + \int_{\mathcal{B}_R(x_0)} \fint_\tau^{\tau+\varepsilon} \psi_\gamma(|u|) 2\eta_\varepsilon\, \chi^{p_0}\, dz,
\end{aligned}
$$

where ψ_γ is defined in (3.1).

We can see that II can be rewritten as

$$
II = \int_{\mathcal{Q}_R(z_0)} \varphi'(|Du|)|Du| T_k^\gamma(\varphi(|u|))\eta_\varepsilon^2\, \chi^{p_0}\, dz
$$

and $III \geq 0$. Regarding IV we can infer

$$
|IV| \leq p_0 \int_{\mathcal{Q}_R(z_0)} T_k^\gamma(\varphi(|u|))\eta_\varepsilon^2 \chi^{p_0-1}\, \varphi'(|Du|)|u||D\chi|\, dz.
$$

Putting together the estimates for I, II, III, and IV, (3.6) becomes

$$\begin{aligned}
&\int_{\mathcal{B}_R(x_0)} \fint_{\tau}^{\tau+\varepsilon} \psi_\gamma(|u|) 2\eta_\varepsilon \chi^{p_0}\, dz + \int_{\mathcal{Q}_R(z_0)} \varphi'(|Du|)|Du| T_k^\gamma(\varphi(|u|))\eta_\varepsilon^2 \chi^{p_0}\, dz \\
&\le \frac{c}{R-\rho} \int_{\mathcal{B}_R(x_0)} \int_{t_0-R^2}^{t_0-\rho^2} \psi_\gamma(|u|)\eta_\varepsilon \chi^{p_0}\, dz \\
&\qquad + p_0 \int_{\mathcal{Q}_R(z_0)} T_k^\gamma(\varphi(|u|))|u|\eta_\varepsilon^2 \chi^{p_0-1} \varphi'(|Du|)|D\chi|\, dz.
\end{aligned} \tag{3.7}$$

Let us observe that by (2.1), (2.2), and (2.3) we deduce that

$$\begin{aligned}
\varphi'(|Du|)\frac{p_0|D\chi|}{\chi}|u| &\le \frac{1}{4}\varphi'(|Du|)|Du| + \frac{1}{4}\varphi'\left(\frac{4p_0|D\chi|}{\chi}|u|\right)\frac{4p_0|D\chi|}{\chi}|u| \\
&\le \frac{1}{4}\varphi'(|Du|)|Du| + \frac{1}{4}p_0\,\varphi\left(\frac{4p_0|D\chi|}{\chi}|u|\right) \\
&\le \frac{1}{4}\varphi'(|Du|)|Du| + \frac{4^{p_0-1}p_0^{p_0+1}|D\chi|^{p_0}}{\chi^{p_0}}\varphi(|u|) \\
&\le \frac{1}{4}\varphi'(|Du|)|Du| + \frac{4^{p_0-1}p_0^{p_0+1}}{\chi^{p_0}|R-\rho|^{p_0}}\varphi(|u|)
\end{aligned}$$

from which

$$\begin{aligned}
p_0 &\int_{\mathcal{Q}_R(z_0)} \varphi'(|u|)\frac{|D\chi|}{\chi}|u|\eta_\varepsilon^2 \chi^{p_0} T_k^\gamma(\varphi(|u|))\, dz \\
&\le \frac{1}{4}\int_{\mathcal{Q}_R(z_0)} \varphi'(|Du|)|Du|\, T_k^\gamma(\varphi(|u|))\eta_\varepsilon^2\chi^{p_0}\, dz \\
&\quad + \frac{4^{p_0-1}p_0^{p_0+1}}{|R-\rho|^{p_0}}\int_{\mathcal{Q}_R(z_0)} \varphi(|u|)\, T_k^\gamma(\varphi(|u|))\eta_\varepsilon^2\, dz.
\end{aligned}$$

Thanks to (2.1) and (2.4) we can see

$$\varphi'(|Du|)|Du| \ge \varphi'(|u|)|Du| - \varphi'(|u|)|u| \ge \varphi'(|u|)|Du| - c\,\varphi(|u|),$$

thus (3.7) can be written as

$$\begin{aligned}
&\int_{\mathcal{B}_R(x_0)} \fint_{\tau}^{\tau+\varepsilon} \psi_\gamma(|u|) 2\eta_\varepsilon \chi^{p_0}\, dz + \int_{\mathcal{Q}_R(z_0)} \varphi'(|u|)|Du|\, T_k^\gamma(\varphi(|u|))\,\eta_\varepsilon^2 \chi^{p_0}\, dz \\
&\quad\le \frac{c}{R-\rho}\int_{\mathcal{B}_R(x_0)} \int_{t_0-R^2}^{t_0-\rho^2} \psi_\gamma(|u|)\eta_\varepsilon \chi^{p_0}\, dz \\
&\qquad + \frac{c}{|R-\rho|^{p_0}}\int_{\mathcal{Q}_R(z_0)} \varphi(|u|)\, T_k^\gamma(\varphi(|u|))\eta_\varepsilon^2 \chi^{p_0}\, dz.
\end{aligned}$$

By passing to the limit as $\varepsilon \to 0$ we have

(3.8)
$$\begin{aligned}
&\sup_{\tau\in(t_0-R^2,t_0)} \int_{\mathcal{B}_R(x_0)} \psi_\gamma(|u|)\eta\,\chi^{p_0}\,dx + \int_{\mathcal{Q}_R(z_0)} \varphi'(|u|)|Du|T_k^\gamma(\varphi(|u|))\eta^2\,\chi^{p_0+1}\,dz \\
&\le \frac{c}{R-\rho}\int_{\mathcal{Q}_R(z_0)} \psi_\gamma(|u|)\eta\,\chi^{p_0}\,dz \\
&\quad + \frac{c}{|R-\rho|^{p_0}}\int_{\mathcal{Q}_R(z_0)} \varphi(|u|)\,T_k^\gamma(\varphi(|u|))\,\eta\,\chi^{p_0}\,dz.
\end{aligned}$$

Let us define the function

$$G(x,t) := \varphi(|u|)\,T_k^\gamma(\varphi(|u|))\,\eta^2(t)\chi^{p_0+1}(x).$$

We can see that

$$\begin{aligned}
|DG(x,t)| \le (\gamma+1)\,\varphi'(|u|)|Du|T_k^\gamma(\varphi(|u|))\eta^2(t)\chi^{p_0+1}(x) \\
+ \frac{p_0+1}{|R-\rho|}\,\varphi(|u|)\,T_k^\gamma(\varphi(|u|))\,\eta^2(t)\,\chi^{p_0}(x),
\end{aligned}$$

from which integrating over $\mathcal{Q}_R(z_0)$, and taking into account (3.8), we have

(3.9)
$$\begin{aligned}
&\int_{\mathcal{Q}_R(z_0)} |DG(x,t)|\,dz \\
&\le (\gamma+1)\int_{\mathcal{Q}_R(z_0)} \varphi'(|u|)|Du|\,T_k^\gamma(\varphi(|u|))\,\eta^2\chi^{p_0+1}\,dz \\
&\quad + \frac{p_0+1}{|R-\rho|}\int_{\mathcal{Q}_R(z_0)} \varphi(|u|)\,T_k^\gamma(\varphi(|u|))\,\eta^2\chi^{p_0}\,dz \\
&\le \frac{c(\gamma+1)}{R-\rho}\int_{\mathcal{Q}_R(z_0)} \psi_\gamma(|u|)\eta\chi^{p_0}\,dz + \frac{\bar{c}(\gamma+1)}{|R-\rho|^{p_0}}\int_{\mathcal{Q}_R(z_0)} \varphi(|u|)\,T_k^\gamma(\varphi(|u|))\,\eta^2\chi^{p_0}\,dz.
\end{aligned}$$

By Lemma 3.1 we get

$$\begin{aligned}
&\int_{\mathcal{B}_\rho(x_0)} T_k^\gamma(\varphi(|u|))\,\eta\,\chi^{p_0}\,dx \\
&\quad = \int_{\mathcal{B}_\rho(x_0)\cap\{|u|\le 1\}} T_k^\gamma(\varphi(|u|))\,\eta\,\chi^{p_0}\,dx + \int_{\mathcal{B}_\rho(x_0)\cap\{|u|\ge 1\}} T_k^\gamma(\varphi(|u|))\,\eta\,\chi^{p_0}\,dx \\
&\quad \le C|\,\mathcal{B}_\rho(x_0)| + \int_{\mathcal{B}_\rho(x_0)} |u|^2 T_k^\gamma(\varphi(|u|))\,\eta\,\chi^{p_0}\,dx \\
&\quad \le C|\,\mathcal{B}_\rho(x_0)| + \int_{\mathcal{B}_\rho(x_0)} \psi_\gamma(|u|)\,\eta\,\chi^{p_0}\,dx.
\end{aligned}$$

By using Hölder and Sobolev inequalities we can infer

$$
\begin{aligned}
&\int_{\mathcal{Q}_\rho(z_0)} \varphi(|u|) T_k^{\gamma+\frac{\gamma}{n}}(\varphi(|u|))\, dz \\
&\quad \le \int_{t_0-\rho^2}^{t_0} \left(\int_{\mathcal{B}_\rho(x_0)} T_k^{\gamma}(\varphi(|u|))\, \eta\, \chi^{p_0}\, dx \right)^{\frac{1}{n}} \\
&\qquad \times \left(\int_{\mathcal{B}_\rho(x_0)} [\varphi(|u|)\, T_k^{\gamma}(\varphi(|u|))\, \eta^2 \chi^{p_0+1}]^{\frac{n}{n-1}}\, dx \right)^{\frac{n-1}{n}} \\
&\quad \le \left(\sup_{t_0-\rho^2<t<t_0} \int_{\mathcal{B}_\rho(x_0)} \psi_\gamma(|u|) \eta \chi^{p_0}\, dx + C|\,\mathcal{B}_\rho(x_0)| \right)^{\frac{1}{n}} \int_{\mathcal{Q}_R(z_0)} |DG|\, dz \qquad (3.10)\\
&\quad \le \Bigg[1 + \Bigg(\frac{c(\gamma+1)}{R-\rho} \int_{\mathcal{Q}_R(z_0)} \psi_\gamma(|u|) \eta \chi^{p_0}\ \ dz \\
&\qquad\qquad + \frac{\bar{c}(\gamma+1)}{|R-\rho|^{p_0}} \int_{\mathcal{Q}_R(z_0)} \varphi(|u|)\, T_k^{\gamma}(\varphi(|u|))\, \eta^2 \chi^{p_0}\, dz \Bigg)^{1+\frac{1}{n}} \Bigg],
\end{aligned}
$$

where we use Lemma 3.1, (3.8), and (3.9).

To estimate the first term in the right-hand side, we use Lemma 3.1 and we get

$$
\begin{aligned}
&\int_{\mathcal{Q}_R(z_0)} \psi_\gamma(|u|) \eta \chi^{p_0}\, dz \\
&\qquad \le C \int_{\mathcal{Q}_R(z_0)} T_k^{\gamma}(\varphi(|u|)) |u|^2 \eta \chi^{p_0}\, dz \\
&\qquad = C \int_{\mathcal{Q}_R(z_0)\cap\{|u|\le\lambda_1\}} T_k^{\gamma}(\varphi(|u|)) |u|^2 \eta \chi^{p_0}\, dz \\
&\qquad\quad + C \int_{\mathcal{Q}_R(z_0)\cap\{|u|\ge\lambda_1\}} T_k^{\gamma}(\varphi(|u|)) |u|^2 \eta \chi^{p_0}\, dz \\
&\qquad \le C|\,\mathcal{Q}_R(z_0)| + C \int_{\mathcal{Q}_R(z_0)\cap\{|u|\ge\lambda_1\}} \varphi^2(|u|)\, T_k^{\gamma}(\varphi(|u|))\, dz, \qquad (3.11)
\end{aligned}
$$

where λ_1 is such that if $|u| \ge \lambda_1$, then $\varphi(|u|) \ge |u|$.

Putting together (3.10) and (3.11), and recalling the properties of η and χ, we have

$$
\begin{aligned}
&\int_{\mathcal{Q}_\rho(z_0)} \varphi(|u|) T_k^{\gamma+\frac{\gamma}{n}}(\varphi(|u|))\, dz \\
&\quad \le C \left[1 + \left(\frac{c(\gamma+1)}{R-\rho} |\,\mathcal{Q}_R(z_0)| + \frac{\bar{c}(\gamma+1)}{|R-\rho|^{p_0}} \int_{\mathcal{Q}_R(z_0)} \varphi(|u|)\, T_k^{\gamma}(\varphi(|u|))\, dz \right)^{1+\frac{1}{n}} \right],
\end{aligned}
$$

and taking the limit as $k \to \infty$ we obtain

(3.12)
$$\int_{\mathcal{Q}_\rho(z_0)} \varphi^{1+\gamma+\frac{\gamma}{n}}(|u|)\,dz \leq C\left[1+\left(\frac{c(\gamma+1)}{R-\rho}|\,\mathcal{Q}_R(z_0)|+\frac{\bar{c}(\gamma+1)}{|R-\rho|^{p_0}}\int_{\mathcal{Q}_R(z_0)}\varphi^{1+\gamma}(|u|)\,dz\right)^{1+\frac{1}{n}}\right].$$

Let $\sigma := 1+\frac{1}{n}$. For some $\gamma_0>0$, we set

$$\gamma_{i+1}=\gamma_i\sigma.$$

In particular,

$$\gamma_{i+1}=\gamma_0\sigma^{i+1}$$

and $\lim_{i\to+\infty}\gamma_i=+\infty$. Define $R_i := \frac{R_0}{2}(1+\frac{1}{2^i})$, and take $\rho=R_{i+1}$ and $R=R_i$ in (3.12). We also define $\Phi_i=\int_{\mathcal{Q}_{R_i}(z_0)}\varphi^{1+\gamma_i}(|u|)\,dz$ and $\beta_i=\gamma_i+1$. Thus we have

$$\Phi_{i+1}\leq 1+C^{i+1}\beta_i^\sigma+C^{i+1}\beta_i^\sigma\Phi_i^\sigma.$$

Iterating we get

$$\Phi_{i+1}\leq 1+C^{\sum_{k=0}^{i}(i-k+1)\sigma^k}\prod_{k=0}^{i}\beta_{i-k}^{\sigma^{k+1}}\,2^{\sum_{k=0}^{i}(\sigma^k-1)}\,\Phi_0^{\sigma^{i+1}} + \sum_{j=1}^{i} C^{\sum_{k=0}^{j}(i-k+1)\sigma^k}\prod_{k=0}^{j}\beta_{i-k}^{\sigma^{k+1}}\,2^{\sum_{k=0}^{j}(\sigma^k-1)}.$$

Now, taking into account that $\beta_{i-k}\leq(\gamma_0+1)\sigma^{i-k+1}$, we have

$$\begin{aligned}\log\left(\prod_{k=0}^{i}\beta_{i-k}^{\sigma^{k+1}}\right)&=\sum_{k=0}^{i}\log(\beta_{i-k}^{\sigma^{k+1}})=\sum_{k=0}^{i}\sigma^{k+1}\log(\beta_{i-k})\\&\leq\sum_{k=0}^{i}\sigma^{k+1}\log((\gamma_0+1)\sigma^{i-k+1})\\&=\log(\gamma_0+1)\sum_{k=0}^{i}\sigma^{k+1}+\log(\sigma)\sum_{k=0}^{i}\sigma^{k+1}(i-k+1)\leq c\sigma^{i+1}\end{aligned}$$

from which

$$\prod_{k=0}^{i}\beta_{i-k}^{\sigma^{k+1}}\leq e^{c\sigma^{i+1}}.$$

So we get

$$\Phi_{i+1}\leq C+K^{\sigma^{i+1}}\Phi_0^{\sigma^{i+1}}+K^{\sigma^{i+1}}(i+1).$$

Now,

$$\Phi_{i+1}^{\frac{1}{\gamma_{i+1}}}\leq C+K^{\frac{\sigma^{i+1}}{\gamma_{i+1}}}\Phi_0^{\frac{\sigma^{i+1}}{\gamma_{i+1}}}+K^{\frac{\sigma^{i+1}}{\gamma_{i+1}}}(i+1)^{\frac{1}{\gamma_{i+1}}}.$$

Recalling the definition of γ_{i+1}, we have that $\gamma_{i+1} \to \infty$, $\frac{\sigma^{i+1}}{\gamma_{i+1}} \to \frac{1}{\gamma_0}$ and $(i+1)^{\frac{1}{\gamma_{i+1}}} \to 1$ as $i \to \infty$. Therefore, we can infer that

$$\sup_{\mathcal{Q}_{\frac{R_0}{2}}(z_0)} \varphi(|u|) \leq c \left(\fint_{\mathcal{Q}_{R_0}(z_0)} \varphi^{\gamma_0+1}(|u|)\, dz \right)^{\frac{1}{\gamma_0}} + c$$

which ends the proof of theorem. □

Acknowledgments

The author warmly thanks the anonymous referee for useful and nice comments on the paper. The manuscript has been carried out under the auspices of the INDAM - Gnampa Project 2017 titled: *Teoria e modelli per problemi non locali.*

References

[1] E. Acerbi and N. Fusco, *Regularity for minimizers of nonquadratic functionals: the case* $1 < p < 2$, J. Math. Anal. Appl. **140** (1989), no. 1, 115–135. MR997847

[2] Verena Bögelein, Frank Duzaar, and Paolo Marcellini, *Existence of evolutionary variational solutions via the calculus of variations*, J. Differential Equations **256** (2014), no. 12, 3912–3942. MR3190487

[3] Verena Bögelein, Frank Duzaar, Paolo Marcellini, and Stefano Signoriello, *Nonlocal diffusion equations*, J. Math. Anal. Appl. **432** (2015), no. 1, 398–428. MR3371243

[4] Dominic Breit, Bianca Stroffolini, and Anna Verde, *A general regularity theorem for functionals with ϕ-growth*, J. Math. Anal. Appl. **383** (2011), no. 1, 226–233. MR2812732

[5] J. Burczak and P. Kaplicky, *Interior regularity of space derivatives to an evolutionary, symmetric φ-Laplacian*, Preprint arXiv:1507.05843 .

[6] Hi Jun Choe, *Hölder continuity for solutions of certain degenerate parabolic systems*, Nonlinear Anal. **18** (1992), no. 3, 235–243. MR1148287

[7] Andrea Dall'Aglio and Elvira Mascolo, *L^∞ estimates for a class of nonlinear elliptic systems with nonstandard growth* (English, with English and Italian summaries), Atti Sem. Mat. Fis. Univ. Modena **50** (2002), no. 1, 65–83. MR1910779

[8] Ennio De Giorgi, *Un esempio di estremali discontinue per un problema variazionale di tipo ellittico* (Italian), Boll. Un. Mat. Ital. (4) **1** (1968), 135–137. MR0227827

[9] Emmanuele DiBenedetto and Avner Friedman, *Regularity of solutions of nonlinear degenerate parabolic systems*, J. Reine Angew. Math. **349** (1984), 83–128. MR743967

[10] Emmanuele DiBenedetto and Avner Friedman, *Hölder estimates for nonlinear degenerate parabolic systems*, J. Reine Angew. Math. **357** (1985), 1–22. MR783531

[11] L. Diening, T. Scharle & S. Schwarzacher, *Regularity for parabolic systems of Uhlenbeck type with Orlicz growth*, Preprint arXiv:1603.05604.

[12] Lars Diening, Bianca Stroffolini, and Anna Verde, *Everywhere regularity of functionals with ϕ-growth*, Manuscripta Math. **129** (2009), no. 4, 449–481. MR2520895

[13] Enrico Giusti and Mario Miranda, *Un esempio di soluzioni discontinue per un problema di minimo relativo ad un integrale regolare del calcolo delle variazioni* (Italian), Boll. Un. Mat. Ital. (4) **1** (1968), 219–226. MR0232265

[14] Gary M. Lieberman, *Boundary regularity for solutions of degenerate parabolic equations*, Nonlinear Anal. **14** (1990), no. 6, 501–524, DOI 10.1016/0362-546X(90)90038-I. MR1044078

[15] Paolo Marcellini, *Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions*, Arch. Rational Mech. Anal. **105** (1989), no. 3, 267–284. MR969900

[16] Paolo Marcellini, *Regularity and existence of solutions of elliptic equations with p,q-growth conditions*, J. Differential Equations **90** (1991), no. 1, 1–30. MR1094446

[17] Paolo Marcellini, *Everywhere regularity for a class of elliptic systems without growth conditions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **23** (1996), no. 1, 1–25. MR1401415

[18] Paolo Marcellini and Gloria Papi, *Nonlinear elliptic systems with general growth*, J. Differential Equations **221** (2006), no. 2, 412–443. MR2196484

[19] Jürgen Moser, *A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations*, Comm. Pure Appl. Math. **13** (1960), 457–468. MR0170091

[20] M. M. Rao and Z. D. Ren, *Theory of Orlicz spaces*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 146, Marcel Dekker, Inc., New York, 1991. MR1113700

[21] K. Uhlenbeck, *Regularity for a class of non-linear elliptic systems*, Acta Math. **138** (1977), no. 3-4, 219–240. MR0474389

[22] Michael Wiegner, *On C_α-regularity of the gradient of solutions of degenerate parabolic systems*, Ann. Mat. Pura Appl. (4) **145** (1986), 385–405. MR886719

[23] Jiangsheng You, *Regularity of solutions of certain parabolic system with nonstandard growth condition*, Acta Math. Sinica (N.S.) **14** (1998), no. 2, 145–160. MR1704826

Dipartimento di Ingegneria Industriale e Scienze Matematiche, Università Politecnica delle Marche, Via Brecce Bianche, 12, 60131 Ancona, Italy

Email address: teresa.isernia@unina.it